Canad. Math. Bull. Vol. **38** (4), 1995 pp. 496–506

DEFINITIZABLE OPERATORS ON A KREIN SPACE

PETR ZIZLER

ABSTRACT. Let A be a bounded linear operator on a Hilbert space H. Assume that A is selfadjoint in the indefinite inner product defined by a selfadjoint, bounded, invertible linear operator G on H; $[x, y] := (Gx, y)$. In the first part of the paper we define two orders of neutrality for the pair (G, A) and a connection is made with the "types" of numbers in the point and approximate point spectrum of A. The main results of the paper are in the second part and they deal with strong and uniform definitizability of a bounded selfadjoint operator on a Pontrjagin space. They state:

A) Let A be a bounded strongly definitizable operator on a Pontrjagin space Π_κ, then A is uniformly definitizable.

B) A bounded selfadjoint operator A on a Pontrjagin space Π_κ is uniformly definitizable if and only if all the eigenvalues of A are of definite type and all the nonisolated eigenvalues of A are of positive type.

Some applications to the theory of linear selfadjoint operator pencils are given.

1. **Introduction and main results.** This paper concerns characterizations and properties of definitizable operators on Krein spaces. We introduce these concepts by first considering a bounded, invertible selfadjoint operator G on a Hilbert space H; the inner product on H is denoted by $(.,.)$. A new, generally indefinite inner product $[.,.]$, is defined on H by letting $[x,y] = (Gx,y)$ for all $x,y \in H$. When G is indefinite the pair $(H,[.,.])$ is a Krein space.

A bounded linear operator A on H is G-selfadjoint if $GA = A^*G$, (here A^* is the Hilbert space adjoint of A). Such an operator is said to be *definitizable* if there exists a nonzero polynomial p such that $Gp(A) \geq 0$, (*i.e.* $Gp(A)$ is a positive semidefinite operator on H), and strongly or uniformly definitizable if $Gp(A) > 0$ or $Gp(A) \gg 0$, respectively. (The latter notation means that there is a $\delta > 0$ such that $\big(Gp(A)x, x\big) = [p(A)x, x] \geq \delta(x,x)$ for all $x \in H$.)

In the case of a finite dimensional Hilbert space H the notion of the *order of neutrality* of a G-selfadjoint operator A was introduced in [6]. It was defined to be the dimension of a maximal A-invariant G-neutral subspaces of H and denoted by $\gamma(G,A)$. Furthermore, it was proven that $\gamma(G,A) = 0$ if and only if the spectrum of A is of definite type. The case of a G-selfadjoint bounded operator A on an infinite dimensional Hilbert space was studied in the work of [5].

Let A be a bounded G-selfadjoint operator on H.

DEFINITION 1.1. Let $\lambda \in \sigma_p(A)$. Then λ is of positive (negative) type if $(Gf,f) > 0$ (< 0) for every nonzero $f \in \ker(\lambda - A)$.

The author would like to thank his PhD supervisor Prof. Lancaster for his guidance and financial support.
Received by the editors August 12, 1994; revised December 8, 1994.
AMS subject classification: Primary: 47B50; secondary: 47B15.

https://doi.org/10.4153/CMB-1995-072-7 Published online by Cambridge University Press

DEFINITION 1.2. Let $\lambda \in \sigma_{ap}(A)$. Then λ is of plus (minus) type if: for every normalized sequence $\{f_n\}$ such that $\|(\lambda - A)f_n\| \longrightarrow 0$ we have that $\underline{\lim}(Gf_n, f_n) > 0$ $\big(\overline{\lim}(Gf_n, f_n) < 0\big)$.

If $\lambda \in \sigma_p(A)$ is of positive or negative type then we say that λ is of definite type. Similarly if $\lambda \in \sigma_{ap}(A)$ is of plus or minus type then we say that λ is of determinate type. It is clear that eigenvalues of definite type have to be real. Points of determinate type have to be real as well and we can see this by the following argument of Langer. Let $\{x_n\}$ be a normalized sequence such that $\|(\lambda - A)x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Then we have that $\underline{\lim}[x_n, x_n] > 0$ or $\overline{\lim}[x_n, x_n] < 0$. We now observe that $\mathrm{Im}(-\lambda[x_n, x_n]) = \mathrm{Im}[(A - \lambda)x_n, x_n] \longrightarrow 0$. Thus $\mathrm{Im}(\lambda) = 0$ and hence $\lambda \in \mathbf{R}$.

Let $\lambda \in \sigma_p(A)$ then it is possible for λ to be of definite type and not of determinate type. But however, if $\lambda \in \sigma_p(A)$ is of determinate type then λ has to be of definite type. (For more details see [5]).

We define the order of neutrality for a G-selfadjoint bounded operator A on an infinite dimensional Hilbert space H and we establish its connection with the type of the point spectrum of A $\big(\sigma_p(A)\big)$ and the approximate point spectrum of A $\big(\sigma_{ap}(A)\big)$. Two orders of neutrality, $\gamma_p(G, A)$ and $\gamma(G, A)$ are defined in Section 2, and, in particular, it will be shown that:

i) $\gamma_p(G, A) = 0$ if and only if $\sigma_p(A)$ is real and of definite type,

ii) $\gamma(G, A) = 0$ if and only if $\sigma(A)$ is real and of determinate type,

where $\gamma_p(G, A)$ is the point order of neutrality and $\gamma(G, A)$ is the order of neutrality associated with the pair (G, A) (two concepts that coincide on a finite dimensional Hilbert space).

Strong and uniform definitizability of a bounded selfadjoint operator A on a Krein space is studied and the following result is obtained (generalization of Proposition 1 in [5] to operators with a k-th power compact):

iii) Let $(H, [\cdot, \cdot])$ be a Krein space. If there exists an operator A such that A is uniformly definitizable and A^k is compact for some positive integer k, then $(H, [\cdot, \cdot])$ is a Pontrjagin space.

The following are the main results of this paper. They are established in Sections 6–8 and are independent of the results in Sections 2–5.

A) Let A be a strongly definitizable bounded operator on a Pontrjagin space Π_κ, then A is uniformly definitizable.

B) A bounded selfadjoint operator A on a Pontrjagin space Π_κ is uniformly definitizable if and only if all the eigenvalues of A are of definite type and all the nonisolated eigenvalues of A are of positive type.

In Section 9 these results are interpreted in the context of selfadjoint operator pencils on H.

The ideas of strong and uniform definitizability have arisen, and are being utilized in the perturbation theory and in a number of other significant applications (see [4], [5], [6], [7], [8], [9] for example). These ideas require a careful study of different characterizations of strong and uniform definitizability. The result A) states that these two types of definitizability are equivalent for a bounded selfadjoint operator on a Pontrjagin space. Let A be a bounded selfadjoint operator on a Pontrjagin space. The result B) provides a nice and simple characterization of uniform definitizability of A just by understanding the eigenvalues of A.

Moreover, the above results, when applied to the theory of linear operator pencils, give a new characterization of certain class of quasihyperbolic (QHP) operator pencils, (see [5]).

All the linear operators considered in this paper will be bounded and G will denote a bounded selfadjoint invertible operator on Hilbert space H.

2. **Order of neutrality for a pair** (G, A). Let A be a G-selfadjoint operator. We begin this section with the following observation. Let $\lambda \in \sigma(A) \cap \mathbf{R}$, then $\lambda \in \sigma_{ap}(A)$. To see this we invoke the Corollary VI.6.2 of [1] stating that $\sigma(A) \cap \mathbf{R} \subset \sigma_c(A) \cup \sigma_p(A)$, where $\sigma_c(A)$ denotes the continuous spectrum of A, (the set of all $\lambda \in \mathbf{C}$ such that $\ker(\lambda I - A) = 0$ and range$(\lambda I - A)$ is dense in H). Clearly we have that $\sigma_c \cup \sigma_p \subset \sigma_{ap}$ and thus $\sigma(A) \cap \mathbf{R} \subset \sigma_{ap}(A)$.

We need the following technical lemma in order to define the order of neutrality for the pair (G, A).

LEMMA 2.1. *A point $\lambda \in \sigma_{ap}(A)$ is of indeterminate type if and only if there exists a sequence $\{x_n\}$ such that $\|x_n\| = 1$, $\|(A - \lambda)x_n\| \longrightarrow 0$ and $(Gx_n, x_n) \longrightarrow 0$.*

PROOF. Suppose $\lambda \in \sigma_{ap}(A)$ is of indeterminate type. Then the only nontrivial case to examine is the case when there exist normalized sequences $\{x_n\}$ and $\{y_n\}$ such that $\|(A - \lambda)x_n\| \longrightarrow 0$, $\|(A - \lambda)y_n\| \longrightarrow 0$ and $\underline{\lim}(Gx_n, x_n) = \delta_1 > 0$, $\overline{\lim}(Gy_n, y_n) = \delta_2 < 0$. Upon extracting subsequences and renaming them we have the existence of $\{x_n\}$, $\{y_n\}$ such that $\|x_n\| = 1$, $\|y_n\| = 1$, $\|(A - \lambda)x_n\| \longrightarrow 0$, $\|(A - \lambda)y_n\| \longrightarrow 0$, $(Gx_n, x_n) \longrightarrow \delta_1$, $(Gy_n, y_n) \longrightarrow \delta_2$ and $(Gx_n, x_n) \in (\delta_1 - \varepsilon, \delta_1 + \varepsilon)$; $(Gy_n, y_n) \in (\delta_2 - \varepsilon, \delta_2 + \varepsilon)$ for every n with $0 < \varepsilon < \frac{\delta_1 - \delta_2}{4}$.

Moreover we can assume that $(x_n, y_n) \in \mathbf{R}$ for every n. To see this we choose $\theta_n \in [0, 2\pi)$ such that $(x_n, e^{i\theta_n} y_n) \in \mathbf{R}$ and consider the sequence $\{e^{i\theta_n} y_n\}$ instead of $\{y_n\}$. Let $w_n' = y_n - (y_n, x_n)x_n$. Note that $w_n' \neq 0$ and $w_n' \in \mathrm{span}_{\mathbf{R}}\{x_n, y_n\}$. Let $w_n = \frac{w_n'}{\|w_n'\|}$ and $\alpha_n(t) = (\cos t)x_n + (\sin t)w_n$.

Now define a map: $f: \mathbf{S} \longrightarrow \mathbf{R}$; $f: x \longmapsto (Gx, x)$, where $\mathbf{S}$ is the unit sphere in the Hilbert space H. Consider $f \circ \alpha_n : [0, \frac{\pi}{2}) \longrightarrow \mathbf{R}$. Note that: $f \circ \alpha_n(0) = (Gx_n, x_n) > 0$ and $f \circ \alpha_n(t_n) = (Gy_n, y_n) < 0$, for some $0 < t_n < \frac{\pi}{2}$. By the intermediate value theorem there exists $h_n \in (0, t_n)$ such that $f \circ \alpha_n(h_n) = 0$. Let $z_n = \alpha_n(h_n)$ and the above states that $(Gz_n, z_n) = 0$. Since $|(Gx_n, x_n) - (Gy_n, y_n)| > \frac{\delta_1 - \delta_2}{2}$, we have that $|(x_n, y_n)|^2 \leq \sigma < 1$ for every n with $0 \leq \sigma < 1$. We now have that $\frac{1}{\|w_n'\|} \leq \frac{1}{1-\sigma}$ and $\|(A - \lambda)z_n\| \leq (1 + \frac{1}{1-\sigma})\big(\|(A - \lambda)x_n\| + \|(A - \lambda)y_n\|\big)$ for every n. This implies that $\{z_n\} \subset \mathbf{S}$, $\|(A - \lambda)z_n\| \longrightarrow 0$ and $(Gz_n, z_n) = 0$ for every n.

The converse direction of this lemma is immediate. ∎

https://doi.org/10.4153/CMB-1995-072-7 Published online by Cambridge University Press

Let us introduce the following Banach space of sequences in H (see [12]):

$$l_\infty(H) = \left\{x = (x_i) \mid x_i \in H \,\forall i \in \mathbf{N} \text{ and } \sup_i \|x_i\| < \infty\right\}$$

with the norm $\|x\| = \sup_i \|x_i\|$. Consider the subspace $\Xi \subset l_\infty(H)$ given by $\Xi = \{x \in l_\infty(H) \mid \lim_i \|x_i\| = 0\}$. We have that Ξ is a closed subspace of $l_\infty(H)$. Hence we can form the quotient Banach space $\tilde{H} = l_\infty(H)/\Xi$. The Banach space $\tilde{H}$ is equipped with the quotient norm $\|\tilde{x}\| = \inf_{y\in\Xi}(\sup_i \|x_i - y_i\|)$, where $\tilde{x} = [(x_i)] \in \tilde{H}$ and $y = (y_i)$. We observe that $\|\tilde{x}\| = \overline{\lim}_i \|x_i\|$, see [12] for example. Consider now the following map:

$$\tilde{A}\colon \tilde{H} \longrightarrow \tilde{H} \text{ given by } \tilde{A}\colon \tilde{x} \longmapsto [(Ax_i)], \qquad \text{where } \tilde{x} = [(x_i)].$$

Observe that this map is well defined and $\tilde{A}$ is a bounded linear operator with the norm $\|\tilde{A}\| = \|A\|$. It is easy to see that the operator A is invertible if and only if $\tilde{A}$ is invertible. This implies that $\sigma(A) = \sigma(\tilde{A})$. It follows from the above construction that $\sigma_{ap}(A) = \sigma_p(\tilde{A})$. Let A be a G-selfadjoint operator on H. The operators A and G determine bounded linear operators $\tilde{A}$ and $\tilde{G}$ on $\tilde{H}$. We make the following definition:

DEFINITION 2.2. A subspace $\tilde{S} \subset \tilde{H}$ is said to be *$\tilde{G}$-neutral* if $\lim_i (Gx_i, x_i) \longrightarrow 0$ for every $\tilde{x} = [(x_i)] \in \tilde{S}$.

This will allow us to define the *order of neutrality* for the pair (G, A) in the following way:

DEFINITION 2.3. Let $\gamma(G, A) := \sup_{\tilde{S}\in\tilde{\Omega}} \dim \tilde{S}$, where $\tilde{\Omega}$ is the set of all $\tilde{A}$-invariant, $\tilde{G}$-neutral, finite dimensional subspaces of $\tilde{H}$.

We can now state:

THEOREM 2.4. *Let A be a selfadjoint operator on a Krein space. The order of neutrality of A with respect to G is zero if and only if $\sigma(A)$ is of determinate type.*

PROOF. Suppose that $\gamma(G, A) \neq 0$, then there exists a finite dimensional subspace $\tilde{S}$ of $\tilde{H}$ which is $\tilde{A}$-invariant and $\tilde{G}$-neutral. Let $\lambda \in \sigma_p(\tilde{A}|\tilde{S})$and it is readily seen that λ is of indeterminate type. Conversely, assume that $\gamma(G, A) = 0$. It follows that if $\lambda \in \sigma_{ap}(A)$ then λ is of determinate type. ∎

To obtain the parallel result for the point spectrum we introduce the *point order of neutrality*. Recall that a subspace $S \subset H$ is said to be *G-neutral* if $(Gx, x) = 0$ for every $x \in S$.

DEFINITION 2.5. Let $\gamma_p(G, A) := \sup_{S\in\Omega} \dim S$, where Ω is the set of all finite dimensional A-invariant G-neutral subspaces of H.

We obtain the following characterization of the point spectrum of A.

THEOREM 2.6. *Let A be a selfadjoint operator on a Krein space. The point order of neutrality of A with respect to G is zero if and only if $\sigma_p(A)$ of definite type.*

PROOF. Suppose $\gamma_p(G,A) \neq 0$. Then there exists a nontrivial finite dimensional subspace $S \subset H$ which is A-invariant and G-neutral. Let $\lambda \in \sigma_p(A|S)$. Observe that λ is of indefinite type. Conversely, let $\lambda \in \sigma_p(A)$ be of indefinite type and consider:

CASE i) $\lambda \notin \mathbf{R}$. Let $S = \operatorname{span}\{x_0\}$, where $x_0 \in \ker(\lambda - A) \setminus \{0\}$. Clearly $S \in \Omega$.

CASE ii) $\lambda \in \mathbf{R}$. If there exists $x_0 \in \ker(\lambda - A) \setminus \{0\}$ such that $(Gx_0, x_0) = 0$ then we are finished. Otherwise there exists $x_1 \in \ker(\lambda - A) \setminus \{0\}$ such that $(Gx_1, x_1) < 0$ and there exists $x_2 \in \ker(\lambda - A) \setminus \{0\}$ such that $(Gx_2, x_2) > 0$. By the intermediate value theorem we conclude that there is a nonzero x_0 such that $(Gx_0, x_0) = 0$ and $x_0 \in \ker(\lambda - A)$. Let $S = \operatorname{span}\{x_0\}$. Clearly $S \in \Omega$. ∎

REMARK. We have the following inequality: $\gamma(G,A) \geq \gamma_p(G,A)$. To see this, let S be a n -dimensional, A-invariant, G-neutral subspace of H. Let $e_1, \ldots, e_n$ be a basis for S and let $\tilde{S}$ be the subspace of $\tilde{H}$ given by $\tilde{S} = \operatorname{span}\{\widetilde{e_1}, \ldots, \widetilde{e_n}\}$, where $\widetilde{e_i} = [(e_i)]$ with $1 \leq i \leq n$. It is easily seen that $\tilde{S}$ is a n-dimensional, $\tilde{A}$-invariant, $\tilde{G}$-neutral subspace of $\tilde{H}$. As the following examples will show, the strict inequality can be achieved.

3. Examples.

We shall illustrate some of the notions with two examples.

EXAMPLE 3.1. Let H be a separable infinite dimensional Hilbert space. Let A and G be bounded linear operators on H given by

$$A = \operatorname{diag}\left[0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right], \quad G = \operatorname{diag}[-1, 1, -1, 1, \ldots]$$

It is seen immediately that A is G-selfadjoint and $\sigma(A) = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\} = \sigma_p(A)$. Let $\tilde{x} = [\{x_i\}]$, where $x_i = (0, 0 \cdots 1, 1, 0, 0)$ with the first "1" in the i-th position. Then $\|\tilde{x}\| = \sqrt{2}$ and $\tilde{A}\tilde{x} = 0$. Moreover $\lim_i (Gx_i, x_i) = 0$. Thus $\operatorname{span}\{\tilde{x}\} \in \Omega$ and $\gamma(G,A) > 0$. However $\gamma_p(G,A) = 0$ since each eigenvalue of A is of definite type.

EXAMPLE 3.2. Let S be the unilateral shift on a separable Hilbert space ([2]). $S\colon (x_1, x_2, \ldots) \longmapsto (0, x_1, x_2, \ldots)$. We have that $\sigma(S) = \bar{\mathbf{D}}$; $\sigma_p(S) = \emptyset$ and $\sigma_{ap}(S) = \partial \mathbf{D}$, where $\mathbf{D}$ is the unit disk in the complex plane $\mathbf{C}$. Note that $S^*\colon (x_1, x_2, \ldots) \longmapsto (x_2, x_3, \ldots)$. As in [10] define the G-selfadjoint operator A on $H \oplus H$ by:

$$A = \begin{bmatrix} S & 0 \\ 0 & S^* \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Note that $GA = A^*G$, $\sigma(A) = \bar{\mathbf{D}}$, $\sigma_p(A) = \mathbf{D}$ and $\sigma_{ap}(A) = \bar{\mathbf{D}} = \sigma(A)$. For $n \geq 2$ consider the sequence of subspaces of H defined by $C_n = \bigoplus_{k=2}^n C_k$, where $C_k = \operatorname{span}\{(0, x_k)\}$ and $x_k = (1, \frac{1}{k}, \frac{1}{k^2}, \ldots)$. We note that $\dim C_n = n - 1$, C_n is A-invariant and G -neutral for every n, and thus $\gamma_p(G,A) = \infty$. Using the remark above we conclude that $\gamma(G,A) = \infty$.

4. **Order of neutrality and strong definitizability.** Suppose H is a finite dimensional Hilbert space. Then A is strongly definitizable with respect to G if and only if all the eigenvalues of A are of definite type (see [6]). If H is an infinite dimensional Hilbert space and A is strongly definitizable with respect to G, then we have that $\sigma(A) \subset \mathbf{R}$ and all the eigenvalues of A are of definite type. To see that the spectrum of A has to be real, we invoke the result of Langer stating that a definitizable operator A on a Krein space has only a finite number of nonreal spectral points and these points are eigenvalues of A (see [10]). Moreover, it is not difficult to see that if A is strongly definitizable then all the eigenvalues of A have to be real and of definite type. Thus we can now conclude that $\sigma(A) \subset \mathbf{R}$ and $\gamma_p(G, A) = 0$.

However, it is not true that the assumptions $\gamma_p(G, A) = 0$ and $\sigma(A) \subset \mathbf{R}$ imply that A is strongly definitizable, even if we additionally assume that $\sigma(A) = \sigma_p(A)$. Let us return to the Example 3.1.

Consider H, G, A as in Example 3.1. Recall that $\gamma_p(G, A) = 0$. We will show that A is not definitizable. Suppose A is definitizable and let $p(\lambda)$ be a definitizing polynomial for A. Then we have

$$Gp(A) = G\operatorname{diag}\left[p(0), p(1), p\left(\frac{1}{2}\right), \ldots\right] = \operatorname{diag}\left[-p(0), p(1), -p\left(\frac{1}{2}\right), \ldots\right] \geq 0,$$

which contradicts the fact that a polynomial has a finite number of zeros.

Having $\gamma_p(G, A) = 0$ for a selfadjoint operator A on a Pontrjagin space also does not imply that A is strongly definitizable either. A simple example for this is $A = \operatorname{diag}[0, 1, \frac{1}{2}, \ldots]$ and $G = \operatorname{diag}[-1, 1, 1, \ldots]$. Note that we have $\sigma(A) = \sigma_p(A)$ in this example as well.

5. **Algebraic operators.** A bounded linear operator A on a Banach space X is said to be *algebraic* if $p(A) = 0$ for some nonzero polynomial p. The following results are natural generalizations of the results of [6] in finite dimensions.

PROPOSITION 5.1. *Let A be a G-selfadjoint operator on H and suppose that A is algebraic. Then the following are equivalent:*

a) A is strongly definitizable;

b) $\gamma_p(G, A) = 0$.

PROOF. We have $H = \ker(\lambda_1 - A)^{m_1} \oplus \cdots \oplus \ker(\lambda_k - A)^{m_k}$ (see [11]). The result then follows from the results of [9] and observing that definite eigenvalues are semisimple. ∎

Note that in this case $\gamma_p(G, A)$ can be calculated in the following way (see [6]). Let $\mathbf{E}_1, \ldots, \mathbf{E}_k$ be the spectral subspaces of A corresponding to either distinct real eigenvalues of mixed type or a conjugate pair of nonreal eigenvalues. Then $\gamma_p(G, A) = \sum_{i=1}^{k} \gamma_{p,j}$, where $\gamma_{p,j} = \gamma_p(G_j, A_j)$ with $G_j = G|\mathbf{E}_j$ and $A_j = A|\mathbf{E}_j$.

The results in the following sections deal with the notions of strong and uniform definitizability of a bounded selfadjoint operator on a Krein space and its applications to linear selfadjoint operator pencils. The notions of the order of neutrality developed in the previous sections are not used here.

6. **Operators with a k-th power compact.** The following lemma is a key step to generalization of Proposition 1 in [5] to operators with a k-power compact.

LEMMA 6.1. *Let A be an uniformly definitizable operator on H (with respect to G). Then for any positive integer k there exists a real polynomial $p(\lambda) = p_0 + p_{k+1}\lambda^{k+1} + \cdots$ such that $Gp(A) \gg 0$.*

PROOF. Let $k \in \mathbf{N}_+$ be given.

CASE i). Suppose that A is invertible. Let $q(\lambda)$ be a real uniformly definitizing polynomial for A (such a choice is possible, see [5]). Thus there exists a $\delta > 0$ such that $\big(Gq(A)f,f\big) \geq \delta\|f\|^2$ for every $f \in H$. Let $f \in H$ and note that $\big(Gq(A)A^{2(k+1)}f,f\big) = \big(Gq(A)A^{k+1}f,A^{k+1}f\big) \geq \delta\|A^{k+1}f\|^2 \geq \delta\frac{1}{\|A^{-(k+1)}\|^2}\|f\|^2$. Thus we have that $Gq(A)A^{2(k+1)} \gg 0$ and the lemma follows with $p(\lambda) = q(\lambda)\lambda^{2(k+1)}$.

CASE ii). Suppose that A is not invertible. Let $q(\lambda) = q_0 + q_1\lambda + \cdots + q_l\lambda^l$ be a real uniformly definitizing polynomial for A, hence there exists a $\delta > 0$ so that $\big(Gq(A)f,f\big) \geq \delta\|f\|^2$ for every $f \in H$. Observe that $q_0 \neq 0$ since $Gq(A)$ is invertible. Suppose k is odd. It is not difficult to see that we can choose a real polynomial $r_0 + \cdots + r_k\lambda^k$ so that for every $a \in \mathbf{R}$ the polynomial $r_a(\lambda) = r_0 + \cdots + r_k\lambda^k + a\lambda^{k+1}$ has the property that the coefficients of $\lambda, \lambda^2, \ldots, \lambda^k$ in $q(\lambda)r_a^2(\lambda)$ are all zero and $r_0 = 1$. Note that the $\deg\big(r_a(\lambda)\big)$ is even. Similarly, in the case of k even we choose a real polynomial $r_0 + \cdots + r_{k+1}\lambda^{k+1}$ so that for every $a \in \mathbf{R}$ the polynomial $r_a(\lambda) = r_0 + \cdots + r_k\lambda^k + r_{k+1}\lambda^{k+1} + a\lambda^{k+2}$ has the property that the coefficients of $\lambda, \ldots, \lambda^k$ in $q(\lambda)r_a^2(\lambda)$ are all zero and $r_0 = 1$. Note that the $\deg\big(r_a(\lambda)\big)$ is even.

Since $r_0 \neq 0$ and the degree of $r_a(\lambda)$ is even in both cases we can choose $a \in \mathbf{R}$ (recall that a is an arbitrary real parameter in the above construction) so that $r_a(\lambda)$ has no real zeros. To see this we consider the polynomial $r_1(\lambda) = 1 + \cdots + r_{l-1}\lambda^{l-1} + \lambda^l$ (where $l = k+1$ if k is odd and $l = k+2$ if k is even). Note that $r_1(0) = 1 > 0$. Clearly we can choose $a \in \mathbf{R}$ large enough so that $r_1(\lambda) + (a-1)\lambda^l = r_a(\lambda)$ has no real zeros.

Since the spectrum of an uniformly definitizable operator is real, (see [5]), it follows that $0 \notin \sigma\big(r_a(A)\big) = r_a\big(\sigma(A)\big)$. Let $f \in H$ and note that $\big(Gq(A)r_a^2(A)f,f\big) = \big(Gq(A)r_a(A)f, r_a(A)f\big) \geq \delta\|r_a(A)f\|^2 \geq \delta\frac{1}{\|(r_a(A))^{-1}\|^2}\|f\|^2$. Thus $Gq(A)r_a^2(A) \gg 0$ and the lemma now follows with $p(\lambda) = q(\lambda)r_a(\lambda)^2$. ∎

We remark that p_{k+1} or p_0 could be zero in the above lemma.

We now state the generalization of Proposition 1 of [5] to operators with a k-power compact.

PROPOSITION 6.2. *Let $(H, [.,.])$ be a Krein space. If there exists an operator A on H such that A is uniformly definitizable and A^k is compact for some positive integer k, then $(H, [.,.])$ is a Pontrjagin space.*

PROOF. Let $p(\lambda)$ be a real uniformly definitizing polynomial for A such that $p(\lambda) = p_0 + p_k\lambda^k + \cdots = p_0 + \lambda^k q(\lambda)$ (see Lemma 6.1). Observe that $A^k q(A)$ is compact. The theorem now follows from Proposition 1 in [5]. ∎

7. **Definitizability of algebraic operators.**

PROPOSITION 7.1. *Let A be a strongly definitizable G-selfadjoint operator on H and assume that $\sigma(A) = \sigma_p(A)$. Suppose that $p(\lambda)$ is a strongly definitizing polynomial for A, then $Gp(A) \gg 0$.*

PROOF. Let $p(\lambda)$ be a strongly definitizing polynomial for A. We claim that $0 \notin \sigma\big(p(A)\big)$. Suppose not, then there exists $\lambda \in \sigma(A) = \sigma_p(A)$ so that $p(\lambda) = 0$. Pick a nonzero $f \in \ker(A - \lambda)$. But then $\big(Gp(A)f, f\big) = p(\lambda)(Gf, f) = 0$ and this contradicts the fact that $Gp(A) > 0$. Thus $p(A)$ is invertible and hence $p(\lambda)$ is an uniformly definitizable polynomial for A. ∎

This yields the following result for algebraic operators:

THEOREM 7.2. *Let A be an algebraic G-selfadjoint operator on H. Assume that A is strongly definitizable, then A is uniformly definitizable.*

PROOF. Note that $\sigma_p(A) = \sigma(A)$ and apply the previous proposition. ∎

The following sections contains the main results of the paper and they are independent of our previous development.

8. **Definitizability on a Pontrjagin space.** Let A be a selfadjoint operator on a Pontrjagin space Π_κ whose inner product is generated by G. Assume without loss of generality that rank $G_- = \kappa < \infty$. It is a well known fact that A is necessarily definitizable (see [1] or [10]). It is also known that, on such a space, strong and uniform definitizability are equivalent for compact operators, see [5] and [3]. Here, this result is generalized to include all selfadjoint operators on Π_κ.

THEOREM 8.1. *Let A be a strongly definitizable operator on a Pontrjagin space Π_κ, then A is uniformly definitizable.*

PROOF. Let A be strongly definitizable. Note that $\sigma(A) = \sigma_{ap}(A)$, since the spectrum of a strongly definitizable operator is real. Let $\lambda \in \sigma(A)$ and consider a sequence $\{f_n\}$ such that $\|f_n\| = 1$ and $\|(A - \lambda)f_n\| \longrightarrow 0$. By the Theorem of Pontrjagin (see Theorem IX.7.2 of [1]), there exists a nonpositive κ-dimensional A-invariant subspace L_- of Π_κ. Moreover, each eigenvalue of A is of definite type and hence L_- is a G-negative subspace. Then its G-orthogonal complement L_+ is G-positive and hence uniformly positive (see Theorem V.6.3 of [1]). We have the following G-orthogonal decomposition of H:

$$H = L_- + L_+$$

with L_+ also an A-invariant subspace. Consider $f_n = f_n^- + f_n^+$, where $f_n^- \in L_-$ and $f_n^+ \in L_+$. Note that $\|f_n^-\| = \|P_- f_n\| \leq \rho$ for every n, where $\rho > 0$ and P_- is the projection on L_- along L_+. Thus $\{f_n^-\}$ is a bounded sequence in the finite dimensional space L_-. Furthermore we have that $\|(A - \lambda)f_n^-\| = \|(A - \lambda)P_- f_n\| = \|P_-(A - \lambda)f_n\| \longrightarrow 0$ and $\|(A - \lambda)f_n^+\| = \|(A - \lambda)P_+ f_n\| = \|P_+(A - \lambda)f_n\| \longrightarrow 0$, where P_+ is the projection on L_+ along L_-.

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Let $\lambda \in \sigma_p(A|L_-)$. Then there exists a nonzero $g_0 \in L_-$ such that $(A-\lambda)g_0 = 0$. Let $p(\lambda)$ be a strongly definitizing polynomial for A and consider $\big(Gp(A)g_0, g_0\big) = p(\lambda)(Gg_0, g_0)$. This implies that $p(\lambda) < 0$ since $(Gg_0, g_0) < 0$. We now have that $\big|\big(Gp(A)f_n^+, f_n^+\big) - p(\lambda)(Gf_n^+, f_n^+)\big| \longrightarrow 0$ and since $\big(Gp(A)f_n^+, f_n^+\big) \geq 0$ for every n and $p(\lambda) < 0$, it follows that $(Gf_n^+, f_n^+) \longrightarrow 0$. This implies that $\|f_n^+\| \longrightarrow 0$ and since $\|f_n\| = 1$ for every n, we have that $\|f_n^-\| \longrightarrow 1$. It follows now that $\overline{\lim}(Gf_n, f_n) = \overline{\lim}(Gf_n^-, f_n^-) < 0$. In particular we have shown that if $\lambda \in \sigma_p(A|L_-)$ then λ is of minus type.

Let $\lambda \in \sigma(A)\setminus\sigma_p(A|L_-)$. We will show that λ is of determinate type. Suppose not, then there exists a normalized sequence $\{f_n\}$ such that $\|(A-\lambda)f_n\| \longrightarrow 0$ and $(Gf_n, f_n) \longrightarrow 0$. Consider $f_n = f_n^+ + f_n^-$ as above. Since $\{f_n^-\}$ is a bounded sequence in the finite dimensional subspace L_- and $\lambda \notin \sigma_p(A|L_-)$, it follows that there exists a subsequence $\{f_{n_k}^-\} \longrightarrow 0$. This implies that $\|f_{n_k}^+\| \longrightarrow 1$ and since $(Gf_{n_k}, f_{n_k}) = (Gf_{n_k}^+, f_{n_k}^+) + (Gf_{n_k}^-, f_{n_k}^-)$ we obtain that (Gf_{n_k}, f_{n_k}) does not converge to zero. But this contradicts the fact that $(Gf_n, f_n) \longrightarrow 0$. Thus λ has to be of determinate type. Moreover, it is now easy to see that λ has to be of plus type. All the spectral points of A are now of determinate type and the uniform definitizability of A follows from Theorem 1 of [5]. ■

The preceding proof reveals some structural properties of the spectrum of a strongly definitizable operator A on a Pontrjagin space. All the spectral points of A are of plus type with the exception of a finite number of eigenvalues of minus type (the actual number does not exceed κ). These eigenvalues are isolated since the distance between the points of plus and minus type must be positive (see [5]). Furthermore, A is d-uniformly definitizable and $d \leq 2\kappa + 1$, (the operator A is said to be *d-uniformly definitizable* if the least degree among all polynomials p for which $Gp(A) \gg 0$ is $d-1$, for reference see [5] and [9]).

It was shown in Section 4 that $\gamma_p(G, A) = 0$ alone does not imply that A is strongly definitizable. The following result provides a necessary and sufficient condition on the type of the point spectrum so that A is strongly (hence uniformly) definitizable.

THEOREM 8.2. *Let A be a selfadjoint operator on a Pontrjagin space. Then the following are equivalent:*

a) All the eigenvalues of A are of definite type and all the nonisolated eigenvalues of A are of positive type;

b) A is uniformly definitizable.

PROOF. b) $\Rightarrow$ a). The spectrum of an uniformly definitizable operator is real and hence $\sigma(A) = \sigma_{ap}(A)$. It is immediate that all the eigenvalues of A are of definite type. It follows from the proof of the preceding theorem that all the nonisolated points in $\sigma(A)$ are of plus type, and hence all the nonisolated eigenvalues of A are of positive type.

a) $\Rightarrow$ b). Recall that A is definitizable. Using a well known result of Langer, see [10], the spectrum of A has only a finite number of nonreal points and furthermore these points are eigenvalues of A . It is clear that a nonreal eigenvalue of A can not be of definite type and hence we have that $\sigma(A) \subset \mathbf{R}$ and $\sigma(A) = \sigma_{ap}(A)$. Let L_-, L_+, P_-, P_+ be as in the preceding theorem. Consider an eigenvalue λ of A of a negative type. It follows that $\lambda \in \sigma_p(A|L_-)$ and by the hypothesis of the theorem, λ is an isolated spectral point of A. Consider the Riesz projector $P_\lambda = \int_\Gamma R_\xi d\xi$, where Γ is a sufficiently small circle centered at λ and R_ξ is the resolvent of A. Note that P_+ commutes with A and hence P_+ commutes with P_λ. Moreover we have that $\operatorname{Im} P_\lambda \cap \operatorname{Im} P_+ = 0$. To see this consider $S = \operatorname{Im} P_\lambda \cap \operatorname{Im} P_+$. Note that S is a closed, A-invariant subspace and $\sigma(A|S) = \{\lambda\}$. Now since L_+ is uniformly positive we have that $A|S$ is a selfadjoint operator on a Hilbert space. The spectral theorem for a selfadjoint operator on a Hilbert space implies that $\lambda \in \sigma_p(A|S)$. Since λ is of negative type we have that $S = \{0\}$. This implies that $P_+P_\lambda = 0$ and since $P_- + P_+ = I$ we obtain that $P_-P_\lambda = P_\lambda$. Now it follows that $\operatorname{Im} P_\lambda \subset \operatorname{Im} P_-$ and hence λ is an eigenvalue of finite type. Referring to the Remark 1 in [5] we conclude that λ is of minus type.

Suppose now that λ is not an eigenvalue of negative type. It follows from the proof of the preceding theorem that λ has to be of plus type. Every spectral point of A is now of determinate type and the result follows from Theorem 1 of [5]. ∎

9. **Selfadjoint linear pencils.** Let $L(\lambda) = \lambda G - A$ be a selfadjoint linear pencil with G and A both selfadjoint operators on a Hilbert space H. Furthermore assume that G is an invertible operator. The notion of quasihyperbolic selfadjoint operator polynomials (QHP) was introduced in the work of [5]. This was defined for monic selfadjoint operator polynomials. It is readily seen that this definition can be extended to selfadjoint operator polynomials with invertible leading coefficient and in particular to a selfadjoint linear pencil with invertible leading coefficient.

DEFINITION 9.1. Let $L(\lambda) = \lambda G - A$ be a selfadjoint linear pencil with G invertible. The linear pencil $L(\lambda)$ is said to be *quasihyperbolic* if its spectrum is real and of determinate type.

For the definition of spectral points of $L(\lambda)$ of determinate type and eigenvalues of definite type see the work of [5] and [6]. The selfadjoint linear pencil of the form $\lambda F - G$ where F is compact and G is invertible was studied in the work of [8]. The results of Section 8 give rise to the following theorem.

THEOREM 9.2. *Let $L(\lambda) = \lambda G - A$ be a selfadjoint linear pencil with G invertible and* rank $G_- < \infty$. *Then the following are equivalent:*

a) All the eigenvalues of $L(\lambda)$ are of definite type and all the nonisolated eigenvalues of $L(\lambda)$ are of positive type;

b) $L(\lambda)$ is quasihyperbolic.

PROOF. The result follows directly from Theorem 8.2 of the preceding section and Theorem 6 of [5]. ∎

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Department of Mathematics and Statistics
The University of Calgary
Calgary, Alberta
T2N 1N4
e-mail: zizler@acs.ucalgary.ca

https://doi.org/10.4153/CMB-1995-072-7 Published online by Cambridge University Press