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PLANARITY AND MINIMAL PATH ALGORITHMS

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Abstract

In 1981 two notions of effective presentation of countable connected graphs were formulated by J. C. E. Dekker—namely, edge recognition algorithm graphs and minimal path algorithm graphs. In this paper we show that every planar graph has a minimal path algorithm presentation but that some graphs have no minimal path algorithm presentations. We introduce the notion of a shortest distance algorithm graph, show that it lies strictly between the two notions of Dekker, and show that every graph has a shortest distance algorithm presentation. Finally, in contrast to Dekker's result about minimal path algorithm graphs, we produce a shortest distance algorithm graph which has no spanning tree which is an edge recognition algorithm graph.

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In [1] J. C. E. Dekker introduced the notions of an edge recognition algorithm graph and a minimal path algorithm graph. In this paper, all graphs will be countable, connected, have no loops or multiple edges, and have a vertex set contained in the natural numbers $N = \{0, 1, 2, \ldots\}$. In [1] Dekker showed that

(i) every graph has a presentation as an edge recognition algorithm graph with an isolated vertex set,
(ii) there are graphs which have a presentation as a minimal path algorithm graph but have no presentation as a minimal path algorithm graph with an isolated vertex set, and
(iii) every minimal path algorithm graph has a spanning tree which is a minimal path algorithm graph.

In addition, he gave a characterization of all graphs which have a presentation as a minimal path algorithm graph with an isolated vertex set.

In this paper, we answer a number of questions which seem to arise naturally from the notions and results above. First, we show that every planar graph has a presentation as a minimal path algorithm graph with an isolated vertex set. Next we show that there are graphs which have no presentations as minimal path algorithm graphs even if we do not require the vertex set to be isolated. Also we introduce the notion of a shortest distance algorithm graph and prove that this notion lies strictly between those of edge recognition algorithm graph and minimal path algorithm graph. We show that every graph has a presentation as a shortest distance algorithm graph with an isolated vertex set and produce a shortest distance algorithm graph, and hence an edge recognition algorithm graph, which has no spanning tree which is an edge recognition algorithm graph.

0. Definitions and conventions

We shall often refer to two (classically) isomorphic graphs as different "presentations" of the same graph. Given a graph $G = (V, E)$ with vertex set V and edge set E, we say that $G_1 = (V_1, E_1)$ is a subgraph of G if $V_1 \subseteq V$ and $E_1 = \{\{x, y\} \subseteq V_1 | \{x, y\} \in E\}$. A *spanning tree* of a connected graph $G = (V, E)$ is a connected tree $T = (V', E')$ where $V' = V$ and $E' \subseteq E$. Note that according to our definitions a spanning tree of G is not necessarily a subgraph of G. A planar graph is a graph which has a presentation with the vertex set consisting of points in the plane, and in which adjacent vertices are connected by curves in the plane which do not intersect each other except at common vertices.

Given a graph $G = (V, E)$, and given $\{x, y\} \subseteq V$, we say that a sequence of vertices $\langle x_0, x_1, \ldots, x_n \rangle$ is a *path* of length n from x to y in G if $x_0 = x$, $x_n = y$, and $\{x_i, x_{i+1}\} \in E$ for $i = 0, \ldots, n-1$. $\langle x_0, x_1, \ldots, x_n \rangle$ is called a *minimal path* from x to y in G if the length of any other path from x to y in G is at least n.

The following two definitions are due to Dekker [1], who referred to these notions as α-graphs and ω-graphs, respectively.

DEFINITION 1. A graph $G = (V, E)$ is an *edge recognition algorithm graph* (ERA) if there is a partial recursive function ε such that for all $\{x, y\} \subseteq V$,

$$\varepsilon(x, y) = \begin{cases} 1 & \text{if } \{x, y\} \in E, \\ 0 & \text{if } \{x, y\} \notin E. \end{cases}$$

Such an ε is called an ERA *function* for G.

Let $N^{<\omega}$ denote the set of all finite sequences of N and let $\langle\ \rangle\colon N^{<\omega} \to N$ be some fixed one to one and onto recursive function.

DEFINITION 2. A graph $G = (V, E)$ is a *minimal path algorithm graph* (MPA) if there is a partial recursive function μ such that for every $\{x, y\} \subseteq V$, $\mu(x, y) = \langle x_0, \ldots, x_n \rangle$, where $x = x_0$, $y = x_n$, and $\langle x_0, \ldots, x_n \rangle$ is some minimal path from x to y in G. (Recall that we are assuming all graphs to be connected.) Such a μ is called an MPA *function* for G.

Finally, a related definition which we shall also consider is the following.

DEFINITION 3. A graph $G = (V, E)$ is a *shortest distance algorithm graph* (SDA) if there is a partial recursive function σ such that for every $\{x, y\} \subset V$, $\sigma(x, y) = n$, where n is the length of a minimal path between x and y in G. Such a σ is called an SDA *function* for G.

It is clear that given any MPA function for G, one can construct an SDA function for G, and given any SDA function for G, one can construct an ERA function for G. Thus the following implications hold:

$$\text{MPA} \Rightarrow \text{SDA} \Rightarrow \text{ERA}.$$

We shall see later that none of the converse implications hold in general.

DEFINITION 4. A subgraph $G_1 = (V_1, E_1)$ of a graph G is called *minimal path closed* if G_1 is connected, and if, for all $\{x, y\} \subseteq V_1$, any minimal path between x and y in G_1 is also a minimal path between x and y in G.

DEFINITION 5. A graph G is called *locally finite* if every finite subgraph G_1 of G is contained in a minimal path closed subgraph G_2 of G, where G_2 is also finite.

1. MPA presentations of planar graphs

The main result of this section is that every planar graph has an MPA presentation. For completeness, we include a direct proof of the following result of [1], which is one of the two keys to the main result of this section.

THEOREM 1.1. *Every locally finite graph has an* MPA *presentation with an isolated vertex set.*

PROOF. We shall construct an MPA U having a minimal path function μ with the property that every locally finite graph G is isomorphic to a subgraph G' of U such that, for all vertices x and y in G', the path $\mu(x, y)$ also lies in G'. The isomorphism between G and G' will not in general be effective.

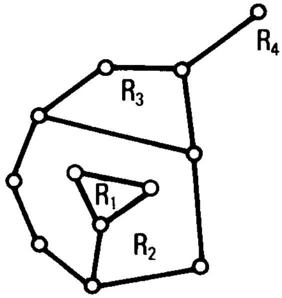
Let $G_0, G_1, \ldots$ be an effective list of all finite graphs. We shall construct U in stages so that at the end of each stage $s > 0$, we will have specified a recursive vertex set V_s such that $N - V_s$ is infinite, and we will have defined μ on $V_s \times V_s$ so that, for all $x, y \in V_s$, the path $\mu(x, y)$ also lies in V_s. At the end of each stage $s > 0$, we will have specified infinitely many distinguished finite subgraphs U_0^s, $U_1^s, \ldots$, such that for each i and for all vertices x, y in U_i^s, the path $\mu(x, y)$ also lies in U_i^s. At stage 0, we will have one distinguished subgraph U_0^0 which consists of a single vertex. To go from stage s to stage $s + 1$, we consider each distinguished subgraph $U_i^s = (V_i^s, E_i^s)$. For each finite graph G_j, we see if there is a way to extend U_i^s to a graph $G_j^{i,s}$, isomorphic to G_j, by adding new edges and vertices to U_i^s in such a way that we can extend our minimal path function μ on U_i^s to a minimal path function on $G_j^{i,s}$. If there is such an extension, we introduce new vertices and edges for $G_j^{i,s}$, we extend our definition of μ to a minimal path function, and we make $G_j^{i,s}$ a distinguished subgraph for stage $s + 1$. We emphasize that if $l \neq k$, then the only vertices which the edges of $G_l^{i,s}$ and $G_k^{i,s}$ have in common will be U_i^s. Similarly, if $i \neq j$, the new vertices and edges of any extensions of U_i^s and U_j^s will be pairwise disjoint. It is not difficult to see that this construction can be carried out in an effective manner in such a way that the final graph U is an MPA with vertex set equal to N. Moreover, by construction, $\mu(x, y)$ will be the index of a minimal path in U between x and y, since once $\mu(x, y)$ is defined at some stage s, we never allow new vertices and edges to be introduced at later stages which would violate the fact that $\mu(x, y) = \langle x_0, \ldots, y_n \rangle$ is a minimal path between x and y in U. Thus U is an MPA with MPA function μ.

Now suppose $G = (V, E)$ is a locally finite graph, where $V \subseteq N$. Clearly, if G is finite, then at stage l some U_i^l is isomorphic to G. If G is infinite, then there exists a sequence $G_1 \subset G_2 \subset G_3 \subset \cdots$ (not necessarily effective) of finite minimal path closed subgraphs of G such that if $G_i = (V_i, E_i)$, then $\bigcup G_i = (\bigcup V_i, \bigcup E_i) = G$. Using the fact that all of the G_i are minimal path closed, it is easy to show by induction that at each stage $s > 0$, there is a distinguished subgraph $U_{i_s}^s$ isomorphic to G_s, and that, moreover, one of the distinguished extensions of $U_{i_s}^s$, $G_j^{i_s, s}$, is isomorphic to G_{s+1}. Thus, if $U' = \bigcup U_{i_s}^s$, then U' is isomorphic to G, and since for all i, and for all vertices x, y in $U_{i_s}^s$, the path with index $\mu(x, y)$ also lies in $U_{i_s}^s$, it follows that μ is a MPA function for U', and hence that G is isomorphic to an MPA. By the same argument, it follows that for any strictly increasing function f: $N \to N$, there is a sequence $U_{i_{f(1)}}^1 \subset U_{i_{f(2)}}^2 \subset U_{i_{f(3)}}^3 \subset \ldots$ of distinguished subgraphs such that $U_{i_{f(n)}}^n$ is isomorphic to $G_{f(n)}$ for all n, and such that $U^f = \bigcup_n U_{i_{f(n)}}$ is an MPA isomorphic to G with MPA function μ. Now if f and g are strictly increasing functions from N into N and if, for some n, $f(n) \neq g(n)$, then clearly $U_{i_{f(n)}} \neq U_{i_{g(n)}}$, and since all extensions of distinguished subgraphs in our construction are pairwise disjoint, it follows that $U^f \cap U^g \subseteq U_{i_{f(n)}} \cap U_{i_{g(n)}}$, and hence that $U^f \cap U^g$ is finite. Thus, since there are $2^{\aleph_0}$ strictly increasing functions from N into N and only countably many infinite r.e sets, it follows that $2^{\aleph_0}$ strictly increasing functions f: $N \to N$ the vertex set of U^f contains no infinite r.e set, i.e., U^f is an MPA presentation of G with an isolated vertex set.

Of course, as Dekker points out, if G is an MPA with MPA function μ, and if G is not locally finite, then we can use μ and some finite subgraph G' of G which is not contained in any minimal path closed finite subgraph of G to generate an infinite r.e subset of the vertices of G. Thus, if G has an MPA presentation with an isolated vertex set, G must be locally finite.

THEOREM 1.2. *Every planar graph is locally finite.*

PROOF. Let A_0 be any finite subset of a planar graph G. If A_0 is not minimal path closed, then there must exist two vertices x_0, y_0 in A_0 between which there exists a shorter path in G such that all vertices of that path except x_0 and y_0 are not in A_0. Let A_1 be the subgraph of G whose vertices are those of A_0 together with the vertices on the shorter path between x_0 and y_0. Since there are at least two disjoint paths in A_1 between x_0 and y_0, A_1 partitions the plane into finitely many regions. Each of these regions is bounded by a cycle of A_1, and each edge of A_1 is counted twice in counting the edges of all the regions bounded by A_1. Figure 1 gives an example of the meaning of the length of boundaries of regions determined by a finite planar graph.



Region	Length
1	3
2	11
3	4
4	10

Figure 1

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If A_1 is not minimal path closed in G, then we may find x_1, y_1 in A_1 with a shorter path in G disjoint from A_1. Let A_2 be the subgraph of G whose vertex set is that of A_1 together with the vertices of a shortest path between x_1 and y_1. Since A_2 is planar, the new path lies entirely within one of the regions determined by A_1. Thus the regions determined by A_2 are those of A_1, except that the region of the new path is now subdivided into at least two new subregions. Each of these new A_2 regions has a length which is strictly less than the A_1 region of which it is a part. To see this, consider first the graph whose vertex set is that of A_2, and whose edges consist only of those A_1 and those on the new path between x_1 and y_1. The former region R_i is now split into two regions, $R_{i,a}$ and $R_{i,b}$. Since the new path between x_1 and y_1 is shorter than any path in A_1, the lengths of both of the regions $R_{i,a}$ and $R_{i,b}$ are less than the length of R_i. As edges are introduced to further divide the regions, the new regions always each have a shorter length than the divided region because there are no multiple edges.

Since each addition of a new and shorter path between two vertices leads to a finer partition of the plane into regions with shorter lengths, we see that the process cannot continue indefinitely. Within each region determined by A_1, we can successively find at most as many new minimal paths as the length of that region. Each such new path leads to finitely many more regions, but since the depth is bounded and the partition of a region is always into only finitely many new regions, there are only finitely many regions which can be added before a minimal path closed subgraph of G is obtained.

THEOREM 1.3. *Every planar graph has an* MPA *presentation with an isolated vertex set.*

Theorem 1.3 follows immediately from Theorems 1.1 and 1.2. As Dekker [1] points out, by a modification of the proof of Theorem 1.1 we can strengthen Theorem 1.3 to assert that the vertex set of the MPA presentation is isolated and also regressive.

2. Some graphs which are not locally finite

We shall examine some ways to construct graphs which are not locally finite, thereby obtaining examples both of graphs which have MPA presentations, but not with isolated vertex sets, and of graphs which have no MPA presentations. In view of the previous section, we know that these examples cannot be planar graphs. Dekker [1] has an example of a not locally finite graph which has an MPA presentation. The example described here is different from his, and it is easier to use to construct a graph with no MPA presentation.

THEOREM 2.1. *There is a graph G which is not locally finite.*

PROOF. For each n, let $R_n = \{a_n, b_n, \ldots, h_n\}$ be a set of eight vertices, and let $E_n = \{(a_n, b_n), (b_n, c_n), \ldots, (g_n, h_n)(h_n, a_n)\}$ be the set of 8 edges so that $G_n = (R_n, E_n)$ is a cycle. The sets R_n will be pairwise disjoint. The vertex set of G will be the union of all the sets R_n. Let

$$A_n = \{(a_n, a_{n+1}), (c_n, e_{n+1}), (e_n, b_{n+1}), (g_n, f_{n+1})\}.$$

The edges of G will be the union of all the edge sets $E_n \cup A_n$. Figure 2, which shows all the edges in $R_n \cup R_{n+1}$, will be useful in analyzing G.

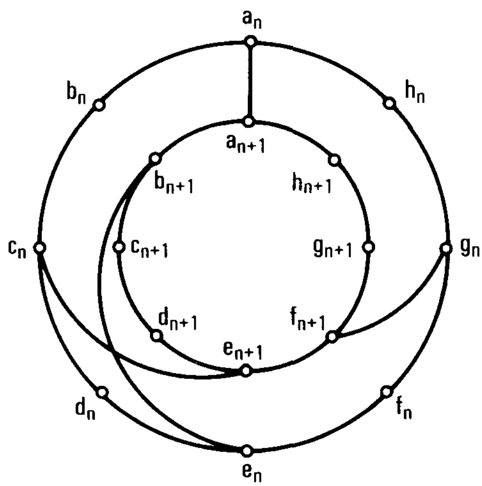

Figure 2

Examination of Figure 2 reveals that there is a unique minimal path in G between each pair of points in R_n, except for the pairs (d_n, h_n) and (b_n, f_n). It is now easy to see that any minimal path closed subgraph H of G which contains R_n must also contain R_{n+1}. Indeed, since $R_n \subset H$, we may successively use the following pairs to obtain all of the points in R_{n+1}:

(a_n, e_n)	gives	a_{n+1}, b_{n+1};
(c_n, g_n)	gives	e_{n+1}, f_{n+1};
(b_{n+1}, e_{n+1})	gives	e_{n+1}, d_{n+1}; and
(a_{n+1}, f_{n+1})	gives	h_{n+1}, g_{n+1}.

Thus G is the only minimal path closed subgraph of G which contains R_0.

We note that G can easily be presented as an MPA. Moreover, in any MPA presentation of G the vertex set must be recursively enumerable, since we can effectively enumerate it from R_0 by using the minimal path algorithm. Indeed, in any MPA presentation, each of the functions of n called $a_n, b_n, \ldots, h_n$ is a recursive function of n.

THEOREM 2.2. *There is a graph G' which has no* MPA *presentation. Moreover, there is such a graph that has an* ERA *presentation with a recursive vertex set.*

PROOF. The desired graph G' will be a modification of the graph G constructed in Theorem 2.1: that is, to construct G' from G, we shall, for certain n, simply add new vertices x_n and y_n and new edges (d_n, x_n), (x_n, y_n), (y_n, h_n) between d_n and h_n so that the length of a minimal path between d_n and h_n becomes 3 instead of 4. Thus, let R_n, E_n, and A_n be as in the proof of Theorem 2.1. Let $R_n^0 = R_n$, $E_n^0 = E_n$, $R_n^1 = R_n \cup \{x_n, y_n\}$, and $E_n^1 = E_n \cup \{(d_n, x_n), (x_n, y_n), (y_n, h_n)\}$ for all n. For each subset $X \subseteq N$, let $G_X = (\bigcup_{n \in X} R_n^1 \cup \bigcup_{n \notin X} R_n^0, \bigcup_{n \in X} E_n^1 \cup \bigcup_{n \notin X} E_n^0 \cup \bigcup_n A_n)$. Thus the graph G used in the previous theorem is $G_\varnothing$.

To see that any minimal path closed subgraph of G_X which contains R_0^0 must also contain $\bigcup_n R_n^0$, we may use the same argument as in the previous proof, since the vertices x_n, y_n, x_{n+1}, and y_{n+1}, if present, do not give shorter paths between any of the points used to obtain R_{n+1}^0 from R_n^0. Using d_n and h_n, we see that any minimal path closed subgraph of G_X which contains R_n^0 must also contain R_n^1 if $n \in X$. Thus, for each G_X, we see that the only minimal path closed subgraph of G_X containing R_0 is G_X.

Suppose that G_X has an MPA presentation. As in the proof of Theorem 2.1, we see that each of d_n and h_n is a recursive function of n. Thus, for each n, we may apply the minimal path algorithm to d_n and h_n to determine whether a minimal path between d_n and h_n has 3 edges or 4 edges. In the former case, $n \in X$, while in the latter, $n \notin X$. Since the process of determining whether or not $n \in X$ is effective, we have shown that if G_X has an MPA presentation, then X is recursive. To prove the first assertion of this theorem, we may use G_X for any non-recursive set X.

To prove the second assertion of the theorem, let X be any recursively enumerable set which is not recursive. Let f be a recursive enumeration of X without repetitions. Present G_X with $a_n = 10n$, $b_n = 10n + 1, \ldots, h_n = 10 + 7$, $x_{f(n)} = 10n + 8$, and $y_{f(n)} = 10n + 0$ for each n. The vertex set of this presentation is the set of all natural numbers. The previous description of G_X shows that this is an ERA presentation. Since X is not recursive, G_X has no MPA presentation, and our proof is complete.

3. Shortest distance algorithm graphs

In this section we shall show that there are SDA presentations of graphs which fail to be MPA presentations, and ERA presentations of graphs which fail to be SDA presentations. Thus the notion of shortest distance algorithm graph lies

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strictly between Dekker's notions of edge recognition graphs and minimal path algorithm graphs. First we shall show that every graph has an SDA presentation. This fact combined with the fact that there are graphs with no MPA presentations will immediately establish the first result mentioned above. Next we will give a direct construction of an ERA presentation which is not an SDA presentation. Finally, we establish another difference between MPA's and SDA's by showing that, in contrast to result (iii) mentioned in the introduction, there is a graph G which is an SDA, but for which no spanning tree of G is an ERA, much less an SDA.

THEOREM 3.1. *Every graph has an* SDA *presentation with an isolated vertex set.*

PROOF. We shall construct an ERA graph U and a partial recursive function σ. It will not be the case that σ is a shortest distance algorithm on all of U, but we will construct σ so that for any graph G, there exists a subgraph U' of U such that U' is isomorphic to G and σ is an SDA function for U'.

Much as in our construction of Theorem 1.1, we shall construct U in stages. U^s will denote the finite subgraph of U constructed by the end of stage s, and within U^s we will specify finitely many connected distinguished subgraphs $U_1^s, \ldots, U_{k_s}^s$. Moreover, for each index i and vertices x, y in U_i^s, $\sigma(x, y)$ will be defined, and $\sigma(x, y)$ will be less than or equal to the actual minimal path length between x and y in U_i^s. At stage 0, $U^0 = U_0^0$ consists of a single vertex. At stage $s + 1$, we shall construct extensions of each distinguished subgraph $U_i^s = (V_i^s, E_i^s)$ as follows. For each subset $S \subseteq V_i^s$, we determine if it is possible to introduce a new vertex x and edges $\{(x, v) | v \in S\}$ and still maintain the fact that, for all u, v in V_i^s, $\sigma(u, v) \leqslant$ the length of a minimal path between u and v in the graph $(\{x\} \cup V_i^s, E_i^s \cup \{(x, v) | v \in S\})$. If so, then for each of the possible ways of defining $\{\sigma(x, v) | v \in V_i^s\}$ such that $\sigma(x, v) \leqslant$ the length of a minimal path between x and v in the graph $(\{x\} \cup V_i^s, E_i^s \cup \{(x, v) | v \in S\})$, we introduce a new vertex x' together with new edges (x', v) for $v \in S$, we define $\sigma(x', v)$ for $v \in V_i^s$ accordingly, and we declare the graph $(\{x'\} \cup V_i^s, E_i^s \cup \{(x', V) | v \in S\})$ to be a distinguished subgraph at stage $s + 1$. U^{s+1} will consist of the union of all these extensions of the distinguished subgraphs $U_1^s, \ldots, U_{k_s}^s$. Note that unless x and y lie in the same distinguished subgraph $U^s i$ at some stage s, there is no edge between x and y, and so $\sigma(x, y)$ is not even defined. It is easy to see that this construction can be carried out effectively so that the vertex set of U equals N, and so that σ is partial recursive.

Let us call a graph isomorphism f: $G \to U$ *distance preserving* if, for all vertices x, y in G, $\sigma(f(x), f(y))$ is the length of a minimal path from x to y in G. To see that every graph G is isomorphic to a subgraph of U via a distance preserving isomorphism, let $g_0, g_1, g_2, \ldots$ be an enumeration of the vertex set of G such that, for each n, the subgraph of G whose vertex set is $\{g_0, \ldots, g_n\}$ is connected. Assume by induction that we can define $f(g_0), \ldots, f(g_n)$ so that $\{f(g_0), \ldots, f(g_n)\}$ is the vertex set of a distinguished subgraph $U_i^n = (V_i^n, E_i^n)$ at stage n, where, for all i and j, $\sigma(f(g_i), f(g_j))$ = length of a minimal path between g_i and g_j in G. Then it is clear that one of the extensions constructed for U_i^n at stage $n+1$ will be of the form $(\{x'\} \cup V_i^n, E_i^n \cup \{(x', v) | v \in S\})$ where $S = \{f(g_i) | i \leqslant n$, and (g_i, g_{n+1}) is an edge in $G\}$, and where $\sigma(x', f(g_i))$ equals the length of a minimal path between g_i and g_{n+1} in G. Then we let $f(g_{n+1}) = x'$. It now easily follows that f defines an isomorphism between G and a subgraph U' of U which is distance preserving. Thus σ is an SDA function for U', and G has an SDA presentation.

Now to ensure that G has an SDA presentation with an isolated vertex set, we need only modify the construction of U given above so that each time we construct an extension $(\{x'\} \cup V_i^s, E_i^s \cup \{(x', v) | v \in S\})$ of a distinguished graph U_i^s as above, we construct another extension $(\{x''\} \cup V_i^s, E_i^s \cup \{(x'', v) | v \in S\})$, where $x' \neq x''$, but where $\sigma(x', v) = \sigma(x'', v)$ for all $v \in V_i^s$. In this way, with G and $g_0, g_1, \ldots$ as in the previous paragraph, we will have two disjoint ways to define $f(g_{n+1})$ at each stage $n + 1 > 0$. It will follow that there are $2^{\aleph_0}$ SDA presentations of G within U, so that any two such presentations have only finitely many vertices in common. As in Theorem 1.1, it then follows that since there are only countably many infinite r.e sets, there are in fact $2^{\aleph_0}$ SDA presentations of G with an isolated vertex set.

Our next two results use techniques very similar to those developed in [2].

THEOREM 3.2. *There is an* ERA *presentation of a graph which is not an* SDA *presentation.*

PROOF. G will be a subgraph of the ERA graph H which we define now and represent in Figure 3. Let $D_n = \{5n, 5n+1, 5n+2, 5n+3, 5n+4\}$, $C_n = \{(5n, 5n+1), (5n+1, 5n+2), (5n+2, 5n+3), (5n+3, 5n+4), (5n+4, 5n)\}$, and $B_n = \{(5n+3, 5(n+1))\}$ for all n.

Let $H = (\bigcup_n D_n, \bigcup_n (C_n \cup B_n))$. Then G is a subgraph of H which contains $\{5n, 5n+1, 5n+2, 5n+3\}$ for each n. To make sure that φ_e is not a shortest distance algorithm for G, we include $5e+4$ in the vertex set of G whenever $\varphi_e(5e, 5e+3) = 3$, and for no other e. It is easy to see that G is an ERA presentation which is not an SDA presentation.

THEOREM 3.3. *There is an* SDA *presented graph with no* ERA *presented spanning subtree.*

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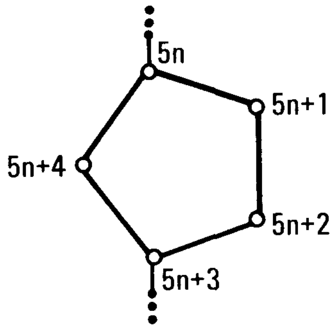

Figure 3

Proof. The proof is quite similar to the proof of Theorem 3.2; G will be a subgraph of the SDA graph H which we define now and represent in Figure 4. Let $D_n = \{4n, 4n + 1, 4n + 2, 4n + 3\}$, $C_n = \{(4n, 4n + 1), (4n + 1, 4n + 2), (4n + 2, 4n + 3), (4n + 3, 4n)\}$, and $B_n = \{(4n + 2, 4(n + 1))\}$ for all n.

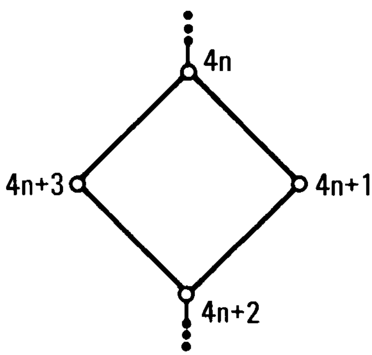

Figure 4

Let H be the graph $(\bigcup_n D_n, \bigcup_n (C_n \cup B_n))$. Clearly, if $\sigma(x, y) =$ the length of a minimal path between x and y in H, then σ is a recursive function, and hence H is a SDA. Let G be a subgraph of H which contains $\{4n, 4n + 2\}$ and at least one of $4n + 1$ and $4n + 3$ for every n. Now any such G will be a SDA since σ will be a SDA function for G.

Note that if T is a spanning tree for G, then T must exclude exactly one of the edges in C_n if both $4n + 1$ and $4n + 3$ are in G, while if $4n + 1$ is in G but $4n + 3$ is not in G, then T must contain both the edges $(4n, 4n + 1)$ and $(4n + 1, 4n + 2)$; similarly, if $4n + 1$ is not in G and $4n + 3$ is in G, then T must contain both the edges $(4n, 4n + 3)$ and $(4n + 2, 4n + 3)$. Now to ensure that the eth partial recursive function φ_e is not an ERA function for a spanning tree T of

G, we construct G as follows. Consider $\varphi_e(4e, 4e+1)$, $\varphi_e(4e+1, 4e+2)$, $\varphi_e(4e+2, 4e+3)$ and $\varphi_e(4e+3, 4e)$. If it is not the case that all these values are defined, and that exactly three of the values are 1 and one of the values is 0, then we include both the vertices $4e+1$ and $4e+3$ in G. Otherwise, if one of $\varphi_e(4e, 4e+1)$ or $\varphi_e(4e+1, 4e+2)$ equals 0, we put $4e+1$ into G and exclude $4e+3$ from G; and if one of $\varphi_e(4e, 4e+3)$ or $\varphi_e(4e+2, 4e+3)$ is 0, we put $4e+3$ into G and exclude $4e+1$ from G. In any case, it is easy to see that φ_e cannot be an ERA function for a spanning tree T of G.

We close this paper with some related questions which we have not answered. Is there a planar MPA U such that every planar graph is isomorphic to a subgraph of U? Are there interesting characterizations of the recursive equivalence types of vertex sets of ERA presentations of specific graphs?

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