Canad. Math. Bull. Vol. 16 (2), 1973

ON A RELATION BETWEEN A THEOREM OF HARTMAN AND A THEOREM OF SHERMAN

BY
A. C. PETERSON(1)

1. **Introduction.** We are concerned with the nth-order linear differential equation

$$y^{(n)}+\sum_{k=0}^{n-1} p_k(x)y^{(k)}=0 \tag{1}$$

where the coefficients are assumed to be continuous. Hartman [1] proved that (see Definition 2) the first conjugate point $\eta_1(t)$ of t satisfies

$$\eta_1(t)=r_{1\cdots 1}(t). \tag{2}$$

Hartman actually proved a more general result which has very important applications in nonlinear differential equations. Opial [2] gives a shorter proof of Hartman's result in the context of (1). Sherman [3], then improved Opial's result by showing that given any $\varepsilon>0$, there is a nontrivial solution of (1) with n zeros on $[t, \eta_1(t)+\varepsilon]$ the first n of which are simple zeros with the first zero being at t. Most recently, Kim [4] gave a shorter proof of this result of Sherman's. Theorem 4 in this paper is an analogue of this result. Another very interesting result which is due to Sherman [5] is that

$$\eta_1(t)=\min\{r_{n-1,1}(t), r_{n-2,2}(t), \ldots, r_{1,n-1}(t)\}. \tag{3}$$

From Hartman's result (2) and Sherman's result (3) we obtain

$$r_{1\cdots 1}(t)=\min\{r_{n-1,1}(t), \ldots, r_{1,n-1}(t)\}. \tag{4}$$

Theorem 3 gives an interesting analogue of (4). Corollary 5 is the analogue of $\eta_1(t)$ being a continuous function of t, and Corollary 6 is the analogue of η_1 being a continuous function of the coefficients of (1). Theorem 1 generalizes some of the results in [6].

2. **Definitions and main results.** Before we define the boundary value functions $r_{i_1\cdots i_k}(t)$, we need the following definition.

DEFINITION 1. A nontrivial solution $y(x)$ of (1) is said to have an $(i_1, \ldots, i_k)$-distribution of zeros, $0\le i_m\le n$, $\sum_{n=1}^{k} i_m=n$, on $[t, b]$ provided there are numbers $t\le t_1<\cdots<t_k\le b$ such that $y(x)$ has a zero at each t_m, $1\le m\le k$, with multiplicity at least i_m.

Received by the editors July 29, 1971.
(1) This work was supported, in part, by NSF Grant GP-17321.

DEFINITION 2. Let $R=\{r>t$: there is a nontrivial solution of (1) having an $(i_1, \ldots, i_k)$-distribution of zeros, $0\le i_m\le n$, $\sum_{m=1}^k i_m=n$, on $[t, r]\}$. If $R\ne\phi$, set $r_{i_1\cdots i_k}(t)=\inf R$. If $R=\phi$, set $r_{i_1\cdots i_k}(t)=\infty$.

REMARK 1. If, in Definition 2, $k=2$ and $R\ne\phi$, then $r_{i_1 i_2}(t)=\min R$. If $k>2$ this is not true in general (see e.g. [7]).

REMARK 2. If $t\le t_1<\cdots<t_k<r_{i_1\cdots i_k}(t)\le\infty$, then there is a unique solution of (1) satisfying

$$y^{(m_j)}(t_j) = A_{m_j j}$$

where $A_{m_j j}$ is a constant, $j=1, \ldots, k$, $m_j=0, \ldots, i_j-1$.

DEFINITION 3. If $r_{n-1,1}(t)=\cdots=r_{k,n-k}(t)=\infty$ we set $\rho_k(t)=\infty$. Otherwise

$$\rho_k(t) = \min\{r_{n-1,1}(t), r_{n-2,2}(t), \ldots, r_{k,n-k}(t)\}$$

$k=1, \ldots, n-1$.

The identity (4) can now be written as

$$\rho_1(t) = r_{1\cdots 1}(t). \tag{5}$$

Before we prove the analogue of (5) we need the following theorem and lemma. Theorem 1 generalizes the first two parts of Theorem 5, [6]. See also Theorem 3.3, [7].

THEOREM 1. *For* $k=1, \ldots, n-1$,

$$r_{i_1\cdots i_m}(t) \ge \rho_k(t)$$

as long as $i_1\ge k$.

Proof. Assume $r_{i_1\cdots i_m}(t)<\rho_k(t)$, $i_1\ge k$. Then there is a nontrivial solution $v(x)$ of (1) with an $(i_1, \ldots, i_m)$ distribution of zeros on $[t, \rho_k(t))$. Let $(t\le)x_1<x_2<\cdots<x_m(<\rho_k(t))$ be the points at which these zeros of $v(x)$ occur. Define a fundamental set of solutions $\{u_j(x, x_1)\}$ of (1) by the initial conditions at $x=x_1$

$$u_j^{(i)}(x_1, x_1) = \delta_{ij} \qquad i, j = 0, \ldots, n-1.$$

Then define the $(n-i_1)$-order operator Q by

$$Q[y] = \begin{vmatrix} y & u_{i_1}(x, x_1) & \cdots & u_{n-1}(x, x_1) \\ y' & u'_{i_1}(x, x_1) & \cdots & u'_{n-1}(x, x_1) \\ \cdots & & & \cdots \\ y^{(n-i_1)} & u_{i_1}^{(n-i_1)}(x, x_1) & \cdots & u_{n-1}^{(n-i_1)}(x, x_1) \end{vmatrix}$$

Since $r_{i_1, n-i_1}(t)\ge\rho_k(t)$, $Q[y]=0$ is a nonsingular linear differential equation on $(x_1, \rho_k(t))$. Note that solutions of $Q[y]=0$ on $(x_1, \rho_k(t))$ are solutions of (1) with zeros of order $\geq i_1$ at x_1 restricted to the interval $(x_1, \rho_k(t))$. But since $r_{j,n-j}(t) \geq \rho_k(t)$, $j=n-1, n-2, \ldots, i_1(\geq k)$, $\{u_{n-1}(x, x_1), \ldots, u_{i_1}(x, x_1)\}$ forms (except for signs) a system of solutions of $Q[y]=0$ with Polya's w-property [8] on $(x_1, \rho_k(t))$. Hence [8] $Q[y]=0$ is disconjugate on $(x_1, \rho_k(t))$. This contradicts the fact that $v(x)$ is nontrivial solution of $Q[y]=0$ with at least $n-i_1$ zeros on $(x_1, \rho_k(t))$.

In analogy with the definition of an extremal solution of (1) for $\eta_1(t)$ we make the definition.

Definition 4. If $\rho_k(t) < \infty$, then any nontrivial solution of (1) with a zero at t of order at least k and a total of n zeros on $[t, p_k(t)]$ is called an extremal solution of (1) for $\rho_k(t)$.

Remark 3. It follows from Theorem 1 that every extremal solution of (1) for $\rho_k(t)$ has a zero at $\rho_k(t)$.

The following lemma will be used in the proof of Theorem 3.

Lemma 2.

$$r_{i_1 \cdots i_m}(t) \geq \min[r_{i_1 1 \cdots 1}(t), r_{i_1, n-i_1}(t)].$$

Proof. Let $b = \min[r_{i_1 1 \cdots 1}(t), r_{i_1 n-i_1}(t)] \leq \infty$, and let $s \in [t, b)$. In Theorem (I) ([1, p. 124]) take F to be the set of all solutions of (1) with a zero at s of order $\geq$ to i_1. It follows from Theorem (I) that no nontrivial solution of (1) has a zero of order i_1 at s and a distribution of zeros on (s, b) whose multiplicities add up to a number $\geq$ to $n-i_1$. Since s is arbitrary in $[t, b)$, no nontrivial solution of (1) has an $(i_1, \ldots, i_m)$-distribution of zeros on $[t, b)$.

In the proof of Theorem 3 we will use notation used by Kim [4]. Let $y_1(x), \ldots, y_n(x)$ be n linearly independent solutions of (1). Define

$$w(x; x_1^{k_1}, \ldots, x_m^{k_m}) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1(x_1) & y_2(x_1) & \cdots & y_n(x_1) \\ y_1'(x_1) & y_2'(x_1) & \cdots & y_n'(x_1) \\ \cdots & & \cdots & \cdots \\ y_1^{(k_1-1)}(x_1) & y_2^{(k_1-1)}(x_1) & \cdots & y_n^{(k_1-1)}(x_1) \\ y_1(x_2) & y_2(x_2) & \cdots & y_n(x_2) \\ \cdots & & \cdots & \cdots \\ y_1(x_m) & y_2(x_m) & \cdots & y_n(x_m) \\ \cdots & & \cdots & \cdots \\ y_1^{(k_m-1)}\ x_m) & y_2^{(k_m-1)}(x_m) & \cdots & y_n^{(k_m-1)}(x_m) \end{vmatrix}$$

$1 \leq m \leq n-1$, $k_1 + \cdots + k_m = n-1$. If no superscript appears on one of the x_i's it

https://doi.org/10.4153/CMB-1973-046-7 Published online by Cambridge University Press

is assumed to be a one. Note, as Kim points out [4], that $w(x; x_1^{k_1}, \ldots, x_m^{k_m})$ is a continuous function of the terms, e.g., $y_2^{(k_1-1)}(x_1)$, appearing in the determinant.

THEOREM 3. *If* $\rho_k(t) < r_{k,n-k}(t)$, *then*

$$r_{k1\cdots 1}(t) = \rho_k(t), \qquad k = 1, \ldots, n-2.$$

Proof. By Theorem 1, $r_{k1\ldots 1}(t) \geq \rho_k(t)$. Assume Theorem 3 is false, then $r_{k1\ldots 1}(t) > \rho_k(t)$. Since $\rho_{k+1}(t) = \rho_k(t) < r_{k,n-k}(t)$, there is an $x_0 \in [t, \rho_k(t))$ such that there is an extremal solution $u(x)$ of (1) for $\rho_{k+1}(x_0)$ (see Remark 1). Also note that $\rho_{k+1}(x_0) = \rho_k(t)$. Let A be the set of all extremal solutions of (1) for $\rho_{k+1}(x_0)$. Note $A \neq \phi$ as $u \in A$. Let p be the maximum number of distinct zeros on $(x_0, \rho_k(t))$ of a member of A. Let $y \in A$ with p distinct zeros at $(t<)x_1 < \cdots < x_p (< \rho_k(t))$ of multiplicities k_i, $i=1, \ldots, p$, respectively. Let $k+k_0 (k_0 \geq 1)$ and $k_{p+1} (\geq 1)$ denote the multiplicities of the zeros of y at x_0 and $x_{p+1} \equiv \rho_k(t)$ respectively. The x_i, $i=0, \ldots, p+1$, will remain fixed in the remainder of this proof. Let

$$M = n-1-k-\sum_{i=0}^{p} k_i \quad (\text{so } 0 \leq M < k_{p+1})$$

and define

$$w_1(x) = w(x; x_0^{k+k_0}, x_1^{k_1}, \ldots, x_p^{k_p}, x_{p+1}^{M}).$$

Let K_i be the exact multiplicity of the zero of $w_1(x)$ at x_i, $i=0, \ldots, p+1$. It is easy to see that $K_0 \geq k+k_0$, $K_i \geq k_i$, $i=1, \ldots, p+1$. Because of the maximality of p, $w_1(x)$ and $y(x)$ have no other zeros on (x_0, x_{p+1}). The claim is that $w_1(x)$ and $y(x)$ are linearly dependent. Assume they are linearly independent, then there are two possibilities. One possibility is that at least one of the zeros, say x_l, of $w_1(x)$ and $y(x)$ has the same multiplicity. In this case there is a nontrivial linear combination of the two solutions with a zero of order at least K_l+1 at x_l. But then it follows from Cramer's rule that $w_1(x)$ has a zero of order K_l+1 at x_l which is a contradiction. The second possibility is that $K_0 > k+k_0$, $K_i > k_i$, $i=1, \ldots, p+1$. But then it is easy to see that there is a nontrivial linear combination of $y(x)$ and $w_1(x)$ which is a member of A with more than p zeros on (x_0, x_{p+1}) which contradicts the maximality of p.

Now let

$$w_2(x) = w(x; x_0^{k+k_0-1}, x_1^{k_1}, \ldots, x_p^{k_p}, x_{p+1}^{P})$$

where $P = M+1$ (so $1 \leq P \leq k_{p+1}$). If $k_0 \geq 2$, then $w_2(x) \not\equiv 0$ by the maximality of p, and if $k_0 = 1$, then $w_2(x) \not\equiv 0$ by Lemma 2. By Cramer's rule $w_2(x)$ has zeros of order $k+k_0$, $k_1, \ldots, k_{p+1}$ at $x_0, \ldots, x_{p+1}$ respectively. Hence, by above argument with $v(x) = w_2(x)$, we see that $w_1(x)$ and $w_2(x)$ are linearly dependent. We now consider two cases.

Case 1. k_{p+1} and P have the same parity: In this case define

$$w_3(x) = w(x; x_0^k, \S_{01}, \ldots, \S_{0k_0}, x_1, \S_{12}, \ldots, \S_{1k_1}, x_2, \ldots, x_p, \S_{p2}, \ldots, \S_{pk_p}, \S_{p+1,1}, \ldots, \S_M)$$

https://doi.org/10.4153/CMB-1973-046-7 Published online by Cambridge University Press

where $x_0<\S_{01}<\cdots<\S_{0k_0}<x_1<\S_{12}<\cdots<\S_{pM}<x_{p+1}$. Since $r_{k1\ldots1}(t)>\S_{pM}$, $w_3(x)\not\equiv 0$. It follows from looking at the Taylor's formula with remainder and the continuity of $w_1(x)$ with respect to elements in its determinate that

$$w_3(x)\to Aw_1(x) \quad (A>0)$$

uniformly on $[x_0, x_{p+1}+1]$ as $\S_{ik_1}\to x_i$, $i=0,\ldots,p$, and $\S_{p+1,1}\to x_{p+1}$. (For the details of writing down the value of A and hence more details on the validity of this statement see [4, p. 559].) Since $r_{i_1\ldots i_m}\geq x_{p+1}$ for $i_1\geq k$ (Theorem 1), $w_3(x)$ has exactly a zero of order k at x_0, simple zeros at the $n-k-1$ points $\S_{01},\ldots,\S_{0k_0}$, $x_1,\ldots,\S_{pM}$ and no other zeros on $[x_0, x_{p+1})$. Since k_{p+1} and P have the same parity, k_{p+1} and M have the opposite parity. Hence for $\S_{ik_i}$, $\S_{p+1,1}$ sufficiently close to x_i, x_{p+1} respectively, $w_3(x)$ has another (odd ordered) zero in $[x_{p+1}, r_{k1\ldots1}(t))$ which is a contradiction.

Case 2. k_{p+1} and P have opposite parity: In this case consider

$$w_4(x)=w(x;x_0^k,\S_{01},\ldots,\S_{0,k_0-1},x_1,\S_{12},\ldots,\S_{1k_1},\\ x_2,\ldots,x_p,\S_{p2},\ldots,\S_{pk_p},\S_{p+1,1},\ldots,\S_{p+1,P})$$

where $x_0<\S_{01}<\S_{02}<\cdots<\S_{p+1,P}<x_{p+1}$. Similar to above

$$w_4(x)\to Lw_2(x) \qquad (L>0)$$

uniformly on $[x_0, x_{p+1}+1]$ as $\S_{0,k_0-1}\to x_0$, $\S_{ik_i}\to x_i$, $i=1,\ldots,p$, and $\S_{p+1,1}\to x_{p+1}$. Similar to above $w_4(x)$ is a nontrivial solution of (1) with a zero at x_0 of order exactly k, simple zeros at the $n-k-1$ points $\S_{01},\ldots,\S_{p+1,P}$, and no other zeros on $[x_0, x_{p+1})$. But k_{p+1} and P have the opposite parity and so $w_4(x)$ must have a (odd ordered) zero in $[x_{p+1}, r_{k1\ldots1}(t))$ which is a contradiction. Hence $\rho_k(t)=r_{k1\ldots1}(t)$.

REMARK 4. Theorem 3 could have been stated: If $\rho_{k+1}(t)<r_{k,n-k}(t)$, then $\rho_{k+1}(t)=r_{k1\ldots1}(t)$. If one lets $k=0$ in this statement one obtains (since $r_{0n}(t)=\infty$) the result (5) of Hartman and Sherman.

EXAMPLE. It is well known (e.g. Theorem 3.2, [9]) that if $p(x)$ is continuous and positive, then for

$$y^{(n)}-p(x)y=0, \tag{6}$$

$r_{ij}(t)=\infty$ if j is odd, and for

$$y^{(n)}+p(x)y=0 \tag{7}$$

$r_{ij}(t)=\infty$ if j is even. Hence for (6) we have $\rho_k(t)=r_{k1\ldots1}(t)$ if n and k have the opposite parity. For equation (7) we have that $\rho_k(t)=r_{k1\ldots1}(t)$ if n and k have the same parity.

THEOREM 4. *If* $\rho_k(t)<r_{k,n-k}(t)$, *then for any* $\varepsilon>0$ *there is a nontrivial solution of* (1) *with a zero of order exactly* k *at some point* $x_0\in[t,\rho_k(t))$, *exactly* $n-k-1$ *simple zeros in* $(x_0,\rho_k(t))$ *and an odd order zero in* $[\rho_k(t),\rho_k(t)+\varepsilon)$.

Proof. This theorem follows from a closer examination of the proof of Theorem 3. Theorem 3 was proved by contradiction. The only place the assumption $r_{k1\cdots 1}(t)>\rho_k(t)$ was needed was to show $w_2(x)\not\equiv 0$ when $k_0=1$. We now show that the assumption

$$w_2(x) = w(x; x_0^k, x_1^{k_1}, \ldots, x_p^{k_p}, x_{p+1}^p) \equiv 0 \tag{8}$$

leads to a contradiction. It follows from (7) that there is an extremal solution of (1) for $\rho_k(t)$ with at least $p+1$ zeros on $(x_0, x_{p+1}=\rho_k(t))$. Let B be the set of all extremal solutions of (1) for $\rho_k(x_0)$. Let q be the maximum number of distinct zeros on (x_0, x_{p+1}) of a member of B. Note that $q\geq p+1$. Since $q>p$, every extremal solution of (1) for $\rho_k(t)$ with q distinct zeros on $(x_0,\rho_k(t))$ has a zero of order exactly k at x_0. Hence there are points $z_1,\ldots,z_q$ such that

$$w_5(x) = w(x; z_0^k, z_1^{l_1}, \ldots, z_q^{l_q}, z_{q+1}^Q),$$

where

$$z_0\equiv x_0, \qquad z_{q+1}\equiv\rho_k(t), \qquad k+\sum_{m=1}^{q} l_m+Q = n-1,$$

is a nontrivial solution (by the maximality of q) with n zeros on $[z_0, z_{q+1}]$. Assume $l_i>1$ for some $i\in\{1,\ldots,q\}$. Let

$$w_6(x) = w(x; x_0^k, z_1, \S_{12}, \ldots, \S_{1,l_1}, z_2, \ldots, z_i, \S_{i2}, \ldots, \S_{i,l_i-1}, \ldots, \S_{ql_q}, \S_{q+1,1}, \ldots, \S_{q+1,Q})$$

where $x_0<z_1<\S_{12}<\cdots<\S_{q+1,Q}<z_{q+1}$. It follows for $\S_{1,l_1},\ldots,\S_{i,l_i-1},\ldots,\S_{q,l_q},\S_{q+1,1}$ sufficiently close to $z_1,\ldots,z_i,\ldots,z_q, z_{q+1}$ respectively, $w_6(x)$ is a nontrivial solution of (1) with a zero of order k at x_0 and $n-k$ zeros on (z_0, z_{q+1}). This contradicts Theorem 1. Hence $l_1=\cdots=l_q=1$ and

$$w_5(x) = w(x; z_0^k, z_1, \ldots, z_q, z_{q+1}^Q).$$

Since $r_{k,n-k}(z_0)>z_{q+1}$ for $\varepsilon>0$ sufficiently small we can define a set of nontrivial solutions $\{u_\varepsilon(x)\}$ of (1) by the boundary conditions

$$u_\varepsilon^{(j)}(z_0) = 0, \qquad j = 0,\ldots,k-1$$

$$u_\varepsilon^{(l)}(z_{q+1}-\varepsilon) = w_5^{(l)}(x_{q+1}), \qquad l = 0,\ldots,n-k-1.$$

Then

$$u_\varepsilon(x)\to w_5(x)$$

uniformly on $[z_0, z_{q+1}]$ as $\varepsilon\to 0$. Since $w_5(x)$ has simple zeros in (z_0, z_{q+1}) we have for $\varepsilon>0$, sufficiently small, $u_\varepsilon(x)$ is a nontrivial solution of (1) with a zero of order k at z_0 and $n-k$ zeros on $[z_0, z_{q+1}-\varepsilon]$. This contradicts Theorem 1.

REMARK 5. By using the type of argument that appears at the end of the proof of Theorem 4 one can show that no extremal solution of (1) for $\rho_k(t)$ has an $(n-k, 1, \ldots, 1)$-distribution of zeros on $[t, \rho_k(t)]$.

Two very interesting corollaries follow from Theorem 4.

COROLLARY 5. *For those values of t for which* $\rho_k(t)<r_{k,n-k}(t)$, $\rho_k(t)$ *is a continuous function of t.*

Proof. The proof of this theorem follows easily from Theorem 4 and the continuous dependence of solutions on the initial point.

REMARK 6. Corollary 5 could have been stated: For those values of t for which $\rho_{k+1}(t)<r_{k+1}(t)$, $\rho_{k+1}(t)$ is a continuous function of t. This statement with $k=0$ is the well known fact that $\eta_1(t)$ is a continuous function of t.

COROLLARY 6. *If* $\rho_k(t)<r_{k,n-k}(t)$, *then* $\rho_k(t)$ *is a continuous function of the coefficients of* (1).

Proof. This theorem follows from Theorem 4 and the use of the continuous dependence of solutions with respect to the coefficients of the differential equation. See the proof of Theorem 3 [10].

REMARK 7. Corollary 6 could have been stated: If $\rho_{k+1}(t)<r_{k,n-k}(t)$, then $\rho_{k+1}(t)$ is a continuous function of the coefficients of (1). This statement with $k=0$ is the fact that $\eta_1(t)$ is a continuous function of the coefficients of (1) (see [3, Theorem 3]).

REFERENCES

1. P. Hartman, *Unrestricted n-parameter families*, Rend. Circ. Mat. Palermo (2) **7** (1958), 123–142.

2. Z. Opial, *On a theorem of O. Arama*, J. Differential Equations **3** (1967), 88–91.

3. T. L. Sherman, *Conjugate points and simple zeros for ordinary differential equations*, Trans. Amer. Math. Soc. **146** (1969), 397–411.

4. W. J. Kim, *Simple zeros of solutions of nth-order linear differential equations*, Proc. Amer. Math. Soc. (2) **28** (1971), 557–561.

5. T. L. Sherman, *Properties of solutions of nth order linear differential equations*, Pacific J. Math. **15** (1965), 1045–1060.

6. A. C. Peterson, *On the ordering of multi-point boundary value functions*, Canad. Math. Bull. (4) **13** (1970), 507–513.

7. ——, *Distribution of zeros of solutions of a fourth order differential equation*, Pacific J. Math. **30** (1969), 751–764.

8. G. Polya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. **24** (1922), 312–324.

9. Z. Nehari, *Disconjugate linear differential operators*, Trans. Amer. Math. Soc. **129** (1967), 500–516.

10. A. C. Peterson, *The distribution of zeros of extremal solutions of a fourth order differential equation for the nth conjugate point*, J. Differential Equations **8** (1970), 502–511.

UNIVERSITY OF NEBRASKA,
LINCOLN, NEBRASKA