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A NOTE ON THE REPRESENTATION TYPE OF POINTED IRREDUCIBLE COALGEBRAS AND UNIPOTENT ALGEBRAS

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1. Introduction. In this paper the representation type of the class of pointed irreducible coalgebras is studied. We refer the reader to [**4**] for the basic definitions. A coalgebra is of *bounded representation type* if there is a bound on the dimension of finite dimensional indecomposable comodules. In Section 1, we show that the representation type is dependent upon the size of the space of primitives. Indeed, a pointed irreducible coalgebra is of bounded type if and only if it is finite dimensional and the space of primitives is one-dimensional, i.e. if and only if it is a coalgebra spanned by a finite sequence of divided powers.

Section 2 examines unipotent algebras, relating them to pointed irreducible coalgebras, and categorizes them as to their representation type. An algebra is of *bounded representation type* if there is a bound on the dimension of finite dimensional indecomposable modules. Two examples show how the results [**2**, 64.1] and [**3**, 4.4] on the representation type of p-groups and p-nilpotent Lie algebras follow from the results in Section 1.

2. Representation type of pointed irreducible coalgebras. Let k be a field of arbitrary characteristic and let (C, Δ, ϵ) be a coalgebra over k. C is called *irreducible* if C has a unique simple subcoalgebra. C is *pointed* if each simple subcoalgebra is one-dimensional. A pointed irreducible coalgebra has a unique *grouplike* element g and kg is its coradical. C is *cocommutative* if for each $c \in C$, $\Delta(c) = T\Delta(c)$ where T is the "twist" map on $C \otimes_k C$, $T(c \otimes d) = d \otimes c$. Let g be a grouplike element of the coalgebra C. The set $P(C, g) = \{c \in C | \Delta(c) = c \otimes g + g \otimes c\}$ is a subspace (in fact a coideal) and is called the *space of g-primitives* of C. If C is pointed irreducible we write $P(C) \equiv P(C, g)$. It is easily seen that $kg + W$ is a subcoalgebra where W is a subspace of $P(C, g)$, and in addition, the sum is direct.

A right C-comodule (M, ψ) is called *indecomposable* if the rational C^*-module structure on M is indecomposable. Clearly if A is an associative algebra, then a locally finite A-module M is indecomposable if and only if M is an indecomposable A^0-comodule, where A^0 is the coalgebra dual of A. Let (M, ψ) be a simple C-comodule for any coalgebra C. M is isomorphic to a minimal right coideal of C and $\psi(M) \subseteq M \otimes_k \mathrm{Corad}(C)$ [**5**]. Thus C is pointed ir-

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reducible if and only if there exists a unique isomorphism class of rational irreducible C^*-modules, a representative of which is one-dimensional.

1.1 *Definition.* A coalgebra C is of *bounded representation type* if there exists $n \in \mathbf{Z}$ such that $\dim_k M < n$ for every indecomposable C-comodule M.

1.2 PROPOSITION. *Let C be a pointed irreducible coalgebra such that $\dim_k P(C) \geqq 2$. Then C is of unbounded type. In particular, C has indecomposable modules of every odd dimension.*

Proof. Let $c_1, c_2 \in P(C)$ be linearly independent and let $D = kg \oplus k\{c_1, c_2\}$ be the coalgebra spanned by c_1, c_2 and the unique grouplike g. Since every D-comodule is a C-comodule it is sufficient to prove the result for D. Let $M = k\{m_0, \ldots, m_n, n_1, \ldots, n_n\}$ be a vector space of dimension $2n + 1$. Let $\psi : M \to M \otimes D$ be defined on a basis by

$$\psi(m_i) = m_i \otimes g \quad i = 0, \ldots, n, \quad \text{and}$$
$$\psi(n_i) = m_i \otimes c_1 + m_{i-1} \otimes c_2 + n_i \otimes g \quad i = 1, \ldots, n.$$

It is easily shown that this defines a C-comodule structure; we show it is indecomposable.

Let $a, b \in D^*$ such that

$$\langle a - 1, c_1 \rangle = 1 = \langle b - 1, c_2 \rangle, \langle a - 1, g \rangle = \langle a - 1, c_2 \rangle$$
$$= \langle b - 1, g \rangle = \langle b - 1, c_1 \rangle = 0.$$

Since D is cocommutative, a and b commute and so the proof of 64.3 of [**2**] may be applied to M and a and b to yield the desired result.

1.3 COROLLARY. *Let C be any coalgebra and let g be a grouplike element of C such that $\dim_k P(C, g) \geqq 2$. Then C is of unbounded type.*

Let C be a coalgebra and let $C_0 = \text{Corad}(C)$. We use the *coradical filtration* $\{C_i\}_{i=0}^{\infty}$ on C defined in [**4**, 9.1] in the sequel. The set $\{c_0, c_1, \ldots\}$ (possibly finite) of elements of C is called a *sequence of divided powers* if c_0 is grouplike and $\Delta(c_n) = \sum_{i=0}^{n} c_i \otimes c_{n-i}$ for all n.

1.4 PROPOSITION. *Let C be a pointed irreducible coalgebra with unique grouplike g. Then C_n is an indecomposable C-comodule for all $n \geqq 0$. If, in addition, $\dim_k P(C) = 1$, then $\dim_k C_{n+1}/C_n \leqq 1$ for all $n \geqq 0$ and every finite dimensional subcoalgebra of C has a basis which is a sequence of divided powers.*

Proof. By the opening remarks of this section, since any subcomodule of C_n contains a minimal right coideal, it must contain kg. Thus C_n is not a direct sum of subcomodules.

Suppose $\dim_k P(C) = 1$. Then let n_0 be the smallest integer such that $\dim_k C_{n_0}/C_{n_0-1} > 1$. Clearly, $n_0 > 1$. Let $c_1, c_2 \in C_{n_0}$ be independent modulo C_{n_0-1} and let C' be the subcoalgebra generated by c_1, c_2 and C_{n_0-1}. Then C' is finite dimensional, pointed irreducible, $\dim_k C' = n_0 + 2$, $\dim_k P(C') = 1$ and $C_k' = C_k$ for $k < n_0$. We may thus assume that $C = C'$.

Let $R = C_0^{\perp}$ in C^*. By [**4**, 11.0.5], $(R^{k+1})^{\perp} = C_k$ and so $\dim_k C^*/R^{i+1} = \dim C_i = i + 1$ for $i = 0, \ldots, n_0$. In particular, $\dim R/R^2 = \dim P(C) = 1$. By choosing $x \in R \backslash R^2$, one can easily show that the set $\{1, x, \ldots, x^{n_0}\}$ is a basis for C^*, contradicting our original assumption. If we now construct a similar basis for each C_n^*, evidently the dual basis for C_n is a sequence of divided powers.

Last, if $D \subseteq C$ is any finite dimensional subcoalgebra, then $D \supseteq kg$. By induction, one can show that $D \supseteq C_n$ for every $n \leqq \dim_k D - 1$. Thus $D = C_k$ for some k and the result is established.

1.5 Theorem. *Let C be a pointed irreducible coalgebra. Then C is of bounded type if and only if* $\dim_k P(C) = 1$ *and C is finite dimensional.*

Proof. If $\dim_k P(C) \geqq 2$, then by 1.2, C is of unbounded type. If $\dim_k P(C) = 1$ and C is infinite dimensional, then 1.4 shows that C is of unbounded type. We shall show that if $\dim_k P(C) = 1$, and M is a finite dimensional indecomposable, then $M \cong C_n$ for some n.

Let (M, ψ) be a finite dimensional indecomposable comodule and let D be a subcoalgebra which is minimal with respect to the property that $\psi(M) \subseteq M \otimes D$. D is finite-dimensional and hence has a basis $\{c_0, \ldots, c_k\}$ of divided powers. Since D is minimal, there exists a $\lambda \in M^*$ and $0 \neq m \in M$ such that $(\lambda \otimes I)\psi(m) = c \notin k\{c_0, \ldots, c_{k-1}\}$. Since $(\lambda \otimes I)\psi$ is a D^*-module map and $D^* \cdot c = D$, $(\lambda \otimes I)\psi(D^* \cdot m) = D$ and $(\lambda \otimes I)\psi|_{D^* \cdot m}$ is injective (by comparing dimensions). Hence $(\lambda \otimes I)\psi$ is a split morphism and so $M = D^* \cdot m = C_k$.

3. Unipotent algebras. Let A be an associative algebra with identity over k. Let A^0 be the coalgebra dual and $R = (\mathrm{Corad}(A^0))^{\perp}$ in A. A is called *unipotent* if $A = kI \oplus R$. The following are easy consequences of the theory.

2.1 Proposition. *Let A be an algebra over a field k.*

(a) *If A is finite dimensional then R is the Jacobson radical.*

(b) *A is unipotent if and only if A^0 is pointed irreducible.*

(c) *If A is unipotent then $(R^n)^{\perp} = A^0_{n-1}$.*

2.2 Theorem. *Let A be a unipotent algebra. Then A is of bounded type if and only if* $\dim_k R/R^2 = 1$ *and A is finite dimensional.*

Proof. A^0 is pointed irreducible and evidently $\dim_k P(A^0) = \dim_k(R/R^2)$. Thus the conclusion follows from 1.5.

We conclude with two applications which yield some known results as corollaries. Let k have characteristic p.

2.3 *Example.* Let G be a finite p-group. Then kG is a Hopf algebra and ker $\epsilon_{kG} = k\{g - 1 | g \in G\}$ is nilpotent. Thus $R = \ker \epsilon_{kG}$ by 2.1 and $kG = kl \oplus R$, i.e. kG is unipotent. It is easily shown that $\dim_k R/R^2 = 1$ if and only if G is cyclic. Thus we have

2.4 COROLLARY. *A finite p-group is of bounded type if and only if it is cyclic.*

2.5 *Example.* Let $\mathscr{L}$ be a finite dimensional p-nilpotent restricted Lie algebra. Let $H = u(\mathscr{L})$ be the restricted enveloping algebra. H is unipotent since ker ϵ_H is generated by $\mathscr{L}$ and is thus nilpotent. Furthermore, $\dim_k R/R^2 = 1$ if and only if $\mathscr{L}$ is cyclic. Thus we obtain the following.

2.6 COROLLARY. *A finite dimensional p-nilpotent restricted Lie algebra is of bounded type if and only if it is cyclic.*

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