4
Space spinors

This chapter discusses a framework for spinors in which a further structure is introduced – a so-called *Hermitian inner product*. The resulting formalism will be referred to as ***space spinors*** or $SU(2,\mathbb{C})$***-spinors***. The space spinor formalism can be used to describe the geometry of three-dimensional Riemannian manifolds and, more generally, foliations of spacetime. Moreover, it can also be used to provide a description of the hyperplanes associated to a congruence of timelike curves.

The notion of space spinors was first introduced in Sommers (1980); see also Sen (1981). It provides a systematic approach to the construction of evolution equations which can be regarded as a spinorial version of the $1+3$ formalism for tensors. Space spinors are used in several other areas of relativity such as quantum gravity (see e.g. Ashtekar (1991)), the construction of quasi-local notions of energy (see e.g. Szabados (2009)) and global aspects of the geometry of 3-manifolds (see e.g. Bäckdahl and Valiente Kroon (2010a); Beig and Szabados (1997); Tod (1984)).

4.1 Hermitian inner products and 2-spinors

Let $(\mathcal{M}, \boldsymbol{g})$ denote a four-dimensional Lorentzian manifold. As in Chapter 3, it is assumed that at each point $p \in \mathcal{M}$ one has a two-dimensional simplectic vector space $\mathfrak{S}|_p(\mathcal{M})$ as given by Definition 3.1. One has the following definition:

Definition 4.1 (***Hermitian inner product***) *A* ***Hermitian inner product*** *on a simplectic two-dimensional vector space* $\mathfrak{S}$ *is a function* $\langle\langle \cdot, \cdot \rangle\rangle : \mathfrak{S} \times \mathfrak{S} \to \mathbb{C}$ *which is:*

(i) ***Hermitian****; that is, given* $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathfrak{S}$

$$\langle\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle\rangle = \overline{\langle\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle\rangle}$$

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(ii) ***linear in the second entry****; that is, given* $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta} \in \mathfrak{S}$, $z \in \mathbb{C}$

$$\langle\langle \boldsymbol{\xi}, \boldsymbol{\eta} + z\boldsymbol{\zeta} \rangle\rangle = \langle\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle\rangle + z\langle\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle\rangle$$

(iii) ***positive definite****; that is, given* $\boldsymbol{\xi} \in \mathfrak{S}$

$$\langle\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle\rangle \geq 0$$

and $\langle\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle\rangle = 0$ *if and only if* $\boldsymbol{\xi} = 0$.

From conditions (i) and (ii) it follows that a Hermitian inner product is *antilinear* in the first entry; that is, given $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta} \in \mathfrak{S}$, $z \in \mathbb{C}$, one has

$$\langle\langle \boldsymbol{\xi} + z\boldsymbol{\zeta}, \boldsymbol{\eta} \rangle\rangle = \langle\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle\rangle + \bar{z}\langle\langle \boldsymbol{\zeta}, \boldsymbol{\eta} \rangle\rangle.$$

4.1.1 Hermitian conjugation

In what follows, given a spacetime $(\mathcal{M}, \boldsymbol{g})$, assume that for each point $p \in \mathcal{M}$, the vector space $\mathfrak{S}|_p(\mathcal{M})$ is endowed with a Hermitian inner product which changes smoothly from point to point. The Hermitian inner product can be expressed in terms of a Hermitian spinor $\varpi_{AA'} \in \mathfrak{S}_{AA'}(\mathcal{M})$ such that

$$\langle\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle\rangle = \varpi_{AA'}\bar{\xi}^{A'}\eta^{A}. \tag{4.1}$$

It can be verified that the right-hand side of the above expression satisfies conditions (i) and (ii) of Definition 4.1. Given a spinor basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$, the components of $\varpi_{AA'}$ with respect to the basis are given by $\varpi_{\boldsymbol{AA'}} \equiv \varpi_{AA'}\epsilon_{\boldsymbol{A}}{}^{A}\bar{\epsilon}_{\boldsymbol{A'}}{}^{A'}$. The components $\varpi_{\boldsymbol{AA'}}$ can be thought of as the entries of a (2×2) matrix $(\varpi_{\boldsymbol{AA'}})$. The *positivity condition* (iii) of Definition 4.1 requires the above matrix to be diagonalisable and to have positive eigenvalues. Thus, it is natural to consider a (not necessarily normalised) basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ for which $(\varpi_{\boldsymbol{AA'}})$ takes the diagonal form

$$(\varpi_{\boldsymbol{AA'}}) = \begin{pmatrix} \varpi_{\boldsymbol{00'}} & 0 \\ 0 & \varpi_{\boldsymbol{11'}} \end{pmatrix}.$$

The scaling of the basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ can be fixed, without loss of generality, so that $(\varpi_{\boldsymbol{AA'}})$ is the identity matrix. *In the rest of the book, whenever a Hermitian inner product is discussed, it will be assumed that a spin basis* $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ *has been chosen so that*

$$(\varpi_{\boldsymbol{AA'}}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.2}$$

A direct consequence of the above normalisation condition is that one can write

$$\varpi_{AA'} = o_A\bar{o}_{A'} + \iota_A\bar{\iota}_{A'} = \epsilon^{\boldsymbol{1}}{}_{A}\bar{\epsilon}^{\boldsymbol{1}'}{}_{A'} + \epsilon^{\boldsymbol{0}}{}_{A}\bar{\epsilon}^{\boldsymbol{0}'}{}_{A'}, \tag{4.3a}$$

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$$\varpi_A{}^{A'} = o_A\bar{o}^{A'} + \iota_A\bar{\iota}^{A'} = \epsilon^{\mathbf{1}}{}_A\bar{\epsilon}_{\mathbf{0}'}{}^{A'} - \epsilon^{\mathbf{0}}{}_A\bar{\epsilon}_{\mathbf{1}}{}^{A'}, \tag{4.3b}$$

$$\varpi^{AA'} = o^A\bar{o}^{A'} + \iota^A\bar{\iota}^{A'} = \epsilon_{\mathbf{0}}{}^A\bar{\epsilon}_{\mathbf{0}'}{}^{A'} + \epsilon_{\mathbf{1}}{}^A\bar{\epsilon}_{\mathbf{1}'}{}^{A'}. \tag{4.3c}$$

From these expressions it follows that

$$\varpi_{AA'}\varpi^{BA'} = o_A\iota^B - \iota_A o^B = \epsilon^{\mathbf{1}}{}_A\epsilon_{\mathbf{1}}{}^B + \epsilon^{\mathbf{0}}{}_A\epsilon_{\mathbf{0}}{}^B.$$

Thus,

$$\varpi_{AA'}\varpi^{A'B} = \delta_A{}^B. \tag{4.4}$$

Notice, in particular, that $\varpi_{AA'}\varpi^{AA'} = 2$.

The spinor $\varpi_{AA'}$ induces an ***operation of Hermitian conjugation*** $^+$: $\mathfrak{S}^\bullet(\mathcal{M}) \to \mathfrak{S}^\bullet(\mathcal{M})$. Given $\mu_A \in \mathfrak{S}(\mathcal{M})$, we define its Hermitian conjugate μ^+_A via

$$\mu^+_A \equiv \varpi_A{}^{A'}\bar{\mu}_{A'}. \tag{4.5}$$

It follows then that one can write

$$\langle\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle\rangle = -\xi^+{}_A\eta^A = \eta_A\xi^{+A}.$$

Observe that as a consequence of the *see-saw rule*, Equations (3.4a) and (3.4b), one has $\mu^{+A} \equiv -\varpi^A{}_{A'}\bar{\mu}^{A'}$. The Hermitian conjugation is extended to higher valence spinors by requiring

$$(\boldsymbol{\mu}\boldsymbol{\lambda})^+ = \boldsymbol{\mu}^+\boldsymbol{\lambda}^+$$

for $\boldsymbol{\mu}$, $\boldsymbol{\lambda} \in \mathfrak{S}^\bullet(\mathcal{M})$. It is a consequence of the normalisation condition (4.2) that

$$\mu^{++}_{A_1\cdots A_k} = (-1)^k\mu_{A_1\cdots A_k}.$$

Furthermore, $\mu^{+A}\mu_A = 0$ if and only if $\mu_A = 0$, the latter as a result of condition (iii) of Definition 3.1. Using the representation of $\varpi_A{}^{A'}$ given by (4.3b) one finds that

$$o^+_A = \iota_A, \qquad \iota^+_A = -o_A, \tag{4.6a}$$

$$o^{+A} = \iota^A, \qquad \iota^{+A} = -o^A. \tag{4.6b}$$

Hence, the normalisation leading to (4.3a)–(4.3c) is equivalent to the normalisation condition

$$o_A o^{+A} = 1,$$

for the spinor $\boldsymbol{o}$. Notice also that, as a consequence of the previous discussion, a non-zero spinor and its Hermitian conjugate are linearly independent. Finally, a calculation using the expression of ϵ_{AB} in terms of o_A and ι_A, yields

$$\epsilon^+{}_{AB} = \varpi_A{}^{A'}\varpi_B{}^{B'}\epsilon_{A'B'} = \epsilon_{AB}.$$

Remark. In the rest of the book, when working with spinor structures endowed with a Hermitian product, it will always be assumed that a spin basis satisfying relations (4.6a) and (4.6b) has been chosen.

4.2 The space spinor formalism

A consequence of the existence of an operation of Hermitian conjugation is that, given a spinor $\boldsymbol{\xi} \in \mathfrak{S}^{\bullet}(\mathcal{M})$, its complex conjugate $\bar{\boldsymbol{\xi}}$ and its Hermitian conjugate $\boldsymbol{\xi}^{+}$ contain the same information. This observation allows the introduction of a ***spinorial formalism*** based entirely on spinors with unprimed indices by contracting all the primed indices in the spinors with $\varpi_A{}^{A'}$. Given $\xi_{A_1\cdots A_p A'_1\cdots A'_q}$, we define its ***space spinor counterpart*** $\xi_{A_1\cdots A_p B_1\cdots B_q}$ as

$$\xi_{A_1\cdots A_p B_1\cdots B_q} \equiv \varpi_{B_1}{}^{A'_1}\cdots\varpi_{B_q}{}^{A'_q}\xi_{A_1\cdots A_p A'_1\cdots A'_q}. \tag{4.7}$$

The above expression can be inverted by recalling the normalisation condition (4.4) to yield

$$\xi_{A_1\cdots A_p A'_1\cdots A'_q} = (-1)^q \varpi^{B_1}{}_{A'_1}\cdots\varpi^{B_q}{}_{A'_q}\xi_{A_1\cdots A_p B_1\cdots B_q}.$$

Thus, the information contained in a spinor with primed indices and its space spinor counterpart is equivalent.

4.2.1 The Hermitian product and three-dimensional vectors

The operation of Hermitian conjugation gives rise to a notion of *reality* for spinors. More precisely, a spinor $\mu_{A_1B_1\cdots A_kB_k}$ with an *even number of indices* will be said to be ***real*** if

$$\mu^{+}_{A_1B_1\cdots A_kB_k} = (-1)^k \mu_{A_1B_1\cdots A_kB_k},$$

and ***imaginary*** if

$$\mu^{+}_{A_1B_1\cdots A_kB_k} = (-1)^{k+1} \mu_{A_1B_1\cdots A_kB_k}.$$

Consider now a *symmetric* valence-2 spinor $v^{AB} \in \mathfrak{S}^{\bullet}(\mathcal{M})$. Given a space spinor basis $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ such that $\boldsymbol{\iota} \equiv \boldsymbol{o}^{+}$, one can write

$$v^{AB} = a o^A o^B + b \iota^A \iota^B + c o^{(A}\iota^{B)}, \tag{4.8}$$

with $a,\ b,\ c \in \mathbb{C}$. The Hermitian conjugate of v^{AB} is given by

$$v^{+AB} = \bar{a}\iota^A\iota^B + \bar{b} o^A o^B - \bar{c}\iota^{(A}o^{B)}.$$

If v^{AB} is *real*, that is, $v^{+AB} = -v^{AB}$, then $\bar{b} = -a$ and $\bar{c} = c$. Thus, *the real spinor v^{AB} has only three real components so that it describes a three-dimensional vector v^i*. This argument can be extended to higher valence *real* space spinors so that

$$\xi_{A_1B_1\cdots A_kB_k}{}^{C_1D_1\cdots C_mD_m} = \xi_{(A_1B_1)\cdots(A_kB_k)}{}^{(C_1D_1)\cdots(C_mD_m)},$$

if real, can be regarded as the space spinor counterpart of a three-dimensional tensor $\xi_{i_1\cdots i_k}{}^{j_1\cdots j_m}$ – every pair of symmetric spinor indices is associated to a spatial tensor index. One can summarise the previous discussion in the following:

Lemma 4.1 (***the distribution associated to a Hermitian product***) *A Hermitian spinor $\varpi_{AA'}$ on $\mathfrak{S}^\bullet(\mathcal{M})$ induces a three-dimensional distribution Π on $\mathcal{M}$.*

Observation. The distribution Π may not possess integrable manifolds.

The consequences of Lemma 4.1 can be further elaborated by considering the spinorial counterpart of the projector $h_a{}^b$ associated to the distribution Π. To this end let

$$h_{AA'}{}^{BB'} \equiv \delta_A{}^B \delta_{A'}{}^{B'} - \frac{1}{2}\varpi_{AA'}\varpi^{BB'}.$$

It can be readily verified that

$$h_{AA'}{}^{BB'} h_{BB'}{}^{CC'} = h_{AA'}{}^{CC'}, \qquad h_{AA'}{}^{BB'}\varpi_{BB'} = 0.$$

Now, given $v_{AA'} \in \mathfrak{S}^\bullet(\mathcal{M})$ denote by v_{AB} its space spinor counterpart. A calculation then shows that

$$v_{(AB)} = \varpi_B{}^{A'} h_{AA'}{}^{CC'} v_{CC'} = \varpi_{(A}{}^{A'} v_{B)A'}.$$

Thus, *the spinor $h_{AA'}{}^{BB'}$ is the projector associated to the distribution induced by $\varpi_{AA'}$*. The space spinor version of $h_{AA'}{}^{BB'}$ is given by $h_{ABCD} \equiv \varpi_B{}^{A'}\varpi_D{}^{C'}$ $h_{AA'CC'}$. It can be readily verified that $h_{ABCD} = h_{(AB)(CD)}$. Using the Jacobi identity (3.5) one can show that

$$h_{ABCD} \equiv -\epsilon_{A(C}\epsilon_{D)B}. \tag{4.9}$$

It can also be verified that

$$h_{ABPQ}h^{PQCD} = h_{AB}{}^{CD} \equiv \epsilon_A{}^{(C}\epsilon_B{}^{D)}, \qquad h_{ABCD}h^{ABCD} = h_{PQ}{}^{PQ} = 3.$$

In addition, one has

$$h^+_{ABCD} = h_{ABCD},$$

so that h_{ABCD} is a *real space spinor*. Moreover, given $v^{AB} = v^{(AB)}$ and $u_{AB} = u_{(AB)}$ one has that

$$v^{AB}h_{ABCD} = v_{CD}, \qquad u_{AB}h^{ABCD} = u^{CD}, \qquad u_{AB}h_{CD}{}^{AB} = u_{CD}.$$

Finally, it is observed that if the spinor v^{AB} is real, then using the decomposition of Equation (4.8) one readily finds that

$$h_{ABCD}v^{AB}v^{CD} = v_{AB}v^{AB} = 2ab - \frac{1}{2}c^2 = -2|b|^2 - \frac{1}{2}c^2 \leq 0.$$

Thus, given $p \in \mathcal{M}$, *the spinor h_{ABCD} gives rise to a negative definite inner product on $\Pi|_p \subset T|_p(\mathcal{M})$*. Accordingly, the vectors in Π are spatial with respect to the metric $\boldsymbol{g}$. As Π may not possess integrable submanifolds, h_{ABCD} is not necessarily the spinorial counterpart of a (negative definite) three-dimensional Riemannian metric of a spacelike submanifold of $\mathcal{M}$.

4.2.2 Spatial Infeld-van der Waerden symbols

The relation between space spinors and three-dimensional vectors can be formalised by means of suitable *soldering objects*. At each point $p \in \mathcal{M}$ consider a $\boldsymbol{g}$-orthonormal basis $\{\boldsymbol{e_i}\}$ of $\Pi|_p$ and let $\{\boldsymbol{\omega^i}\}$ denote the associated cobasis. One has then that $\boldsymbol{g}(\boldsymbol{e_i}, \boldsymbol{e_j}) = -\delta_{\boldsymbol{ij}}$. In addition, let $\{\epsilon_{\boldsymbol{A}}{}^A\}$ be a spin basis satisfying $\epsilon_{\boldsymbol{1}}^{+A} = \epsilon_{\boldsymbol{0}}{}^A$; compare Equation (4.6b). ***Spatial Infeld-van der Waerden symbols*** can be defined from the spacetime Infeld-van der Waerden symbols $\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}$ and $\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}$ through the relations

$$\begin{aligned} \sigma_{\boldsymbol{i}}{}^{\boldsymbol{AB}} &\equiv -\varpi^{(\boldsymbol{A}}{}_{\boldsymbol{A'}}\sigma_{\boldsymbol{i}}{}^{\boldsymbol{B})\boldsymbol{A'}}, \qquad \sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}} \equiv \varpi_{(\boldsymbol{A}}{}^{\boldsymbol{A'}}\sigma^{\boldsymbol{i}}{}_{\boldsymbol{B})\boldsymbol{A'}}. \\ h_{\boldsymbol{ABCD}} &= -\sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}}\sigma^{\boldsymbol{j}}{}_{\boldsymbol{CD}}\delta_{\boldsymbol{ij}}, \qquad \sigma_{\boldsymbol{i}}{}^{\boldsymbol{AB}}\sigma^{\boldsymbol{j}}{}_{\boldsymbol{AB}} = \delta_{\boldsymbol{i}}{}^{\boldsymbol{j}}. \end{aligned} \tag{4.10}$$

The explicit expressions for the spatial Infeld-van der Waerden symbols are given by

$$\sigma_{\boldsymbol{1}}{}^{\boldsymbol{AB}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{\boldsymbol{2}}{}^{\boldsymbol{AB}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad \sigma_{\boldsymbol{3}}{}^{\boldsymbol{AB}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$\sigma^{\boldsymbol{1}}{}_{\boldsymbol{AB}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^{\boldsymbol{2}}{}_{\boldsymbol{AB}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad \sigma^{\boldsymbol{3}}{}_{\boldsymbol{AB}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Given the above, the components of a three-dimensional vector $\boldsymbol{v}$ and a three-dimensional covector $\boldsymbol{\xi}$ can be put in correspondence with symmetric valence-2 real spinors via the formulae

$$v^{\boldsymbol{i}} \mapsto v^{\boldsymbol{AB}} = v^{\boldsymbol{i}}\sigma_{\boldsymbol{i}}{}^{\boldsymbol{AB}}, \qquad \xi_{\boldsymbol{i}} \mapsto \xi_{\boldsymbol{AB}} = \xi_{\boldsymbol{i}}\sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}},$$

or more explicitly

$$(v^1, v^2, v^3) \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} -v^1 - \mathrm{i}v^2 & v^3 \\ v^3 & v^1 - \mathrm{i}v^2 \end{pmatrix}, \tag{4.11a}$$

$$(\xi_1, \xi_2, \xi_3) \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} -\xi_1 + \mathrm{i}\xi_2 & \xi_3 \\ \xi_3 & \xi_1 + \mathrm{i}\xi_2 \end{pmatrix}. \tag{4.11b}$$

It can be verified that

$$\langle \boldsymbol{\xi}, \boldsymbol{v} \rangle = \xi_{\boldsymbol{i}} v^{\boldsymbol{i}} = \xi_{\boldsymbol{AB}} v^{\boldsymbol{AB}} = -\xi_1 v^1 - \xi_2 v^2 - \xi_3 v^3.$$

The above correspondence between space spinors and three-dimensional vectors can be readily extended to higher rank tensors. For example, given the components $T_{\boldsymbol{ij}}{}^{\boldsymbol{k}}$ of a tensor $T_{ij}{}^{k}$ one has the correspondence

$$T_{\boldsymbol{ij}}{}^{\boldsymbol{k}} \mapsto T_{\boldsymbol{ABCD}}{}^{\boldsymbol{EF}} \equiv \sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}}\sigma^{\boldsymbol{j}}{}_{\boldsymbol{CD}}\sigma_{\boldsymbol{k}}{}^{\boldsymbol{EF}}T_{\boldsymbol{ij}}{}^{\boldsymbol{k}}.$$

Finally, it is observed that $\sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}}$ and $\sigma_{\boldsymbol{i}}{}^{\boldsymbol{AB}}$ can be used to define an alternative set of basis and cobasis $\{\boldsymbol{e_{AB}}\}$, $\{\boldsymbol{\omega^{AB}}\}$ for $\Pi|_p$ through the relations

$$\boldsymbol{e_{AB}} \equiv \sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}}\boldsymbol{e_i}, \qquad \boldsymbol{\omega^{AB}} \equiv \sigma_{\boldsymbol{i}}{}^{\boldsymbol{AB}}\boldsymbol{\omega^i}. \tag{4.12}$$

In terms of these definitions the symbols $\sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}}$ are the components of the frame $\{\boldsymbol{e_{AB}}\}$ with respect to itself. It can be verified that

$$\langle \boldsymbol{\omega^{AB}}, \boldsymbol{e_{CD}}\rangle = h_{\boldsymbol{CD}}{}^{\boldsymbol{AB}}, \qquad \boldsymbol{g}(\boldsymbol{e_{AB}}, \boldsymbol{e_{CD}}) = h_{\boldsymbol{ABCD}}.$$

4.2.3 Changes of basis and $SU(2,\mathbb{C})$ transformations

To characterise the class of transformations preserving the structure of the Hermitian inner product it is convenient to consider a change of spin basis

$$\tilde{\epsilon}_{\boldsymbol{A}}{}^{A} = O_{\boldsymbol{A}}{}^{\boldsymbol{P}}\epsilon_{\boldsymbol{P}}{}^{A}, \qquad \tilde{\epsilon}^{\boldsymbol{A}}{}_{A} = O^{\boldsymbol{A}}{}_{\boldsymbol{P}}\epsilon^{\boldsymbol{P}}{}_{A},$$

where $(O_{\boldsymbol{A}}{}^{\boldsymbol{P}})$ and $(O^{\boldsymbol{A}}{}_{\boldsymbol{P}})$ are $SL(2,\mathbb{C})$ matrices such that

$$O_{\boldsymbol{A}}{}^{\boldsymbol{P}}O^{\boldsymbol{B}}{}_{\boldsymbol{P}} = \delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}.$$

It follows that

$$\begin{aligned}\tilde{\varpi}_{\boldsymbol{AA'}} &= \varpi_{AA'}\tilde{\epsilon}_{\boldsymbol{A}}{}^{A}\bar{\tilde{\epsilon}}_{\boldsymbol{A'}}{}^{A'}\\ &= \varpi_{AA'}\left(O_{\boldsymbol{A}}{}^{\boldsymbol{P}}\epsilon_{\boldsymbol{P}}{}^{A}\right)\left(\bar{O}_{\boldsymbol{A'}}{}^{\boldsymbol{P'}}\bar{\epsilon}_{\boldsymbol{P'}}{}^{A'}\right) = \varpi_{\boldsymbol{PP'}}O_{\boldsymbol{A}}{}^{\boldsymbol{P}}\bar{O}_{\boldsymbol{A'}}{}^{\boldsymbol{P'}}.\end{aligned}$$

Hence, if one requires $O_{\boldsymbol{A}}{}^{\boldsymbol{P}}$ to be such that both $\varpi_{\boldsymbol{AA'}}$ and $\tilde{\varpi}_{\boldsymbol{AA'}}$ are the identity matrix – compare Equation (4.2) – then $O_{\boldsymbol{A}}{}^{\boldsymbol{P}}$ and $\bar{O}_{\boldsymbol{A'}}{}^{\boldsymbol{P'}}$ have to be inverses of each other; that is, the transformation described by the matrix $(O_{\boldsymbol{A}}{}^{\boldsymbol{B}})$ is an $SU(2,\mathbb{C})$ ***transformation***. This property explains the alternative name of $SU(2,\mathbb{C})$ spinors used to describe spinorial structures endowed with a Hermitian product; see, for example, Ashtekar (1991). It is a direct consequence of the previous discussion that the notions of real and imaginary space spinors as discussed in Section 4.2.1 are *invariant under* $SU(2,\mathbb{C})$ *transformations*.

It can be readily verified that $SU(2,\mathbb{C})$ *transformations are related to three-dimensional rotations*, that is, $O(3)$ ***transformations***. As $SU(2,\mathbb{C})$ transformations are a special case of $SL(2,\mathbb{C})$ transformations, it follows that $\epsilon_{\boldsymbol{AB}} = O_{\boldsymbol{A}}{}^{\boldsymbol{C}}O_{\boldsymbol{B}}{}^{\boldsymbol{D}}\epsilon_{\boldsymbol{CD}}$; that is, $\tilde{\epsilon}_{\boldsymbol{AB}} = \epsilon_{\boldsymbol{AB}}$. From the latter one has that

$$h_{\boldsymbol{ABCD}} = O_{\boldsymbol{A}}{}^{\boldsymbol{E}}O_{\boldsymbol{B}}{}^{\boldsymbol{F}}O_{\boldsymbol{C}}{}^{\boldsymbol{G}}O_{\boldsymbol{D}}{}^{\boldsymbol{H}}h_{\boldsymbol{EFGH}}.$$

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Contracting the above expression with spatial Infeld-van der Waerden symbols one obtains

$$\delta_{\boldsymbol{ij}} = O_{\boldsymbol{i}}{}^{\boldsymbol{k}} O_{\boldsymbol{j}}{}^{\boldsymbol{l}} \delta_{\boldsymbol{kl}},$$

where the matrix $(O_{\boldsymbol{i}}{}^{\boldsymbol{k}})$ with elements given by

$$O_{\boldsymbol{i}}{}^{\boldsymbol{k}} \equiv \sigma_{\boldsymbol{i}}{}^{\boldsymbol{AB}} \sigma^{\boldsymbol{k}}{}_{\boldsymbol{EF}} O_{\boldsymbol{A}}{}^{\boldsymbol{E}} O_{\boldsymbol{B}}{}^{\boldsymbol{F}}$$

is an $O(3)$-transformation, that is, a three-dimensional real matrix preserving the identity matrix.

4.2.4 Spinors on three-dimensional manifolds

In this section it is assumed that the distribution Π associated to the Hermitian spinor $\varpi_{AA'}$ has an integral manifold $\mathcal{S}$. It follows that $\mathcal{S}$ is a spacelike hypersurface of the spacetime manifold $\mathcal{M}$, and the restriction of Π to $\mathcal{S}$ coincides with the tangent bundle $T(\mathcal{S})$. Vectors and covectors in $T(\mathcal{S})$ are associated to symmetric valence-2 real spinors.

Under the assumptions of the previous paragraph, consistently with the discussion of Section 4.2.1, one has that the spinor $h_{ABCD} = -\epsilon_{A(C}\epsilon_{D)B}$ is the spinorial counterpart of the three-dimensional (negative definite) Riemannian metric $\boldsymbol{h}$ induced on $\mathcal{S}$ by $\boldsymbol{g}$. One can write

$$\boldsymbol{h} = -\delta_{\boldsymbol{ij}} \boldsymbol{\omega}^{\boldsymbol{i}} \otimes \boldsymbol{\omega}^{\boldsymbol{j}} = h_{\boldsymbol{ABCD}} \boldsymbol{\omega}^{\boldsymbol{AB}} \otimes \boldsymbol{\omega}^{\boldsymbol{CD}}$$

with the coframe $\{\boldsymbol{\omega}^{\boldsymbol{AB}}\}$ defined as in Equation (4.12). In a natural manner, the spinor $\varpi_{AA'}$ can be identified with the normal to $\mathcal{S}$.

An alternative point of view

The discussion of spinors on three-dimensional manifolds outlined in the previous paragraphs assumes that $\mathcal{S}$ is a spacelike hypersurface of a spacetime $(\mathcal{M}, \boldsymbol{g})$. A more intrinsic perspective can be obtained by *postulating* the existence of a spinorial structure over $\mathcal{S}$, to be denoted by $\mathfrak{S}(\mathcal{S})$, endowed with an operation of Hermitian conjugation $^{+} : \mathfrak{S}(\mathcal{S}) \rightarrow \mathfrak{S}(\mathcal{S})$ satisfying the properties discussed in Section 4.1.1. This point of view leads one to consider conditions ensuring the existence of this ***space spinor structure***. An example of a sufficient condition is that the vacuum Einstein constraint equations (see Chapter 11) can be solved on $\mathcal{S}$. If this is the case, the three-dimensional manifold can be regarded as a spacelike hypersurface of a spacetime $(\mathcal{M}, \boldsymbol{g})$; see Chapter 14. The spacetime $(\mathcal{M}, \boldsymbol{g})$ is globally hyperbolic and, thus, admits a spinor structure; see Proposition 3.4. The $\boldsymbol{g}$-normal to $\mathcal{S}$ in $\mathcal{M}$ induces the required operation of Hermitian conjugation.

In what follows, the notational conventions of Section 3.1.4 are adopted, and one writes $\mathfrak{S}^{\bullet}(\mathcal{S})$, $\mathfrak{S}_{A}(\mathcal{S})$, $\mathfrak{S}^{A}(\mathcal{S}), \ldots$ to denote the various bundles associated to $\mathfrak{S}(\mathcal{S})$.

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Totally symmetric spinors

Spinors provide a simple representation of the operation of taking the *symmetric trace-free part of a three-dimensional tensor.*

Proposition 4.1 (***space spinor representation of trace-free three-dimensional tensors***) *Let* $T_{A_1B_1\cdots A_pB_p}$ *denote the spinorial counterpart of a three-dimensional real tensor* $T_{i_1\cdots i_p}$. *One has that*

$$T_{A_1B_1\cdots A_pB_p} = T_{(A_1B_1\cdots A_pB_p)} \qquad \text{if and only if} \qquad T_{i_1\cdots i_p} = T_{\{i_1\cdots i_p\}}.$$

Proof Any possible contraction of $T_{(A_1B_1\cdots A_pB_p)}$ with h^{ABCD} must vanish so that $T_{i_1\cdots i_p}$ must be trace free. Conversely, if $T_{i_1\cdots i_p} = T_{\{i_1\cdots i_p\}}$, then one has that

$$h^{A_1B_1A_2B_2}T_{A_1B_1A_2B_2\cdots A_pB_p} = T^{PQ}{}_{PQ\cdots A_pB_p} = 0.$$

Using the decomposition (3.8) together with the symmetries of $T_{A_1B_1A_2B_2\cdots A_pB_p}$ in the indices ${}_{A_1B_1A_2B_2}$ one concludes that

$$T_{A_1B_1A_2B_2\cdots A_pB_p} = T_{(A_1B_1A_2B_2)\cdots A_pB_p}.$$

Now, considering the contraction of the pair ${}_{A_2B_2}$ with pairs outside the symmetrisation bracket and repeating the previous argument as many times as necessary one concludes that $T_{A_1B_1A_2B_2\cdots A_pB_p}$ must be completely symmetric. □

4.2.5 Timelike congruences and Hermitian products

Assume now that the spacetime $(\mathcal{M}, \boldsymbol{g})$ has some *privileged future directed timelike vector* $\boldsymbol{\tau}$ with parameter τ.[1] The vector $\boldsymbol{\tau}$ does not need to be hypersurface orthogonal. Let $\tau^{AA'}$ denote the spinorial counterpart of $\boldsymbol{\tau}$ and consider the normalisation $\boldsymbol{g}(\boldsymbol{\tau}, \boldsymbol{\tau}) = 2$. As discussed in Section 2.7.1, $\boldsymbol{\tau}$ defines a distribution on $\mathcal{M}$. Let $\mathcal{S}_\tau$ denote the hyperplanes generated by $\boldsymbol{\tau}$; as $\boldsymbol{\tau}$ is not hypersurface orthogonal, the hyperplanes are not, in general, the tangent bundles to the leaves of a foliation of $\mathcal{M}$. The timelike spinor $\tau_{AA'}$ induces a Hermitian product $\langle\langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle\rangle = \tau_{AA'}\bar{\xi}^{A'}\eta^A$ for ξ^A, $\eta^A \in \mathfrak{S}(\mathcal{S})$. Indeed, as $\tau^{AA'}$ is the spinorial counterpart of a spacetime vector, it is a Hermitian spinor, so that

$$\overline{\tau_{AA'}\bar{\xi}^{A'}\eta^A} = \tau_{AA'}\bar{\eta}^{A'}\xi^A.$$

Furthermore, given that $\tau^{AA'}$ is timelike future directed and $\xi^A\bar{\xi}^{A'}$ describes a future-directed null vector, it follows that $\tau_{AA'}\xi^A\bar{\xi}^{A'} \geq 0$. Thus, formula (4.1)

[1] It is possible to construct a "space spinor" formalism adapted to spacelike congruences with tangent vector $\rho^{AA'}$; see e.g. Szabados (1994). This requires adapting some of the formulae given in the preceding sections. In particular, the associated Hermitian product needs to be negative definite. Moreover, the analogue of Equation (4.13) is given by $\rho_{AA'}\rho^{BA'} = \epsilon^B{}_A$ so that $\rho_{AA'}\rho^{AA'} = -2$.

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and the various subsequent expressions in Section 4.1 can be used for the choice $\varpi_{AA'} = \tau_{AA'}$.

From the discussion in Section 4.1.1 it follows that there exists a spin basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ such that

$$\tau^{AA'} = \epsilon_{\mathbf{0}}{}^{A}\epsilon_{\mathbf{0}'}{}^{A'} + \epsilon_{\mathbf{1}}{}^{A}\epsilon_{\mathbf{1}'}{}^{A'}.$$

In particular, one has that

$$\tau_{AA'}\tau^{BA'} = \epsilon_{A}{}^{B}. \tag{4.13}$$

Space spinor split of general spacetime spinors

The tensorial counterpart of a spinor $\mu_{A_1A'_1\cdots A_kA'_k}$ can be expanded in terms of the spatial frame $\{\boldsymbol{e_{AB}}\}$ if and only if it is *spatial* with respect to $\boldsymbol{\tau}$, that is, if the k conditions

$$\tau^{A_1A'_1}\mu_{A_1A'_1\cdots A_kA'_k} = 0, \qquad \cdots \qquad \tau^{A_kA'_k}\mu_{A_1A'_1\cdots A_kA'_k} = 0$$

hold. In this case, its space spinor counterpart is given by

$$\mu_{A_1B_1\cdots A_kB_k} = \tau_{B_1}{}^{A'_1}\cdots\tau_{B_k}{}^{A'_k}\mu_{A_1A'_1\cdots A_kA'_k} = \mu_{(A_1B_1)\cdots(A_kB_k)}.$$

To deal with the spinorial counterparts of tensors which are not spatial in the sense described above, one makes use of the ***projector***

$$h^{BB'}{}_{AA'} \equiv \epsilon_{A}{}^{B}\epsilon_{A'}{}^{B'} - \frac{1}{2}\tau_{AA'}\tau^{BB'},$$

which takes a spinor $\xi_{A_1A'_1\cdots A_kA'_k}$ onto the *spatial spinor*

$$\xi_{A_1A'_1\cdots A_kA'_k}h^{A_1A'_1}{}_{B_1B'_1}\cdots h^{A_kA'_k}{}_{B_kB'_k}.$$

The space spinor version of the above spatial spinor is obtained by contracting the primed indices with $\tau_{A}{}^{A'}$ as in formula (4.7). In particular, this procedure applied to the projector $h_{AA'BB'}$ yields h_{ABCD}. The non-spatial components of $\xi_{A_1A'_1\cdots A_kA'_k}$ can be obtained by a full contraction of a primed-unprimed pair of indices with $\tau^{AA'}$.

An alternative way of looking at the projection procedure described in the previous paragraphs is the following: given the spinorial counterpart $\xi_{A_1A'_1\cdots A_1A'_k}$ of a (in principle non-spatial) tensor, define

$$\xi_{A_1B_1\cdots A_kB_k} \equiv \tau_{B_1}{}^{A'_1}\cdots\tau_{B_k}{}^{A'_k}\xi_{A_1A'_1\cdots A_1A'_k}.$$

Then $\xi_{(A_1B_1)\cdots(A_kB_k)}$ encodes the spatial part of $\xi_{A_1A'_1\cdots A_1A'_k}$, while $\xi_{P_1}{}^{P_1}\cdots{}_{P_k}{}^{P_k}$ corresponds to its pure time component. Mixed time-spatial components have the form $\xi_{P}{}^{P}{}_{(A_2B_2)\cdots(A_kB_k)}$, and so on.

As a particular example of the previous discussion one has that for a Hermitian spinor $v_{AA'} \in \mathfrak{S}^{\bullet}(\mathcal{M})$ it holds that

$$v_{AA'} = \frac{1}{2}\tau_{AA'}v - \tau^{Q}{}_{A'}v_{(QA)}$$

where $v \equiv v_{PP'}\tau^{PP'}$ and $v_{AB} \equiv \tau_{B}{}^{A'}v_{AA'}$. Observing that $v = v_{Q}{}^{Q}$ one can write, alternatively, that

$$v_{AB} = \frac{1}{2}\epsilon_{AB}v + v_{(AB)}.$$

The $1+3$ *split of frame and the metric*

Given a $\boldsymbol{g}$-orthonormal frame $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ and its coframe $\{\boldsymbol{\omega}^{\boldsymbol{AA'}}\}$, the discussion of the previous paragraphs implies that they can be written as

$$\boldsymbol{e}_{\boldsymbol{AA'}} = \frac{1}{2}\tau_{\boldsymbol{AA'}}\boldsymbol{e} - \tau^{\boldsymbol{B}}{}_{\boldsymbol{A'}}\boldsymbol{e}_{\boldsymbol{AB}},$$

$$\boldsymbol{\omega}^{\boldsymbol{AA'}} = \frac{1}{2}\tau^{\boldsymbol{AA'}}\boldsymbol{\omega} + \tau_{\boldsymbol{C}}{}^{\boldsymbol{A'}}\boldsymbol{\omega}^{\boldsymbol{CA}},$$

where the various vectors and covectors in the decomposition are given by

$$\boldsymbol{e} \equiv \tau^{\boldsymbol{PP'}}\boldsymbol{e}_{\boldsymbol{PP'}}, \qquad \boldsymbol{e}_{\boldsymbol{AB}} \equiv \tau_{(\boldsymbol{A}}{}^{\boldsymbol{P'}}\boldsymbol{e}_{\boldsymbol{B})\boldsymbol{P'}},$$
$$\boldsymbol{\omega} \equiv \tau_{\boldsymbol{PP'}}\boldsymbol{\omega}^{\boldsymbol{PP'}}, \qquad \boldsymbol{\omega}^{\boldsymbol{AB}} \equiv -\tau^{(\boldsymbol{A}}{}_{\boldsymbol{P'}}\boldsymbol{\omega}^{\boldsymbol{B})\boldsymbol{P'}}.$$

Now, recalling that $\langle \boldsymbol{\omega}^{\boldsymbol{AA'}}, \boldsymbol{e}_{\boldsymbol{BB'}}\rangle = \epsilon_{\boldsymbol{B}}{}^{\boldsymbol{A}}\epsilon_{\boldsymbol{B'}}{}^{\boldsymbol{A'}}$, one obtains that

$$\langle \boldsymbol{\omega}, \boldsymbol{e}\rangle = 2, \qquad \langle \boldsymbol{\omega}, \boldsymbol{e}_{\boldsymbol{AB}}\rangle = 0,$$
$$\langle \boldsymbol{\omega}^{\boldsymbol{AB}}, \boldsymbol{e}\rangle = 0, \qquad \langle \boldsymbol{\omega}^{\boldsymbol{AB}}, \boldsymbol{e}_{\boldsymbol{CD}}\rangle = h^{\boldsymbol{AB}}{}_{\boldsymbol{CD}}.$$

Using the above pairings together with expression (3.29) one obtains the following $1+3$ *split of* $\boldsymbol{g}$:

$$\boldsymbol{g} = \frac{1}{2}\boldsymbol{\omega}\otimes\boldsymbol{\omega} + h_{\boldsymbol{ABCD}}\boldsymbol{\omega}^{\boldsymbol{AB}}\otimes\boldsymbol{\omega}^{\boldsymbol{CD}}. \tag{4.14}$$

In particular, one has that

$$h_{\boldsymbol{ABCD}} \equiv \boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{AB}}, \boldsymbol{e}_{\boldsymbol{CD}}) = -\epsilon_{\boldsymbol{A}(\boldsymbol{C}}\epsilon_{\boldsymbol{D})\boldsymbol{B}}.$$

If $\boldsymbol{\tau}$ is hypersurface orthogonal, then the vectors $\{\boldsymbol{e}_{\boldsymbol{AB}}\}$ and the covectors $\{\boldsymbol{\omega}^{\boldsymbol{AB}}\}$ are intrinsic to the hypersurfaces $\mathcal{S}_{\boldsymbol{\tau}}$ orthogonal to $\boldsymbol{\tau}$; thus, they can be regarded as belonging to $T(\mathcal{S}_{\boldsymbol{\tau}})$ and $T^*(\mathcal{S}_{\boldsymbol{\tau}})$, respectively. In addition,

$$\boldsymbol{h} \equiv h_{\boldsymbol{ABCD}}\boldsymbol{\omega}^{\boldsymbol{AB}}\otimes\boldsymbol{\omega}^{\boldsymbol{CD}}$$

corresponds to the (negative definite, Riemannian) metric induced by $\boldsymbol{g}$ on $\mathcal{S}_{\boldsymbol{\tau}}$. Let ϵ_{ijk} denote the volume form of the three-dimensional metric $\boldsymbol{h}$, and let

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ϵ_{ABCDEF} be its spinorial counterpart. Using the antisymmetry properties of ϵ_{ABCDEF} it can be expressed in terms of ϵ_{AB} as

$$\epsilon_{ABCDEF} = \frac{\mathrm{i}}{\sqrt{2}}(\epsilon_{AC}\epsilon_{BE}\epsilon_{DF} + \epsilon_{BD}\epsilon_{AF}\epsilon_{CE}). \tag{4.15}$$

Furthermore, it can be checked that

$$\epsilon_{ABCDEF}\epsilon^{ABCDEF} = -6.$$

Alternatively, Equation (4.15) can be obtained from Equation (3.25), by suitable contactions with $\tau^{AA'}$. More precisely, one has that

$$\epsilon_{CDEFGH} = \frac{1}{\sqrt{2}}\tau^{AA'}\tau_D{}^{C'}\tau_F{}^{E'}\tau_H{}^{G'}\epsilon_{AA'CC'EE'GG'}.$$

4.3 Calculus of space spinors

This section discusses the notion of covariant derivative in the context of the space spinor formalism. For simplicity of the presentation, it is assumed that one has a situation as described in Section 4.2.5 where the spinor $\varpi_{AA'}$ is given by the spinorial counterpart $\tau_{AA'}$ of the tangent vector $\boldsymbol{\tau}$ to a timelike congruence in $(\mathcal{M}, \boldsymbol{g})$. Moreover, it is also assumed that $\nabla_{AA'}$ is the spinorial counterpart of the Levi-Civita connection of the metric $\boldsymbol{g}$.

4.3.1 The Sen connection

The spinor $\tau^{AA'}$ can be used to obtain a *space spinor version* of the spacetime spinorial covariant derivative $\nabla_{AA'}$. More precisely, one can define

$$\nabla_{AB} \equiv \tau_B{}^{A'}\nabla_{AA'}.$$

The latter, in turn, can be written in terms of its irreducible components as

$$\nabla_{AB} = \frac{1}{2}\epsilon_{AB}\mathcal{P} + \mathcal{D}_{AB}, \tag{4.16}$$

where

$$\mathcal{P} \equiv \tau^{AA'}\nabla_{AA'}, \qquad \mathcal{D}_{AB} \equiv \tau_{(B}{}^{A'}\nabla_{A)A'}.$$

The operator $\mathcal{P}$ is the ***directional derivative*** of the connection $\boldsymbol{\nabla}$ in the direction of $\boldsymbol{\tau}$. The differential operator $\mathcal{D}_{AB}$ is the so-called ***Sen connection*** of $\boldsymbol{\nabla}$ relative to the vector field $\boldsymbol{\tau}$. In view of these definitions one can also write

$$\nabla_{AA'} = \frac{1}{2}\tau_{AA'}\mathcal{P} - \tau_{A'}{}^{Q}\mathcal{D}_{AQ}.$$

The timelike vector $\boldsymbol{\tau}$ is completely arbitrary; in particular, *it is not assumed to be hypersurface orthogonal*. This has several consequences; most notably, the Sen connection has, in general, a non-vanishing torsion which can be expressed in terms of the covariant derivative of $\tau^{AA'}$. Furthermore, even in the case when $\boldsymbol{\tau}$ is hypersurface orthogonal, $\mathcal{D}_{AB}$ does not coincide with the Levi-Civita connection $\boldsymbol{D}$ of the intrinsic 3-metric of the hypersurfaces $\mathcal{S}_\tau$ orthogonal to $\boldsymbol{\tau}$. Finally, it is pointed out that $\mathcal{D}_{AB}$ *is not a real differential operator* in the sense that $\mathcal{D}^+_{AB} \neq -\mathcal{D}_{AB}$.

The derivative of $\tau^{AA'}$

For future use, it is convenient to define

$$\chi_{ABCD} \equiv \frac{1}{\sqrt{2}}\tau_D{}^{C'}\nabla_{AB}\tau_{CC'}. \tag{4.17}$$

Using the split (4.16) one obtains the decomposition

$$\chi_{ABCD} = \frac{1}{2}\epsilon_{AB}\chi_{CD} + \chi_{(AB)CD}$$

where

$$\chi_{AB} \equiv \frac{1}{\sqrt{2}}\tau_B{}^{A'}\mathcal{P}\tau_{AA'}, \qquad \chi_{(AB)CD} \equiv \frac{1}{\sqrt{2}}\tau_D{}^{C'}\mathcal{D}_{AB}\tau_{CC'}.$$

It can be verified that the above spinors satisfy the following symmetry and reality properties:

$$\chi_{AB} = \chi_{(AB)} = -\chi^+_{AB}, \qquad \chi_{(AB)CD} = \chi_{(AB)(CD)} = \chi^+_{(AB)CD}.$$

The spinor χ_{AB} corresponds to the *acceleration vector* of $\boldsymbol{\tau}$, while $\chi_{(AB)CD}$ is related to the *Weingarten tensor* of the distribution defined by $\boldsymbol{\tau}$. It can be checked that the distribution is integrable if and only if $\chi^Q{}_{(BC)Q} = 0$. In this case $\boldsymbol{\tau}$ is hypersurface orthogonal, and χ_{ABCD} corresponds to the extrinsic curvature of the orthogonal hypersurfaces $\mathcal{S}_\tau$.

The hypersurface orthogonal case

If $\boldsymbol{\tau}$ is hypersurface orthogonal, given a spinor μ_C, the covariant derivative D_{AB} defined by

$$D_{AB}\mu_C \equiv \mathcal{D}_{AB}\mu_C + \frac{1}{\sqrt{2}}\chi_{(AB)C}{}^Q\mu_Q \tag{4.18}$$

can be verified to be torsion-free. As $\mathcal{D}_{AB}\epsilon_{CD} = 0$ and using that $\chi_{ABCD} = \chi_{AB(CD)}$, one concludes that $D_{AB}\epsilon_{CD} = 0$. Thus, D_{AB} *is metric and must coincide with the (spinorial counterpart of the) Levi-Civita connection of the leaves of the foliation defined by* $\boldsymbol{\tau}$. It can be further verified from Equation (4.18) that

$$\big(D_{AB}\mu_C\big)^+ = -D_{AB}\mu^+_C,$$

so that D_{AB} is a *real* differential operator in the sense of Section 4.2.1.

Remark. The derivative D_{AB} as defined in Equation (4.18) provides an explicit example of the notion of space spinor covariant derivative to be introduced in Section 4.3.3.

4.3.2 Space spinor split of the spacetime connection coefficients

Following the notation of Section 3.2.2, let $\Gamma_{\boldsymbol{AA'CD}}$ denote the spin connection coefficients of a Levi-Civita connection $\nabla_{AA'}$ with respect to some spin basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$. Its space spinor counterpart $\Gamma_{\boldsymbol{ABCD}}$ is defined by

$$\Gamma_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\Gamma_{\boldsymbol{AA'CD}}.$$

The spin coefficients $\Gamma_{\boldsymbol{ABCD}}$ satisfy no specific reality conditions. However, sometimes it is convenient to have a split of $\Gamma_{\boldsymbol{ABCD}}$ into real and imaginary parts. One has that

$$\begin{aligned}
\tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}(\nabla_{\boldsymbol{AA'}}\tau_{\boldsymbol{CC'}})\tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}} &= -\tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{C}}\tau_{\boldsymbol{QC'}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}} - \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\bar{\Gamma}_{\boldsymbol{A'A}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{C'}}\tau_{\boldsymbol{CQ'}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}} \\
&= -\Gamma_{\boldsymbol{ABCD}} + \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\tau_{\boldsymbol{C}}{}^{\boldsymbol{Q'}}\tau_{\boldsymbol{A'}}{}^{\boldsymbol{E}}\tau_{\boldsymbol{Q'}}{}^{\boldsymbol{F}}\tau_{\boldsymbol{C'}}{}^{\boldsymbol{G}}\Gamma^{+}_{\boldsymbol{EAFG}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}} \\
&= -\Gamma_{\boldsymbol{ABCD}} - \delta_{\boldsymbol{B}}{}^{\boldsymbol{E}}\delta_{\boldsymbol{C}}{}^{\boldsymbol{F}}\delta_{\boldsymbol{D}}{}^{\boldsymbol{G}}\Gamma^{+}_{\boldsymbol{EAFG}} \\
&= -\Gamma_{\boldsymbol{ABCD}} - \Gamma^{+}_{\boldsymbol{BACD}},
\end{aligned}$$

where it has been used that $\boldsymbol{e}_{\boldsymbol{AA'}}(\tau_{\boldsymbol{CC'}}) = 0$, the identity

$$\bar{\Gamma}_{\boldsymbol{A'AB'C'}} = -\tau_{\boldsymbol{A'}}{}^{\boldsymbol{E}}\tau_{\boldsymbol{B'}}{}^{\boldsymbol{F}}\tau_{\boldsymbol{C'}}{}^{\boldsymbol{G}}\Gamma^{+}_{\boldsymbol{EAFG}},$$

and the identity (4.13). Hence, it follows that $\chi_{\boldsymbol{ABCD}}$ corresponds, essentially, to the *real part* of $\Gamma_{\boldsymbol{ABCD}}$; that is,

$$\chi_{\boldsymbol{ABCD}} = -\frac{1}{\sqrt{2}}(\Gamma_{\boldsymbol{ABCD}} + \Gamma^{+}_{\boldsymbol{ABCD}}).$$

The reality of the above expression follows from $\Gamma^{++}_{\boldsymbol{ABCD}} = \Gamma_{\boldsymbol{ABCD}}$. The *imaginary part* of $\Gamma_{\boldsymbol{ABCD}}$ is given by

$$\xi_{\boldsymbol{ABCD}} = \frac{1}{\sqrt{2}}(\Gamma_{\boldsymbol{ABCD}} - \Gamma^{+}_{\boldsymbol{ABCD}}).$$

Inverting the definitions of $\chi_{\boldsymbol{ABCD}}$ and $\xi_{\boldsymbol{ABCD}}$ it follows then that

$$\begin{aligned}
\Gamma_{\boldsymbol{ABCD}} &= \frac{1}{\sqrt{2}}(\xi_{\boldsymbol{ABCD}} - \chi_{\boldsymbol{ABCD}}), \\
&= \frac{1}{\sqrt{2}}(\xi_{\boldsymbol{ABCD}} - \chi_{(\boldsymbol{AB})\boldsymbol{CD}}) - \frac{1}{2\sqrt{2}}\epsilon_{\boldsymbol{AB}}\chi_{\boldsymbol{CD}}.
\end{aligned}$$

Observe the symmetry conditions

$$\chi_{\boldsymbol{ABCD}} = \chi_{\boldsymbol{AB}(\boldsymbol{CD})}, \qquad \xi_{\boldsymbol{ABCD}} = \xi_{(\boldsymbol{AB})(\boldsymbol{CD})}.$$

4.3.3 Intrinsic derivatives

When working with a three-dimensional Riemannian $(\mathcal{S}, \boldsymbol{h})$ it is convenient to make use of an intrinsic notion of covariant derivative, a so-called ***space spinor covariant derivative*** D_{AB}, compatible with operation of Hermitian conjugation, which is the spinorial counterpart of the Levi-Civita connection $\boldsymbol{D}$ of the metric $\boldsymbol{h}$. One regards D_{AB} as a map

$$D_{AB} : \mathfrak{S}^{C\cdots D}{}_{E\cdots F}(\mathcal{S}) \to \mathfrak{S}^{C\cdots D}{}_{ABE\cdots F}(\mathcal{S}).$$

The properties of the operator D_{AB} have to be consistent with those of the operator defined in Equation (4.18). It is required to satisfy:

(i) ***Symmetry.*** Given $\zeta^{C\cdots D}{}_{E\cdots F} \in \mathfrak{S}^{\bullet}(\mathcal{S})$ one has

$$D_{AB}\zeta^{C\cdots D}{}_{E\cdots F} = D_{(AB)}\zeta^{C\cdots D}{}_{E\cdots F}.$$

(ii) ***Linearity.*** Given $\zeta^{C\cdots D}{}_{E\cdots F}$, $\eta^{C\cdots D}{}_{E\cdots F} \in \mathfrak{S}^{\bullet}(\mathcal{S})$ one has

$$D_{AB}(\zeta^{C\cdots D}{}_{E\cdots F} + \eta^{C\cdots D}{}_{E\cdots F}) = D_{AB}\zeta^{C\cdots D}{}_{E\cdots F} + D_{AB}\eta^{C\cdots D}{}_{E\cdots F}.$$

(iii) ***Leibnitz rule.*** Given $\zeta^{C\cdots D}{}_{E\cdots F}$, $\xi^{G\cdots H}{}_{P\cdots Q} \in \mathfrak{S}^{\bullet}(\mathcal{S})$ one has

$$\begin{aligned} D_{AB}(\zeta^{C\cdots D}{}_{E\cdots F}\xi^{G\cdots H}{}_{P\cdots Q}) = {} & \xi^{G\cdots H}{}_{P\cdots Q}D_{AB}\zeta^{C\cdots D}{}_{E\cdots F} \\ & + \zeta^{C\cdots D}{}_{E\cdots F}D_{AB}\zeta^{G\cdots H}{}_{P\cdots Q}. \end{aligned}$$

(iv) ***Reality.*** Given $\zeta^{C\cdots D}{}_{E\cdots F} \in \mathfrak{S}^{\bullet}(\mathcal{S})$ one has

$$\big(D_{AB}\zeta^{C\cdots D}{}_{E\cdots F}\big)^{+} = -D_{AB}\zeta^{+C\cdots D}{}_{E\cdots F}$$

(v) ***Action on scalars.*** Given a scalar $\phi \in \mathfrak{X}(\mathcal{S})$, then $D_{AB}\phi$ is the spinorial counterpart of $D_i\phi$.

(vi) ***Representation of derivations.*** Given a derivation $\mathcal{D}$, there exists a spinor $\xi^{AB} \in \mathfrak{S}^{\bullet}(\mathcal{S})$ such that

$$\mathcal{D}\zeta^{C\cdots D}{}_{E\cdots F} = \xi^{PQ}D_{PQ}\zeta^{C\cdots D}{}_{E\cdots F}$$

for all $\zeta^{C\cdots D}{}_{E\cdots F} \in \mathfrak{S}^{C\cdots D}{}_{E\cdots F}$.

(vii) ***Compatibility with the ϵ-spinor.*** The operator D_{AB} satisfies $D_{AB}\epsilon_{CD} = 0$ so that, in addition, $D_{AB}h_{CDEF} = 0$.

(viii) ***No torsion.*** For $\phi \in \mathfrak{X}(\mathcal{S})$ one has that $D_{AB}D_{CD}\phi = D_{CD}D_{AB}\phi$.

The space spinor spin coefficients

Let $\{\boldsymbol{e_i}\}$ and $\{\boldsymbol{\omega^i}\}$ denote, respectively, an $\boldsymbol{h}$-orthonormal basis and cobasis on $\mathcal{S}$. One defines the ***spatial connection coefficients*** $\gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$ via the equation

$$D_{\boldsymbol{i}}\boldsymbol{e_j} = \gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}\boldsymbol{e_k}. \tag{4.19}$$

In what follows, it will be assumed that the connection $\boldsymbol{D}$ has spinorial counterpart D_{AB} satisfying conditions (i)–(viii) of the previous section. Let $\{\boldsymbol{e_{AB}}\}$ and $\{\boldsymbol{\omega^{AB}}\}$ denote, respectively, the vector basis and cobasis obtained from $\{\boldsymbol{e_i}\}$ and $\{\boldsymbol{\omega^i}\}$ through the correspondences in (4.12) and let $D_{\boldsymbol{AB}}$ denote the associated covariant directional derivative.

The spinorial counterpart of the spatial connection coefficients $\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}}$ can be obtained by contraction of $\gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$ with the spatial Infeld-van der Waerden symbols so that the reality condition

$$\gamma^{+}_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}} = -\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}} \tag{4.20}$$

holds. Now, defining the ***space spinor directional covariant derivative*** $D_{\boldsymbol{AB}} \equiv \sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}}D_{\boldsymbol{i}}$, the spinorial counterpart of (4.19) can be written as

$$D_{\boldsymbol{AB}}\boldsymbol{e_{EF}} = \gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}}\boldsymbol{e_{CD}}.$$

Hence, one has

$$\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}} = \langle \boldsymbol{\omega^{CD}}, D_{\boldsymbol{AB}}\boldsymbol{e_{EF}}\rangle,$$

so that $\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}}$ has the symmetries

$$\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}} = \gamma_{(\boldsymbol{AB})}{}^{(\boldsymbol{CD})}{}_{(\boldsymbol{EF})}.$$

Now, as $D_{\boldsymbol{AB}}h_{\boldsymbol{CDEF}} = 0$, it follows then from

$$D_{\boldsymbol{AB}}h_{\boldsymbol{CDEF}} = \boldsymbol{e_{AB}}(h_{\boldsymbol{CDEF}}) - \gamma_{\boldsymbol{AB}}{}^{\boldsymbol{PQ}}{}_{\boldsymbol{CD}}h_{\boldsymbol{PQEF}} - \gamma_{\boldsymbol{AB}}{}^{\boldsymbol{PQ}}{}_{\boldsymbol{EF}}h_{\boldsymbol{CDPQ}},$$

that

$$\gamma_{\boldsymbol{ABCDEF}} = -\gamma_{\boldsymbol{ABEFCD}}.$$

This antisymmetry can be exploited to obtain the decomposition

$$\begin{aligned}\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}} &= \frac{1}{2}\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{PD}}{}_{\boldsymbol{PF}}\delta_{\boldsymbol{E}}{}^{\boldsymbol{C}} + \frac{1}{2}\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CP}}{}_{\boldsymbol{EP}}\delta_{\boldsymbol{F}}{}^{\boldsymbol{D}} \\ &= \gamma_{\boldsymbol{AB}}{}^{\boldsymbol{D}}{}_{\boldsymbol{F}}\delta_{\boldsymbol{E}}{}^{\boldsymbol{C}} + \gamma_{\boldsymbol{AB}}{}^{\boldsymbol{C}}{}_{\boldsymbol{E}}\delta_{\boldsymbol{F}}{}^{\boldsymbol{D}},\end{aligned}$$

where the ***space spinor spin coefficients***, $\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{D}}{}_{\boldsymbol{F}} = \gamma_{(\boldsymbol{AB})}{}^{\boldsymbol{D}}{}_{\boldsymbol{F}}$, have been defined by

$$\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{D}}{}_{\boldsymbol{F}} \equiv \frac{1}{2}\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{PD}}{}_{\boldsymbol{PF}}. \tag{4.21}$$

Observing the reality condition (4.20) and that $\epsilon^{+}_{AB} = \epsilon_{AB}$, it follows that

$$\gamma^{+}_{\boldsymbol{AB}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}} = -\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}};$$

that is, the space spinor connection coefficients are imaginary. A computation similar to the one performed in Section 3.2.2 to express the spacetime spin coefficients in terms of derivatives of the spin basis shows that

$$\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}} = \epsilon^{\boldsymbol{C}}{}_Q D_{\boldsymbol{AB}} \epsilon_{\boldsymbol{D}}{}^Q = -\epsilon_{\boldsymbol{D}}{}^Q D_{\boldsymbol{AB}} \epsilon^{\boldsymbol{C}}{}_Q.$$

From these expressions, it can be shown that given spinors κ_A and μ^A with components $\kappa_{\boldsymbol{A}}$ and $\mu^{\boldsymbol{A}}$ with respect to the space spinor basis $\{\epsilon_{\boldsymbol{A}}{}^A\}$, one has

$$D_{\boldsymbol{AB}}\kappa_{\boldsymbol{C}} = \boldsymbol{e}_{\boldsymbol{AB}}(\kappa_{\boldsymbol{C}}) - \gamma_{\boldsymbol{AB}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{C}}\kappa_{\boldsymbol{Q}},$$
$$D_{\boldsymbol{AB}}\mu^{\boldsymbol{C}} = \boldsymbol{e}_{\boldsymbol{AB}}(\mu^{\boldsymbol{C}}) + \gamma_{\boldsymbol{AB}}{}^{\boldsymbol{C}}{}_{\boldsymbol{Q}}\mu^{\boldsymbol{Q}},$$

where $D_{\boldsymbol{AB}}\kappa_{\boldsymbol{C}} \equiv \epsilon_{\boldsymbol{C}}{}^Q D_{\boldsymbol{AB}}\kappa_Q$ and $D_{\boldsymbol{AB}}\mu^{\boldsymbol{C}} \equiv \epsilon^{\boldsymbol{C}}{}_Q D_{\boldsymbol{AB}}\mu^Q$.

The three-dimensional curvature spinors

As $\boldsymbol{D}$ is being assumed to be the Levi-Civita connection of a three-dimensional negative definite metric $\boldsymbol{h}$, it follows that the spinorial counterpart $r_{ABCDEFGH}$ of the Riemann tensor r_{ijkl} of $\boldsymbol{D}$ satisfies the antisymmetry property

$$r_{ABCDEFGH} = -r_{CDABEFGH}.$$

Hence, one has the decomposition

$$r_{ABCDEFGH} = -r_{ACEFGH}\epsilon_{BD} - r_{BDEFGH}\epsilon_{AC}, \tag{4.22}$$

with

$$r_{ACEFGH} \equiv \frac{1}{2} r_{AQC}{}^Q{}_{EFGH}, \qquad r_{ACEFGH} = r_{(AC)EFGH}.$$

Now, as $r_{ABCDEF} = -r_{ABEFCD}$ one has further that

$$r_{ABCDEF} = r_{ABCE}\epsilon_{DF} + r_{ABDF}\epsilon_{CE},$$

with

$$r_{ABCE} = \frac{1}{2} r_{ABCQE}{}^Q, \qquad r_{ABCE} = r_{AB(CE)}.$$

As a consequence of the symmetry $r_{ABCDEFGH} = r_{EFGHABCD}$, the spinor r_{ABCD} inherits the symmetry $r_{ABCD} = r_{CDAB}$. Taking into account all these symmetries in the general decomposition for a general valence-4 spinor, Equation (3.8), one concludes that

$$r_{ABCD} = r_{(ABCD)} + \frac{1}{3} r_{PQ}{}^{PQ} h_{ABCD}.$$

In what follows let s_{ABCD} and r denote, respectively, the ***spinorial counterpart of the trace-free Ricci tensor*** and the ***Ricci scalar*** of the connection $\boldsymbol{D}$. One has that

$$s_{ABCD} = r_{(ABCD)}, \qquad r = -4 r_{PQ}{}^{PQ}.$$

Hence, one finds that r_{ABCDEF} can be written as

$$r_{ABCDEF} = \left(\frac{1}{2}s_{ABCE} - \frac{1}{12}rh_{ABCE}\right)\epsilon_{DF} + \left(\frac{1}{2}s_{ABDF} - \frac{1}{12}rh_{ABDF}\right)\epsilon_{CE}. \tag{4.23}$$

Using an argument similar to the one employed in Section 3.2.3 one finds that the commutator of the covariant derivative D_{AB} satisfies

$$(D_{AB}D_{CD} - D_{CD}D_{AB})\mu^E = r^E{}_{FABCD}\mu^F.$$

Finally, for completeness it is noticed that the three-dimensional second Bianchi identity takes, in the present context, the form

$$D^{PQ}s_{PQAB} = \frac{1}{6}D_{AB}r.$$

This last expression can be obtained from multiplying by ϵ^{ijk} the tensorial Bianchi identity

$$D_i r_{jklm} + D_j r_{kilm} + D_k r_{ijlm} = 0,$$

and considering its spinorial counterpart using Equations (4.22) and (4.23).

4.4 Further reading

The notions of space spinor and space spinor splits were originally introduced in Sommers (1980); see also Sen (1981). A monograph on space spinors is Torres del Castillo (2003). An alternative discussion, having applications in quantum gravity in mind, is given in an appendix of Ashtekar (1991). The space spinor formalism was first used in Friedrich (1988, 1991) to analyse the conformal field equations. Further developments can be found in Friedrich (1995, 1998c), and a slightly different perspective on these ideas is given in Frauendiener (1998a).