ON THE REPRESENTATION OF BOOLEAN ALGEBRAS

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1. Let B be a Boolean algebra and let $\mathfrak{m}$ and $\mathfrak{n}$ be two systems of subsets of B, both containing all finite subsets of B. Let us assume further that the join[1] $\bigvee M$ of every set $M \in \mathfrak{m}$ and the meet $\bigwedge N$ of every set $N \in \mathfrak{n}$ exist. Several authors[2] have treated the question under which conditions there exists an isomorphism φ between B and a field $\mathfrak{F}$ of sets, satisfying the conditions:

$$\text{if } M \in \mathfrak{m}, \text{ then } \varphi(\bigvee M) = \cup \varphi(M),$$

$$\text{if } N \in \mathfrak{n}, \text{ then } \varphi(\bigwedge N) = \cap \varphi(N).$$

An obvious necessary condition for the existence of such an isomorphism is the following distributive law:

If $\{x_{ij} \mid j \in J_i\} \in \mathfrak{m}$ for all $i \in I$, $\{\bigvee_{j \in J_i} x_{ij} \mid i \in I\} \in \mathfrak{n}$ and $\{x_{i\alpha(i)} \mid i \in I\} \in \mathfrak{n}$ for all $\alpha \in \prod_{i \in I} J_i$, then

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} x_{ij} = \bigvee_{\alpha \in \pi J_i} \bigwedge_{i \in I} x_{i\alpha(i)}.$$

However, this distributive law is - in general - not sufficient. In fact, there exist $\mathfrak{m}$-complete ($\mathfrak{m}$ a certain transfinite cardinal) Boolean algebras[3] which satisfy this distributive law for all

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[1] We denote by "$\bigvee, \bigwedge$" the lattice theoretical operations, by "$\cup$, $\cap$" the corresponding set theoretical operations.

[2] See Sikorski, Boolean algebras, Berlin-Göttingen-Heidelberg 1960, pp. 79 ff and the literature cited there.

[3] See Sikorski, loc. cit. p. 93, D).

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families $(x_{ij})_{i \in I, j \in J_i}$ with $|I| \leq \mathfrak{m}$ and $|J_i| \leq \mathfrak{m}$ for all $i \in I$, and, yet, have no representation of the above mentioned type. So far, it seems to have remained unnoticed that a slight modification of the distributive law yields a necessary and sufficient condition for the existence of such a representation.

2. For an arbitrary set $N \subseteq B$ let $N' = \{x' \mid x \in N\}$ be the set of all complements x' of elements $x \in N$. For a system $\mathfrak{n}$ of subsets of B define $\mathfrak{n}' = \{N' \mid N \in \mathfrak{n}\}$. There is no loss of generality if we treat our problem only in the case $\mathfrak{m} = \mathfrak{n}'$. For, as is easily seen, if the equality $\varphi(\bigvee M) = \bigcup \varphi(M)$ holds, we also have $\varphi(\bigwedge M') = \bigcap \varphi(M')$, and dually. Therefore, if $\mathfrak{n}$ is an arbitrary system of subsets of B, we mean by an $\mathfrak{n}$ representation of B an isomorphism φ between B and a field $\mathfrak{F}$ of sets which satisfies the condition:

$$\text{if } N \in \mathfrak{n} \text{ then } \varphi(\bigwedge N) = \bigcap \varphi(N).$$

An $\mathfrak{n}$-representation then automatically fulfills also the condition:

$$\text{if } M \in \mathfrak{n}' \text{ then } \varphi(\bigvee M) = \bigcup \varphi(M).$$

A system $\mathfrak{n}$ of subsets of B is called closed if it has the following properties:

(a) If $N_1, N_2 \in \mathfrak{n}$ then $N_1 \cup N_2 \in \mathfrak{n}$,

(b) If $\bigwedge A = \bigwedge N$, $N \in \mathfrak{n}$ and for each $x \in N$ there exists an $a \in A$ with $a \leq x$, then also $A \in \mathfrak{n}$.

Obviously the intersection $\bigcap_k \mathfrak{n}_k$ of an arbitrary family of closed systems $\mathfrak{n}_k$ is again closed. Therefore, for an arbitrary system $\mathfrak{n}$ of subsets of B there exists a smallest closed system $\overline{\mathfrak{n}}$ which contains $\mathfrak{n}$. Moreover, if the meet of every set $N \in \mathfrak{n}$ exists then $\overline{\mathfrak{n}}$ has the same property.

Finally, we need the distributive law

$(D_{\mathfrak{n}})$ If the family $(x_{ij})_{i \in I, j \in J_i}$ has the properties $\{x_{ij} \mid j \in J_i\} \in \mathfrak{n}'$ for all $i \in I$, $\{\bigvee_{j \in J_i} x_{ij} \mid i \in I\} \in \mathfrak{n}$ and $\{x_{i\alpha(i)} \mid i \in I\} \in \mathfrak{n}$ for all

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$\alpha \in \prod_{i\in I} J_i$, then the join $\bigvee_{\alpha\in\pi J_i} \bigwedge_{i\in I} x_{i\alpha(i)}$ exists and the equality

$$\bigwedge_{i\in I} \bigvee_{j\in J_i} x_{ij} = \bigvee_{\alpha\in\pi J_i} \bigwedge_{i\in I} x_{i\alpha(i)}$$

holds.

3. Now we obtain the announced necessary and sufficient condition by postulating distributivity not only for the system $\mathfrak{N}$ but for the system $\overline{\mathfrak{N}}$. That is, we prove the following

Theorem. Let B be a Boolean algebra and $\mathfrak{N}$ a system of subsets of B which contains all finite subsets and has the property that the meet $\bigwedge N$ of every set $N \in \mathfrak{N}$ exists. Then the following two conditions are equivalent:

(1) B has an $\mathfrak{N}$-representation,

(2) B satisfies the distributive law $(D_{\overline{\mathfrak{N}}})$.

Proof. (1) → (2). Let φ be an $\mathfrak{N}$-representation. Define the system $\mathfrak{A}$ by $\mathfrak{A} = \{N \mid N \subseteq B,\ \bigwedge N$ exists and $\varphi(\bigwedge N) = \bigcap \varphi(N)\}$. Obviously the system $\mathfrak{A}$ is closed in the above defined sense. Since $\mathfrak{N} \subseteq \mathfrak{A}$ holds, we have $\overline{\mathfrak{N}} \subseteq \mathfrak{A}$. Therefore the image of the meet of any set $N \in \overline{\mathfrak{N}}$ under the mapping φ is the set theoretical intersection of the system $\varphi(N)$. If the family $(x_{ij})_{i\in I, j\in J_i}$ fulfills the assumptions of $(D_{\overline{\mathfrak{N}}})$, we infer:

$$\varphi(\bigwedge_{i\in I} \bigvee_{j\in J_i} x_{ij}) = \bigcap_{i\in I} \bigcup_{j\in J_i} \varphi(x_{ij}) = \bigcup_{\alpha\in\pi J_i} \bigcap_{i\in I} \varphi(x_{i\alpha(i)})$$

$$= \bigcup_{\alpha\in\pi J_i} \varphi(\bigwedge_{i\in I} x_{i\alpha(i)}).$$

By this equality the union $\bigcup_{\alpha\in\pi J_i} \varphi(\ {}_{i\in I}\ x_{i\alpha(i)})$ belongs to $\varphi(B)$.

It follows: $\bigcup_{\alpha\in\pi J_i} \varphi(\bigwedge x_{i\alpha(i)}) = \bigvee_{\alpha\in\pi J_i} \varphi(\bigwedge_{i\in I} x_{i\alpha(i)}) = \varphi(\bigvee_{\alpha\in\pi J_i} \bigwedge_{i\in I} x_{i\alpha(i)})$.

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If we apply the isomorphism φ^{-1} to the equality $\varphi(\bigwedge_{i \in J} \bigvee_{j \in J_i} x_{ij}) = \varphi(\bigvee_{\alpha \in \pi J_i} \bigwedge_{i \in I} x_{i\alpha(i)})$ we obtain the conclusion of the distributive law $(D_{\bar{\mathfrak{N}}})$.

(2) → (1). To prove (1), it is sufficient[4] to show that for each $b \neq o$ in B there exists a maximal (proper) filter F containing b and $\mathfrak{N}$-closed in the following sense: if $N \in \mathfrak{N}$ and $N \subseteq F$ then $\bigwedge N \in F$. We first show that there exists a maximal filter F in B which is $\mathfrak{N}$-closed, in fact, even $\bar{\mathfrak{N}}$-closed. To do this, we write the set of all those two-element subsets of B which consist of two complementary elements $a, a' \in B$ as a family: $(\{a_i, a_i'\})_{i \in I}$. Let Φ be the set of all mappings α, which attach to each element $i \in I$ one of the elements a_i or a_i'. We assert: if for a fixed $\alpha \in \Phi$ the set $A_\alpha = \{x | x \geqslant \alpha(i)$ for some $i \in I\}$ is not a maximal filter which is $\bar{\mathfrak{N}}$-closed, then we have $\{\alpha(i) | i \in I\} \in \bar{\mathfrak{N}}$ and $\bigwedge_{i \in I} \alpha(i) = o$. Let us assume first that the set A_α is not $\bar{\mathfrak{N}}$-closed. Then the set $\bar{A}_\alpha = \{x |$ there exists a set $N \in \bar{\mathfrak{N}}$, $N \subseteq A_\alpha$ with $x \geqslant \bigwedge N\}$ is, $\bar{\mathfrak{N}}$ satisfying the condition a), a filter which properly contains A_α. But A_α already contains one of each two complementary elements of B. So $\bar{A}_\alpha = B$ must hold. This implies the existence of a set $N \in \bar{\mathfrak{N}}$ with $N \subseteq A_\alpha$ and $\bigwedge N = o$. By condition b) we obtain $\{\alpha(i) | i \in I\} \in \bar{\mathfrak{N}}$ and $\bigwedge_{i \in I} \alpha(i) = o$. Let us assume next that the set A_α is $\bar{\mathfrak{N}}$-closed. Thus, in particular, A_α is a filter. By hypothesis, A_α is not a proper filter, i.e. $o \in A_\alpha$. Therefore there exists an element $i \in I$ with $\alpha(i) = o$, which again implies $\{\alpha(i) | i \in I\} \in \bar{\mathfrak{N}}$ and $\bigwedge_{i \in I} \alpha(i) = o$. We conclude: if A_α is not a maximal (proper) filter which is $\bar{\mathfrak{N}}$-closed then we have $\{\alpha(i) | i \in I\} \in \bar{\mathfrak{N}}$ and $\bigwedge_{i \in I} \alpha(i) = o$. Now, if none of the sets A_α were a maximal (proper) filter which is $\bar{\mathfrak{N}}$-closed, our

[4] See Sikorski, loc. cit., p. 80, 24.1.

distributive law ($D_{\bar{\mathfrak{N}}}$) would lead to the contradiction:
$1 = \bigwedge_{i \in I} (a_i \vee a_i') = \bigvee_{\alpha \in \Phi} \bigwedge_{i \in I} \alpha(i) = o$. We infer: there exists a maximal (proper) filter in B which is $\bar{\mathfrak{N}}$-closed. We still have to show that each element $b \neq o$ is contained in such a filter. To do this, we consider the new Boolean algebra $[o, b]$ and the set $\bar{\mathfrak{N}}_b$ of all those elements of $\bar{\mathfrak{N}}$ which are contained in $[o, b]$. Obviously the system $\bar{\mathfrak{N}}_b$ is closed with respect to $[o, b]$ and the Boolean algebra $[o, b]$ satisfies the distributive law ($D_{\bar{\mathfrak{N}}_b}$). As we have just shown, there exists a maximal (proper) filter F_b in $[o, b]$ which is $\bar{\mathfrak{N}}_b$-closed. We complete the proof by showing that the filter F generated by F_b in B is $\bar{\mathfrak{N}}$-closed. Let $N \subseteq F$ be an arbitrary element of $\bar{\mathfrak{N}}$. By property a) the set $\{b\} \cup N$ also belongs to $\bar{\mathfrak{N}}$. But the set $\{b\} \cup N$ has the same meet as the set $\{b \wedge x \mid x \in N\}$, and every element of the first has a lower bound belonging to the second. So by property b) the set $\{b \wedge x \mid x \in N\}$ belongs to $\bar{\mathfrak{N}}$. But this set is obviously contained in F_b. By hypothesis the meet $\bigwedge_{x \in N} (b \wedge x)$ belongs to F_b, and from this we obtain that $\bigwedge N \geq \bigwedge_{x \in N} (b \wedge x)$ belongs to F, completing the proof.