Compositio Mathematica **105:** 95–108, 1997.

Seeking invariants for blow-analytic equivalence*

TOSHIZUMI FUKUI
Nagoya Institute of Technology, Gokiso-cho, Showa-ku, Nagoya, 466 Japan
Current address: Department of Mathematics, Faculty of Science, Saitama University, 255 Shimo-Okubo, Urawa, 338 Japan. e-mail: tfukui@rimath.saitama-u.ac.jp

Received 20 July 1995; accepted in final form 27 November 1995

Abstract. We introduce some blow-analytic invariants of real analytic function-germs and discuss their properties. As a consequence, we obtain, for instance, the multiplicity of function-germs is a blow-analytic invariant.

Key words: real analytic function-germ blowing-up, resolution, blow-analytic equivalence.

0. Introduction

We consider the classification problem of real function-germs. At the beginning of this theory, H. Whitney showed in [22, (13.1)] that the diffeomorphism type of the zero locus of $W_t(x,y) = xy(x-y)(x-ty)$, $(t \geqslant 2)$ near 0 in $\mathbf{R}^2$ varies, when t varies. In general, there are modulus near 'non-simple' germs for the differentiable equivalence, then the situation is very complicated and seems to cause many problems. Speaking topological equivalence, it does not seems to cause modulus, see [4], but appears some pathology: e.g. $f_k(x,y) = y^2 - x^{2k-1}$ $(k = 1, 2, \dots,)$ determine the same topological type near 0 in $\mathbf{R}^2$. Such pathology is not desirable to classify singularities.

Thus, we are interested in the following observation due to T.-C. Kuo ([14]). Let $\pi\colon M \to \mathbf{R}^2$ be the blowing up at the origin. There is a family of real analytic isomorphism H_t of M which induces a family of homeomorphisms h_t of $\mathbf{R}^2$ with $W_t \circ h_t = W_2$, whenever $t \geqslant 2$. This suggests the notion of blow-analytic equivalence for real analytic functions, which is reviewed in Section 2. In [16], T.-C. Kuo introduced the notion of blow-analytic equivalence, and showed a satisfactory finite classification theorem. In [14, 5, 6], proved were some theorems which asserts several families are blow-analytically trivial. The next problem we have to consider is to find criterions that two function-germs are not blow-analytically equivalent. This is our subject.

In this paper, we present an idea to show that two real analytic function-germs are not blow-analytically equivalent. The first two sections devote some fundamental facts on blowing up. In Section 3, we define the blow-analytic invariant $A_n(f)$, and work on them in the next three sections. We next define blow-analytic equivalence

* This work is partly supported by Grand-in-Aid for encouragement of young scientists

for coherent subspace-germs and work subspaces defined by function-germs. I think these results are satisfactory as a first step on this problem.

1. Blowing-up

In this section, we review some basic definitions and facts of blowing-ups from H. Hironaka's papers [8, 9, 10].

(1.1) Let us denote by **K** either the field of real numbers **R** or that of complex numbers **C**. For a local-ringed space X, we denote $|X|$ the underlying topological space of X, and $\mathcal{O}_X$ its structure sheaf. See the first paragraph of Chapter 0. Section 1 in [8], for the definition of local-ringed spaces. By a **K**-analytic space, we mean an analytic **K**-space in the sense of the fifth paragraph of Chapter 0. Section 1 in [8]. For a coherent sheaf I of ideals on X, we have a local ringed space $Y = (|Y|, \mathcal{O}_Y)$, where $|Y|$ is the zero set of I in X and $\mathcal{O}_Y$ is the restriction of $\mathcal{O}_X/I$ to $|Y|$. We call such a space Y a *coherent subspace* of X.

(1.2) Let X be a **K**-analytic space, and D a coherent subspace of X defined by some coherent sheaf J of ideals on X. Then a morphism $\pi : \widetilde{X} \to X$ is said to be the *blowing-up* of X along D (or J), or with center D (or J), if the following conditions satisfied.

(i) $J\mathcal{O}_{\widetilde{X}}$ is invertible as $\mathcal{O}_{\widetilde{X}}$-module.
(ii) For any morphism of **K**-analytic spaces $f : X' \to X$, if $J\mathcal{O}_{X'}$ is invertible, then there exists a unique morphism $f' : X' \to \widetilde{X}$ with $\pi \circ f' = f$.

The existence of the blowing-up of X along D was shown in [8]. See the tenth paragraph of Section 2 in Chapter 0 *ibid.*

(1.3) Let $f : Z \to X$ be a **K**-analytic map and Y a subspace of X defined by the coherent sheaf I of ideals on X. We denote $f^{-1}(Y)$ the subspace of Z defined by the ideal sheaf $I\mathcal{O}_Z$ on Z. If f is the blowing-up of X along D, then $f^{-1}(Y)$ is called the *total transform* of Y by π.

(1.4) Let $\pi : \widetilde{X} \to X$ be the blowing-up of X along D, and Y a subspace of X. If $q : \widetilde{Y} \to Y$ is the blowing up of Y along $Y \cap D$, then there exists a unique isomorphism of $\widetilde{Y}$ to a subspace Y' of $\widetilde{X}$ such that q is induced by π. Y' is called the *strict transform* of Y by π.

(1.5) A morphism obtained by a finite succession of blowing-ups can be also obtained by a single blowing-up with suitably chosen center. For a proof, see [8, p. 132].

(1.6) Let $D_\alpha (\alpha = 1, 2)$ be coherent subspaces of **K**-analytic space X, and J_α $(\alpha = 1, 2)$ the ideal sheaf of D_α on X. If D_3 is the coherent subspace of X defined by $J_1 J_2$, and $\pi_\beta : X_\beta \to X$ are blowing-ups along D_β, $\beta = 1, 2, 3$, then there exist morphisms $q_\alpha : X_3 \to X_\alpha$ $(\alpha = 1, 2)$ with $\pi_3 = \pi_\alpha \circ q_\alpha$ $(\alpha = 1, 2)$. See [9, (2.10)], for a proof. Suppose that there exist an invertible sheaf I of ideals containing J. Since I is principal, $(J : I)I = J$. Thus J and $J : I$ give isomorphic blowing-ups. Therefore, any blowing-up of X is isomorphic to that along some sheaf of ideals not contained in any invertible sheaf of proper ideals.

(1.7) Let Λ be a well-ordered set with a minimal element 0 and a maximal element γ. For $\lambda \in \Lambda$, we denote the successor of λ by $\lambda + 1$. By a *succession of blowing-ups*, we mean a system of morphisms $\{f_{\lambda,\mu} : X_\lambda \to X_\mu ; \lambda > \mu, \lambda, \mu \in \Lambda\}$ which satisfies the following properties.

(i) $f_{\lambda,\mu} \circ f_{\mu,\nu} = f_{\lambda,\nu}$, for $\lambda, \mu, \nu \in \Lambda$ with $\lambda > \mu > \nu$.
(ii) $f_\lambda := f_{\lambda+1,\lambda}$ is a blowing-up of X_λ with some center for each $\lambda \in \Lambda$ with $\lambda + 1 \in \Lambda$.
(iii) X_λ is the projective limit of the system $\{f_\mu : X_{\mu+1} \to X_\mu, \mu < \lambda\}$ for each $\lambda \in \Lambda$ with $\lambda + 1 \notin \Lambda$.

We often abbreviate the above a succession of blowing-ups $f_\lambda : X_{\lambda+1} \to X_\lambda$ for $\lambda \in \Lambda$.

We say that the succession of blowing-ups above is *locally finite*, if each point of X_0 has a neighborhood N in $|X_0|$ such that the center of f_λ meets $f^{-1}_{\lambda,0}(N)$ only finite number of $\lambda \in \Lambda$.

(1.8) For the sake of convenience to refer, we quote the real analytic version of the H Hironaka's resolution theorem in [8]. See Section 5 of [9], also.

RESOLUTION THEOREM FOR REAL ANALYTIC SPACES ([8, p. 158]). *Let $X = X_0$ be a reduced* **R**-*analytic space. Then there exists a locally finite succession of blowing-ups $f_\lambda : X_{\lambda+1} \to X_\lambda$ with centers D_λ for $\lambda \in \Lambda$, which has the following properties.*

(i) *D_λ is nonsingular and does not contain any simple point of X_λ for $\lambda \in \Lambda$.*
(ii) *X_λ are normally flat along D_λ for $\lambda \in \Lambda$.*
(iii) *X_γ is nonsingular.*

We call the resulting morphism $f : X_\gamma \to X_0 = X$ a *resolution* of X.

SIMPLIFICATION THEOREM FOR IDEALS ([8, p. 158]). *Let $X = X_0$ be a nonsingular* **R**-*analytic space, $I = I_0$ a coherent sheaf of non-zero ideals on X, and E_0 a reduced analytic subspace everywhere of codimension* 1 *in X which has* *only normal crossings. Then there exists a locally finite succession of blowing-ups* $f_\lambda\colon X_{\lambda+1} \to X_\lambda$ *with centers* D_λ *for* $\lambda \in \Lambda$*, which has the following properties.*

(i) D_λ *is nonsingular and irreducible for* $\lambda \in \Lambda$.
(ii) *If* $I_{\lambda+1}$ *is the weak transform of* I_λ *by* f_λ *for* $\lambda \in \Lambda$*, then* $\nu(I_{\lambda,y})$ *is a positive constant for* $y \in D_\lambda$.
(iii) *If* $E_{\lambda+1}$ *is the reduced analytic space* $\mathrm{red}(f_\lambda^{-1}(E_\lambda) \cup f_\lambda^{-1}(D_\lambda))$ *for* $\lambda \in \Lambda$*, then* E_λ *has only normal crossings with* D_λ.
(iv) E_γ *has only normal crossings, and* $I_\gamma = \mathcal{O}_{X_\gamma}$.

We call the resulting morphism $f\colon X_\gamma \to X_0 = X$ a *simplification* of I.

In this paper, we consider germs of real analytic spaces at some compact real analytic sets. Resolutions (or simplifications) of such objects always exist.

Here we quickly review some definitions. Let J be a coherent sheaf of ideals on X defining a subspace D. Then X is *normally flat* along D, if J^p/J^{p+1} is a sheaf of free $\mathcal{O}_D$-modules for each non-negative integer p. For a coherent sheaf I of ideals on X, we denote $\nu(I_x)$ the maximal integer m such that the mth power of the maximal ideal of $\mathcal{O}_{X,x}$ includes I_x. If $f : \widetilde{X} \to X$ is the blowing-up along nonsingular irreducible D, and $m = \nu(I_x)$ for the generic point x of D, then the sheaf $I\mathcal{O}_{\widetilde{X}}$ is divisible by the mth power of the sheaf of ideals of $f^{-1}(D)$ on $\widetilde{X}$. By this division, we obtain the *weak transform* of I by f. Let E and D be subspaces of X. We say that E has only *normal crossings* with D, if for each $x \in E$ there exists a local coordinate system $(z_1, \dots, z_n)$ at x such that the ideal of E is generated by a monomial in z_i's, and that that of D is generated by some of z_i's. In the case $D = X$, we simply say that E has only *normal crossings*.

2. Definition of blow-analytic maps

Following [16], we define the notion of blow-analytic maps. Let $f\colon X \to Y$ be a continuous map between **R**-analytic spaces. According to T.-C. Kuo, the following conditions are equivalent.

(i) There exists a surjective blowing-up $\pi_1 : X_1 \to X$ along some coherent subspace D so that $f \circ \pi_1$ is a real analytic morphism.
(ii) There exists a succession of blowing-ups $\pi_2 : X_2 \to X$ with nonsingular centers so that $f \circ \pi_2$ is a real analytic morphism.
(iii) There exists a proper modification $\pi_3^*\colon X_3^* \to X^*$ of complex spaces, which is a complexification of a real morphism $\pi_3\colon X_3 \to X$, so that $f \circ \pi_3$ is a real analytic morphism.

Proof. (i) $\Longrightarrow$ (ii): Let $\pi_2\colon X_2 \to X$ be a simplification of the sheaf of ideals of D. Because of the universal property of π_1, π_2 factors through π_1.
(ii) $\Longrightarrow$ (i): Since the composition of blowing-ups is a blowing-up, this is obvious.
(i) $\Longrightarrow$ (iii): This is obvious, since a blowing up admits a complexification which is a proper modification.

(iii) $\Longrightarrow$ (i): Consequence of the real version of Hironaka's Chow's lemma [11, p. 504]. See [16], also. □

A mapping $f: X \to Y$ of real spaces is called *blow-analytic* if it satisfies one of the equivalent conditions above. In [13, 14, 15, 19, 20, 5, 6] etc., the word 'modified analytic' or 'almost analytic' were used instead of 'blow-analytic'. Following [16], we use the word 'blow-analytic' here, because of importance of roles of blowing-ups in our discussions.

3. Blow-analytic equivalence for function-germs

Let (X_α, E_α) $(\alpha = 1, 2)$ be germs of **R**-analytic spaces X_α at compact closed connected subspaces E_α of X_α, and $f_\alpha: (X_\alpha, E_\alpha) \to (\mathbf{R}, 0)$ $(\alpha = 1, 2)$ germs of real analytic functions. We say that f_1 is *blow-analytic equivalent* to f_2 if there exist some surjective blowing-ups $\pi_\alpha : \widetilde{X}_\alpha \to X_\alpha$ $(\alpha = 1, 2)$ with some centers D_α, and a **R**-analytic isomorphism-germ $H : (\widetilde{X}_1, \pi_1^{-1}(E_1)) \to (\widetilde{X}_2, \pi_2^{-1}(E_2))$ with $f_2 \circ \pi_2 \circ H = f_1 \circ \pi_1$. We denote it by $f_1 \overset{\text{b.a.}}{\sim} f_2$. We also denote $[f]$ the equivalence class of $f : (X, E) \to (\mathbf{R}, 0)$. Thus $f_1 \overset{\text{b.a.}}{\sim} f_2$ is equivalent to $[f_1] = [f_2]$.

Let A_n denote the set of blow-analytic equivalence class of **R**-analytic function-germs on germs of n-dimensional nonsingular irreducible **R**-analytic space X at a compact closed connected subspace E, which is not a zero divisor.

Let f be a germ of a **R**-analytic function of an irreducible **R**-analytic space X at a compact closed connected subspace E. Let $\varphi: (Y, E') \to (X, E)$ be a germ of a proper **R**-analytic map with $E' = \varphi^{-1}(E)$. If Y is n-dimensional, nonsingular, and irreducible, E' is connected, and $f \circ \varphi$ is not a zero divisor in $\mathcal{O}_{X'}$, then the germ $f \circ \varphi: (Y, E') \to (\mathbf{R}, 0)$ determines a class in A_n. We denote $A_n(f)$ the set of all such classes in A_n.

THEOREM 3.1 *If $f_1 \overset{\text{b.a.}}{\sim} f_2$, then $A_n(f_1) = A_n(f_2)$ for each n.*

We prepare a lemma to show this theorem.

LEMMA 3.2 *Let $f : (X, E) \to (\mathbf{R}, 0)$ be a real analytic function-germ of a **R**-analytic space-germ (X, E), and $(D, D \cap E)$ a **R**-analytic subspace-germ of X of everywhere codimension more than or equal to one. For any class $[\Phi]$ in $A_n(f)$, there exists a proper real analytic map $\varphi: (Y, E') \to (X, E)$ so that $[f \circ \varphi] = [\Phi]$, $E' = \varphi^{-1}(E)$, and that $\varphi^{-1}(D)$ is a proper subspace of Y.*

Proof. By abuse of language, we do not distinguish germs and their representatives. Let $\varphi_0: Y \to X$ be a proper morphism with $[f \circ \varphi_0] = [\Phi]$.

Remark that $f \circ \varphi_0$ is not a zero divisor in $\mathcal{O}_Y$. Let $\pi_1: X_1 \to X$ be a resolution of X, and $\pi_2: X' \to X_1$ a simplification of the sheaf of ideals generated by $f \circ \pi_1$. Then the composition $\pi = \pi_2 \circ \pi_1: X' \to X$ is the blowing up along some subspace B. We sometimes call π a simplification of f. We may assume that B is in

$f^{-1}(0)$. Let $\varpi\colon Y' \to Y$ be the blowing up along $\varphi^{-1}(B)$. Then there is a unique morphism $\varphi'\colon Y' \to X'$. Let $\mathcal{F}$ be the sheaf of germs of real analytic vector fields tangent to each level surface of $f \circ \pi$, $\boldsymbol{v}$ a global section of $\mathcal{F}$ which is not tangent to $\pi^{-1}(D)$. Because of Theorem 3 in [3], such $\boldsymbol{v}$ always exists. Let $h_t : X' \to X'$ denote the one-parameter family of analytic isomorphisms generated by $\boldsymbol{v}$. Then the map $\varphi = \pi \circ h_t \circ \varphi'$ has the desired properties. □

Proof of (3.1). By abuse of language, we do not distinguish germs and their representatives. Let $\pi_\alpha\colon \widetilde{X}_\alpha \to X_\alpha$ ($\alpha = 1, 2$) be the blowing-ups along D_α. We assume that there is a real analytic isomorphism $h\colon X_1 \to X_2$ with $f_1 \circ \pi_1 = f_2 \circ \pi_2 \circ h$. For each $[\Phi]$ in $A_n(f_1)$, there is a proper morphism $\varphi\colon Y \to X_1$ so that $\varphi^{-1}(D_1)$ is a proper subspace of Y, and that $[f_1 \circ \varphi] = [\Phi]$. Let $\varpi\colon \widetilde{Y} \to Y$ be the blowing-up along $\varphi(D_1)$ and denote $\widetilde{\varphi}\colon \widetilde{Y} \to \widetilde{X}$ the unique morphism. Obviously $[f_2 \circ \pi_2 \circ h \circ \widetilde{\varphi}]$ defines a class of $A_n(f_2)$, which is $[\Phi]$. This implies $A_n(f_1) \subset A_n(f_2)$, and vice versa. □

4. A_1 and $A_1(f)$

Since a blowing-up of a nonsingular real analytic curve is an isomorphism, a class in A_1 is expressed by $(\mathbf{R}, 0) \ni t \mapsto \pm t^k \in (\mathbf{R}, 0)$, which we denote by $[k^\pm]$. Since $[(2k+1)^+] = [(2k+1)^-]$, we often denote it by $[2k+1]$. Obviously $A_1(f)$ is a class of real analytic map $(\mathbf{R}, 0) \to (\mathbf{R}, 0)$ which factors through $f\colon (X, E) \to (\mathbf{R}, 0)$. Let $\mathbf{N}$ denote the set of non-negative integers, and $\mathbf{R}_+$ the set of non-negative real numbers. Let $x = (x_1, \ldots, x_n)$ be a coordinate system of $(\mathbf{R}^n, 0)$.

LEMMA 4.1 *Let* $f\colon (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ *be the map defined by* $f(x) = \pm x_1^{m_1} \cdots x_n^{m_n}$. *We then have that* $A_1(f) = \{[(\sum_{i=1}^n k_i m_i)^\pm] \in A_1 : k_i \in \mathbf{N}$ *for* $i = 1, \ldots, n\}$.

Proof. Elementary computation: Consider an analytic map $\varphi\colon (\mathbf{R}, 0) \to (\mathbf{R}^n, 0)$. If we write $\varphi(t) = (c_1 t^{k_1}, \ldots, c_n t^{k_n})+$higher order terms, ($c_i \neq 0$), then $f \circ \varphi(t) = c_1^{m_1} \cdots c_n^{m_n} t^{\sum_{i=1}^n k_i m_i} +$ higher order terms. This completes the proof. □

Let $x = (x_1, \ldots, x_n)$ be a coordinate system at the origin 0 of $\mathbf{R}^n$, $f\colon (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ a real analytic function-germ, and $\sum_{\nu \in \mathbf{N}^n} c_\nu x^\nu$ the Taylor expansion of f at 0, where $x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$ for $\nu = (\nu_1, \ldots, \nu_n) \in \mathbf{N}^n$. The *Newton polygon* $\Gamma_+(f)$ of f means the convex hull of the set $\{\nu + \mathbf{R}_+^n : c_\nu \neq 0\}$. For $a = (a_1, \ldots, a_n) \in \mathbf{R}^n$ and $\nu = (\nu_1, \ldots, \nu_n) \in \mathbf{R}_+^n$, we set $\langle a, \nu \rangle = a_1\nu_1 + \cdots + a_n\nu_n$, $\ell(a) = \min\{\langle a, \nu\rangle : \nu \in \Gamma_+(f)\}$, and $\gamma(a) = \{\nu \in \Gamma_+(f) : \langle a, \nu \rangle = \ell(a)\}$. We set $f_\gamma(x) = \sum_{\nu \in \gamma} c_\nu x^\nu$ for a subset γ of $\mathbf{R}_+^n$. For $a \in \mathbf{N}^n$, we define $[\ell(a)^\sigma]$ by

$$[\ell(a)^\sigma] = \begin{cases} [\ell(a)^+] & \text{if } f_{\gamma(a)} \text{ is positive semi-definite near } 0,\\ [\ell(a)^-] & \text{if } f_{\gamma(a)} \text{ is negative semi-definite near } 0,\\ [\ell(a)^\pm] & \text{otherwise.}\end{cases}$$

LEMMA 4.2 *For a function-germ* $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$, *we have* $A_1(f) \supset \{[\ell(a)^\sigma] \in A_1 : a \in \mathbf{N}^n\}$.

Proof. Consider the map $\varphi : (\mathbf{R}, 0) \ni t \mapsto (c_1 t^{a_1}, \dots, c_n t^{a_n}) \in (\mathbf{R}^n, 0)$ for generic $c = (c_1, \dots, c_n)$. Then we have $f \circ \varphi(t) = f_{\gamma(a)}(c) t^{\ell(a)} +$higher order terms, which shows the lemma. □

We say that $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ is *nondegenerate* if the gradient of $f_\gamma(x)$ has no zeros in $(\mathbf{R} - 0)^n$ for each compact face γ of $\Gamma_+(f)$.

PROPOSITION 4.3 *For a nondegenerate function-germ* $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$, *we have* $A_1(f) = \{[\ell(a)^\sigma] \in A_1 : a \in \mathbf{N}^n\} \cup \{[p^\pm] : p \geqslant p_0\}$. *Here,* $p_0 = \min\{\ell(a) : a \in \mathbf{N}^n, \dim \gamma(a) = n - 1, f_{\gamma(a)}$ *is not semi-definite near* $0\}$.

Proof. It is well-known that there is a toric modification $\pi : (X, E) \to (\mathbf{R}^n, 0)$ which is a simplification of the ideal generated by f, if f is nondegenerate. (See [12], [1, pp. 234–250], [5], etc.) For any $a \in \mathbf{N}^n$, there is a map $\varphi : (\mathbf{R}, 0) \to (\mathbf{R}^n, 0)$ with $[f \circ \varphi] = [\ell(a)^\sigma]$. Let $\tilde{\varphi} : (\mathbf{R}, 0) \to (X, E)$ be the lift of φ. Without loss of generality, we may assume that the image of $\tilde{\varphi}$ is in some coordinate patch $(\mathbf{R}^n, y = (y_1, \dots, y_n))$ of X, and that the map π is expressed by $\pi(y) = (y_1^{a_1^1} \cdots y_n^{a_1^n}, \dots, y_1^{a_n^1} \cdots y_n^{a_n^n})$. Then we obtain that $f \circ \pi(y) = f'(y) y_1^{\ell(a^1)} \cdots y_n^{\ell(a^n)}$ and the zero locus of f' is nonsingular and transverse to each coordinate spaces. Here $a^j = {}^t(a_1^j, \dots, a_n^j)$. By (4.1), this completes the proof. □

Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be a real analytic function-germ. Since the minimal number k with $[k^+]$ (or $[k^-]$) $\in A_1(f)$ is the multiplicity $\mathrm{mult}_0(\mathrm{f})$ of f at 0, the degree of the leading term of a Taylor expansion of f at 0, we obtain the following.

COROLLARY 4.4 *Let* $f_\alpha : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ $(\alpha = 1, 2)$ *be real analytic function-germs. If* $f_1 \overset{\text{b.a.}}{\sim} f_2$, *then* $\mathrm{mult}_0(f_1) = \mathrm{mult}_0(f_2)$.

A similar result was also obtained by another method due to M. Suzuki [18].

EXAMPLE 4.5 Here, we consider some polynomial germs $(\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$. Since $[3] \notin A_1([x^5 + y^2])$, we have $[x^3 + y^2] \neq [x^5 + y^2]$. Such discussion shows that $[f] = [g]$ iff $f = g$, for $f, g \in \{x^{2k-1} + y^2 : k = 1, 2, \dots, \}$. But the use of $A_1(f)$ is restrictive, since $A_1([x^4 + y^2]) = A_1([x^2 + y^2])$.

5. A_2 and graphs

Let $\pi : (X, E) \to (\mathbf{R}^2, 0)$ be a blowing up along some coherent subspace D. We may assume that D is of codimension 2, since we may do that I_D is not contained in a proper invertible ideal in $\mathcal{O}_{\mathbf{R}^2, 0}$. Then, there is a coordinate system (x_1, x_2) of $(\mathbf{R}^2, 0)$ so that I_D is generated by polynomials in x_1, x_2, because of [17] or [21]. Thus we may assume that π is an algebraic map. By the discussion in [7, pp. 510–512], if X is nonsingular, then $\pi : (X, E) \to (\mathbf{R}^2, 0)$ is isomorphic to a sequence of blowing-ups along some real points. Thus, if $\widetilde{X} \to X$ is a blowing up between some nonsingular surfaces, then it is isomorphic to a composition of blowing-ups at some points.

Let $f : (X, E) \to (\mathbf{R}, 0)$ be a real analytic function-germ, and $\pi : (\widetilde{X}, \widetilde{E}) \to (X, E)$ a simplification of f, where $\widetilde{E} = \pi^{-1}(E)$. Then the zero locus of f is a divisor with only normal crossings, and we denote it by $\sum_{i=1}^{s} m_i D_i$ where D_i $(i = 1, \dots, s)$ denote its irreducible components, and m_i the multiplicity of f along D_i. It is often convenient to consider a 'graph' of a simplification $\pi : (\widetilde{X}, \widetilde{E}) \to (X, E)$ of f obtained by the following way: To each D_i such that $m_i \neq 0$ there corresponds a vertex '$\bigcirc$'. If D_i and D_j intersect, then we draw a line connecting the corresponding vertices. We record the multiplicity m_i by placing that integer above the corresponding vertex i.e. $\overset{m_i}{\bigcirc}$. If $f \circ \pi$ is positive (resp. negative) semi-definite near D_i, we assign the sign $+$ (resp. $-$) to the corresponding vertex and denote it by $\overset{m_i}{\oplus}$ (resp. $\overset{m_i}{\ominus}$).

Given the graph of a simplification $\pi : \widetilde{X} \to X$ of f admits operations induced by more blowing-ups of $\widetilde{X}$. For example, we can replace

(something)$\to \overset{a}{\bigcirc} - \overset{b}{\bigcirc} \leftarrow$(something) by (something)$\to \overset{a}{\bigcirc} - \overset{a+b}{\bigcirc} - \overset{b}{\bigcirc} \leftarrow$(something), (something)$\to \overset{a}{\oplus} - \overset{b}{\oplus} \leftarrow$(something) by (something)$\to \overset{a}{\oplus} - \overset{a+b}{\oplus} - \overset{b}{\oplus} \leftarrow$(something), $\dots$, (something)$\to \overset{a}{\bigcirc}$ by (something)$\to \overset{a}{\bigcirc} - \overset{a}{\bigcirc}$, and so on. We say a vertex in such a graph is *contractible* if it corresponds to the exceptional set of the blowing up above. The inverses of the operations above are *contraction* of graphs. These operations generate an equivalence relation on the set of all such graphs. Let $G(f)$ be the equivalence class of the graphs of $f \in A_2$. For $f_\alpha \in A_2$ $(\alpha = 1, 2)$, $[f_1] = [f_2]$ implies $G(f_1) = G(f_2)$, by the discussion *ibid*.

PROPOSITION 5.1 *If two graphs belong to the same equivalent class, and each has the minimum number of vertices for graphs in the class, they are isomorphic.*

Proof. Let G_1 be a graph and G_2 a graph obtained from G_1 by a succession of contractions above. Assume that G_2 has no contractible vertices. It then is not hard to see that a contractible vertex of G_1 cannot survive in G_2 except the case $G_2 = \overset{a}{\bigcirc}, \overset{a}{\oplus}, \overset{a}{\ominus}$. This completes the proof. □

PROPOSITION 5.2 *Let* $f : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ *be a nondegenerate real analytic function-germ. Then* $G(f)$ *is the class obtained by the following way:*

(i) *Set* $V(\Gamma_+(f)) = \{v = \binom{a}{b} \in \mathbf{N}^2 : \mathrm{GCD}(a, b) = 1, \dim \gamma(v) = 1\}$.

(ii) *Take a sequence of lattice points* $v_1, \dots, v_n$ *in the first quadrant* $\mathbf{R}^2_+$ *such that the successive pairs generate the lattice* $\mathbf{N}^2$ *and that* $\{v_1, \dots, v_n\} \supset V(\Gamma_+(f))$.

(iii) *Assign the vertex* $\overset{\ell(v_i)}{\bigcirc}$ *to each* v_i *whenever* $\ell(v_i) \neq 0$, *and the sign* $+$ *(resp.* $-$*) to that vertex if* $f_{\gamma(v_i)}(x)$ *is positive (resp. negative) semi-definite near* 0.

(iv) *Draw lines connecting vertices corresponding to the successive pairs of these lattice points.*

(v) *If the zero locus of $f_{\gamma(v_i)}$ has m irreducible components near 0 except the axes, assign m vertices $\overset{1}{\bigcirc}$, and draw m lines connecting these m vertices and $\overset{\ell(v_i)}{\bigcirc}$.*

(vi) *$G(f)$ is the class of this graph we obtained.*

Proof. We set $v_i = \binom{a_i}{b_i}$ $(i = 1, \ldots, n)$. Let $\mathbf{R}_i^2$ be a copy of $\mathbf{R}^2$ with a coordinate system (x_i, y_i). Define the map $\pi_i : \mathbf{R}_i^2 \to \mathbf{R}^2$ $(i = 1, \ldots, n-1)$ by $\pi_i(x_i, y_i) = (x_i^{a_i} y_i^{a_{i+1}}, x_i^{b_i} y_i^{b_{i+1}})$. Then we can glue $\pi_i : \mathbf{R}_i^2 \to \mathbf{R}^2$ together and obtain a map $\pi : X \to \mathbf{R}^2$. If $f : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ is nondegenerate, then π is a simplification of the ideal generated by f. This gives our assertion. □

EXAMPLE 5.3 Using (5.2), we can distinguish many polynomial-germs in 2 variables. For example, we have $[f] = [g]$ iff $f = g$ for $f, g \in \{\pm(x^{2k-1} \pm y^2), \pm(x^{2k} \pm y^2), x^2y \pm y^{2+k}, x^3 \pm y^4, x^3 + xy^3, x^3 + y^5, x^3 \pm xy^4, x^3 \pm (x^2y^2 + y^{2k}), x^3 \pm (x^2y^2 - y^{2k}), x^3 \pm x^2y^2 + y^{2k+1}, x^3 + y^7, x^3 + xy^5, x^3 \pm y^8, \pm(x^4 + y^4), xy(x - y)(x - 2y), x^4 - y^4\}$. It is not hard to extend this list, using (5.2), (7.1) and (7.2).

6. P.o.sets of f

Let $\mathbf{N}_+$ denote the set of positive integers. Let $\mathcal{P}$ be a triple (P, ν, σ) where P is a partially ordered set, ν is a map of P to the set of nonempty additive sub-semigroups of $\mathbf{N}_+$, and σ is a map of P to $\{\{+1\}, \{-1\}, \{+1, -1\}\}$ satisfying the following conditions.

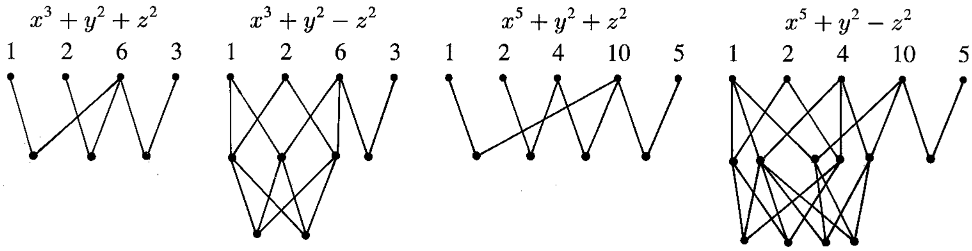
(i) $\nu(e) = \sum_{e' > e} \mathbf{N}_+ \nu(e')$ for $e \in P$.

(ii) $\sigma(e) = \{-1, +1\}$ if and only if there exists an $e_1 \geqslant e$ with $\nu(e_1) \not\subset 2\mathbf{N}_+$.

Let $\mathcal{P}_\alpha = (P_\alpha, \nu_\alpha, \sigma_\alpha)$ $(\alpha = 1, 2)$ be two such triples. A *morphism* φ of $\mathcal{P}_1$ to $\mathcal{P}_2$, we often denote it by $\varphi\colon \mathcal{P}_1 \to \mathcal{P}_2$, means a morphism $\varphi\colon P_1 \to P_2$ as partially ordered sets which satisfies $\nu_1(e) \subset \nu_2(\varphi(e)), \sigma_1(e) \subset \sigma_2(\varphi(e))$ for each $e \in P_1$.

Let $f\colon (X, E) \to (\mathbf{R}, 0)$ be a real analytic function-germs, and $\pi : (\widetilde{X}, \widetilde{E}) \to (X, E)$ a simplification of f. Then the zero locus of $f \circ \pi$ is a divisor with only normal crossings and denote it by $\sum_{i=1}^{s} m_i D_i$, where m_i is the multiplicity of $f \circ \pi$ along an irreducible component D_i. Setting $e_I = \bigcap_{i \in I} D_i$ for $I \subset \{1, \ldots, s\}$ and P the set of connected components of e_I's for nonempty $I \subset \{1, \ldots, s\}$, P forms a partially ordered set by the order defined by the inclusion. Let $e \in P$ be a connected component of e_I. Setting $\nu(e) = \{\sum k_i m_i : k_i \in \mathbf{N}_+, i \in I\}$, and $\sigma(e) =$ the possible signs of values of $f \circ \pi$ near e, $\mathcal{P} = (P, \nu, \sigma)$ is a triple satisfying the conditions above. We say that $\mathcal{P}$ is a *p.o.set* belonging to f.

EXAMPLE 6.1 The Hasse diagrams of some (simplest) p.o.sets belonging to the function-germ defined by $x^3 + y^2 \pm z^2$ (or $x^5 + y^2 \pm z^2$) near 0 are the following.



PROPOSITION 6.2 *Let* $f_\alpha : (X_\alpha, E_\alpha) \to (\mathbf{R}, 0)$ *(*$\alpha = 1, 2$*) be real analytic function-germs with* $[f_1] \in A_n(f_2)$. *For each poset* $\mathcal{P}_2$ *belonging to* f_2, *there exist some poset* $\mathcal{P}_1$ *belonging to* f_1 *and a morphism of* $\mathcal{P}_1$ *to* $\mathcal{P}_2$.

Proof. By assumption, there is a proper morphism $\varphi : (X_1, E_1) \to (X_2, E_2)$ with $[f_1] = [f_2 \circ \varphi]$. Let $\pi_2 : X_2' \to X_2$ be a simplification of f_2. Then π_2 is a blowing up with some center, say B. Let $\pi_1 : X_1' \to X_1$ be the blowing up along $\varphi^{-1}(B)$, $\varphi' : X_1' \to X_2'$ the unique morphism, and $\pi' : \widetilde{X_1} \to X_1'$ a simplification of $f \circ \pi_1$. We write the zero locus of $f \circ \pi_1$ by $\sum_{i=1}^{s} m_i D_i$ and that of $f \circ \varphi \circ \pi_2$ by $\sum_{i=1}^{s'} m_i' D_i'$. Setting e' an irreducible component of $\bigcap_{i \in I'} D_i'$, we define $\varphi(e')$ the intersection of D_i's containing $\varphi' \circ \pi'(e')$. This φ is the desired morphism. □

EXAMPLE 6.3 After some routine calculation using (5.1), we show that there are no morphism of poset belonging to germ defined by $x^3 + y^2$ to that belonging to the function-germ defined by $x^5 + y^2 \pm z^2$ near 0, and $[x^3 + y^2] \notin A_2([x^5 + y^2 \pm z^2])$. This shows that $[x^3 + y^2 \pm z^2] \neq [x^5 + y^2 \pm z^2]$. Since $[y^2 - z^2] \notin A_2([x^3 + y^2 + z^2])$, we have that $[x^3 + y^2 + z^2] \neq [x^3 + y^2 - z^2]$. Such discussion shows that $[f] = [g]$ iff $f = g$ for $f, g \in \{x^{2k+1} + y^2 \pm z^2\ k = 1, 2, \ldots\}$.

7. Blow-analytic equivalence for coherent subspace-germs

Let (X_α, E_α) $(\alpha = 1, 2)$ be **R**-analytic space-germs, and $(V_\alpha, V_\alpha \cap E_\alpha)$ $(\alpha = 1, 2)$ are subspace-germs of (X_α, E_α). We say that $(X_1, V_1; E_1)$ is *blow-analytic equivalent* to $(X_2, V_2; E_2)$ if there exist some surjective blowing-ups $\pi_\alpha : \widetilde{X}_\alpha \to X_\alpha$ $(\alpha = 1, 2)$ with some centers D_α, and an **R**-analytic isomorphism-germ $H : (\widetilde{X}_1, \pi_1^{-1}(E_1)) \to (\widetilde{X}_2, \pi_2^{-1}(E_2))$ so that $H(\pi_1^{-1}(V_1), \pi_1^{-1}(E_1)) = (\pi_2^{-1}(V_2), \pi_2^{-1}(E_2))$. We denote it by $(X_1, V_1; E_1) \overset{\text{b.a.}}{\sim} (X_2, V_2; E_2)$. We also denote $[(X, V; E)]$ the equivalence class of $(X, V; E)$. Thus $(X_1, V_1; E_1) \overset{\text{b.a.}}{\sim} (X_2, V_2; E_2)$ is equivalent to $[(X_1, V_1; E_1)] = [(X_2, V_2; E_2)]$.

Let $f : (X, E) \to (\mathbf{R}, 0)$ be a real analytic function-germ. We denote $f_1 \overset{\text{b.a.}-V}{\sim} f_2$ if f_1 and f_2 define subspaces which are blow-analytic equivalent.

Let V be a coherent subspace of X defined by the coherent sheaf I of ideals on X. A blowing-up $\pi : \widetilde{X} \to X$ is said to be a *simplification* of V, if $\widetilde{X}$ is nonsingular and the space $\pi^{-1}(V)$ is a divisor with only normal crossings. The following (7.1) is a consequence of the existence of a simplification of any coherent subspace of nonsingular analytic spaces.

PROPOSITION 7.1 *Let* $(X_\alpha, V_\alpha; E_\alpha)$ *(*$\alpha = 1,2$*) be subspace-germs defined by some coherent sheaves of ideals in some nonsingular* $\mathbf{R}$*-analytic spaces* X_α*. Then* $[(X_1, V_1; E_1)] = [(X_2, V_2; E_2)]$*, if and only if,* $(X_\alpha, V_\alpha; E_\alpha)$ *(*$\alpha = 1,2$*) admit isomorphic simplifications of* V_α*.*

Let $f_\alpha : (X_\alpha, E_\alpha) \to (\mathbf{R}, 0)$ $(\alpha = 1,2)$ be real analytic function-germs on real analytic manifolds X_α. It is easy to see that $f_1 \overset{\text{b.a.}-\text{V}}{\sim} f_2$, if $f_1 \overset{\text{b.a.}}{\sim} \pm f_2$. We show the converse.

PROPOSITION 7.2 $f_1 \overset{\text{b.a.}-\text{V}}{\sim} f_2$ *implies* $f_1 \overset{\text{b.a.}}{\sim} \pm f_2$.

The proof is essentially same to the discussion in [16, Sect. 3].

Proof. To save notations, we do not distinguish germs and their representatives. Since $f_1 \overset{\text{b.a.}-\text{V}}{\sim} f_2$, there exist blowing-ups $\pi_\alpha : \widetilde{X}_\alpha \to X_\alpha$ $(\alpha = 1,2)$ and analytic isomorphism $H : (\widetilde{X}_1, \pi_1^{-1}(E_1)) \to (\widetilde{X}_2, \pi_2^{-1}(E_2))$ which induces an isomorphism of $\pi_2^{-1}((f_2))$ to $\pi_1^{-1}((f_1))$. Let $\pi' : X \to \widetilde{X}_1$ be a simplification of $f_1 \circ \pi_1$. Then for each point P of X there exists a coordinate system $y = (y_1, \dots, y_n)$ of X near P so that $f_1 \circ \pi_1 \circ \pi'(y) = y_1^{m_1} \cdots y_n^{m_n}$ for some $m_1, ..., m_n$. Since $f_2 \circ \pi_2 \circ H$ generate the ideal generated by $f_1 \circ \pi_1$, $f_2 \circ \pi_2 \circ H \circ \pi'(y) = u y_1^{m_1} \cdots y_n^{m_n}$ for some unit function u near P. Changing sign of f_2, if necessary, we may assume that $u > 0$. Let I be an open interval $(-\varepsilon, 1+\varepsilon)$ for small positive number ε. Define a map $F : X \times I \to \mathbf{R}$ by $F(y,t) = t(f_1 \circ \pi_1 \circ \pi'(y)) + (1-t)(f_2 \circ \pi_2 \circ H \circ \pi'(y))$. We have that $F(y,t) = (t + (1-t)u) y_1^{m_1} \cdots y_n^{m_n}$ near P. Replacing y_1 by $(t + (1-t)u)^{1/m_1} y_1$, we obtain that $F(y,t) = y_1^{m_1} \cdots y_n^{m_n}$ near P. Let $p : X \times I \to I$ be the natural projection. Then the vector field $\partial/\partial t$ on I has a local lift near each point in $X \times I$.

Let $\mathcal{F}$ denote the sheaf of germs of analytic vector fields on $X \times I$ which are consistent with the canonical stratification of $F^{-1}(0)$ and tangent to each level surfaces of F, and $\mathcal{F}_0$ the subsheaf of those germs which vanish under dp. Then , by Theorem 3 in [3], $0 \to H^0(\mathcal{F}_0) \to H^0(\mathcal{F}) \to H^0(\mathcal{F}/\mathcal{F}_0) \to 0$ is exact. The local lifting of $\partial/\partial t$, constructed above, together yield an element in $H^0(\mathcal{F}/\mathcal{F}_0)$, which, by exactness, is the image of a global section $\boldsymbol{v}$ of $\mathcal{F}$. Integration of $\boldsymbol{v}$ gives the desired isomorphism of X. □

Let B_n denote the set of blow-analytic equivalence class of $\mathbf{R}$-analytic proper coherent subspace-germs of n-dimensional nonsingular irreducible $\mathbf{R}$-analytic space germ (X, E), where E is a compact closed connected subspace of X.

Let $(V, V \cap E)$ be a coherent subspace-germ of an $\mathbf{R}$-analytic space-germ (X, E), and I_V the coherent sheaf of ideals on X defining V. Let $\varphi : (X', E') \to (X, E)$ be a germ of a proper $\mathbf{R}$-analytic map with $E' = \varphi^{-1}(E)$. If X' is n-dimensional, nonsingular, and irreducible, E' is connected, and $I_V \mathcal{O}_{X'}$ is not identically zero, then the germ $\varphi^{-1}(X,V;E) = (X',\varphi^{-1}(V);E')$ determines a class in B_n. We denote $B_n(X,V;E)$ the set of all such classes in B_n. We set $B_n(f) = B_n(X,V;E)$ where V is the subspace defined by the ideal generated by function-germ $f\colon (X,E) \to (\mathbf{R},0)$.

THEOREM 7.3 *If* $(X_1,V_1;E_1) \overset{\text{b.a.}}{\sim} (X_2,V_2;E_2)$, *then* $B_n(X_1,V_1;E_1) = B_n(X_2,V_2;E_2)$ *for each* n.

We prepare a lemma to show this theorem.

LEMMA 7.4 *Let* I *be a coherent sheaf of ideals on* X, D *a coherent proper subspace of* X *of everywhere codimension more than or equal to one. For any class* $[(Y,V';E')]$ *in* $B_n(X,V,E)$, *there exist a* $\mathbf{R}$*-analytic map* $\varphi\colon (Y,E') \to (X,E)$ *so that* $[\varphi^{-1}(X,V;E)] = [(Y,V';E')]$ *and* $\varphi^{-1}(D)$ *is a proper subspace of* Y.

Proof. By abuse of language, we do not distinguish germs and their representatives. Let $\varphi_0\colon Y \to X$ be a proper morphism with $[f\circ\varphi_0] = [\Phi]$. Remark that $\varphi_0^{-1}(V)$ is a proper subspace of Y. Let $\pi_1\colon X_1 \to X$ be a resolution of X, and $\pi_2\colon X' \to X_1$ a simplification of the sheaf of ideals of $\pi_1^{-1}(V)$. Then the composition $\pi = \pi_2\circ\pi_1\colon X' \to X$ is the blowing up along some subspace B. We may assume that B is in V. Let $\varpi\colon Y' \to Y$ be the blowing up along $\varphi^{-1}(B)$. Then there is a unique morphism $\varphi'\colon Y' \to X'$. Let $\mathcal{F}$ be the sheaf of germs of real analytic vector fields tangent to $\pi^{-1}(V)$, $\boldsymbol{v}$ a global section of $\mathcal{F}$ which is not tangent to $\pi^{-1}(D)$. Because of Theorem 3 in [3], such $\boldsymbol{v}$ always exists. Let $h_t\colon X' \to X'$ denote the one-parameter family of analytic isomorphisms generated by $\boldsymbol{v}$. Then the map $\varphi = \pi\circ h_t\circ\varphi'$ has the desired properties. □

Proof of (7.3). By abuse of language, we do not distinguish germs and their representatives. Let $\pi_\alpha\colon \widetilde{X}_\alpha \to X_\alpha$ $(\alpha = 1,2)$ be the blowing-ups along D_α. We assume that there is a real analytic isomorphism $h\colon X_1 \to X_2$ with $f_1\circ\pi_1 = f_2\circ\pi_2\circ h$. For each $[(Y,V';E')]$ in $B_n(X_1,V_1;E_1)$, there is a proper morphism $\varphi\colon Y \to X_1$ so that $\varphi^{-1}(D_1)$ is a proper subspace of Y, and that $[\varphi^{-1}(X_1,V_1;E_1)] = [(Y,V';E')]$. Let $\varpi\colon \widetilde{Y} \to Y$ be the blowing-up along $\varphi(D_1)$ and denote $\widetilde{\varphi}\colon \widetilde{Y} \to \widetilde{X}$ the unique morphism. Obviously $[(\pi_2\circ h\circ\widetilde{\varphi})^{-1}(X_2,V_2;E_2)]$ defines a class of $B_n(f_2)$, which is $[(Y,V';E')]$. This implies $B_n(X_1,V_1;E_1) \subset B_n(X_2,V_2;E_2)$, and vice versa. □

8. B_1, graphs, and p.o.sets for the subspaces defined by function-germs

By (7.2), forgetting about signs from $A_n(f)$, we obtain some results on $B_n(f)$ from that of $A_n(f)$.

Since a blowing-up of nonsingular real analytic curve is an isomorphism, a class in B_1 is generated by $(\mathbf{R},0) \ni t \mapsto t^k \in (\mathbf{R},0)$, which we denote by $[k]$. By discussions similar to Section 4, we obtain the followings.

LEMMA 8.1 *Let* $f:(\mathbf{R}^n,0)\to(\mathbf{R},0)$ *be a real analytic function-germ defined by* $f(x)=x_1^{m_1}\cdots x_n^{m_n}$. *Then,* $B_1(f)=\{[(\sum_{i=1}^n k_i m_i)]\in B_1 : k_i\in\mathbf{N}$ *for* $i=1,\ldots,n\}$.

LEMMA 8.2 *Let* $f:(\mathbf{R}^n,0)\to(\mathbf{R},0)$ *be a real analytic function-germ. Then, we have* $B_1(f)\supset\{[\ell(a)]\in B_1 : a\in\mathbf{N}^n\}$.

PROPOSITION 8.3 *Let* $f:(\mathbf{R}^n,0)\to(\mathbf{R},0)$ *be a nondegenerate real analytic function-germ. We then have* $B_1(f)=\{[\ell(a)]\in B_1 : a\in\mathbf{N}^n\}\cup\{[p] : p\geqslant p_0\}$. *Here,* $p_0=\min\{\ell(a) : a\in\mathbf{N}^n, \dim\gamma(a)=n-1, f_{\gamma(a)}$ *is not semi-definite near* $0\}$.

COROLLARY 8.4 *Let* $f_\alpha:(\mathbf{R}^n,0)\to(\mathbf{R},0)$ *(*$\alpha=1,2$*) be real analytic function-germs, and* V_α *the subspaces defined by the ideals generated by* f_α*. If* $(\mathbf{R}^n,V_1;0)\overset{\text{b.a.}}{\sim}(\mathbf{R}^n,V_2;0)$*, then* $\mathrm{mult}_0(\mathrm{f}_1)=\mathrm{mult}_0(\mathrm{f}_2)$.

(8.5) Let $f:(X,E)\to(\mathbf{R},0)$ be a real analytic function-germ on real analytic surface X. Let $\pi:(\widetilde{X},\widetilde{E})\to(X,E)$ be a simplification of f. Forgetting the signs in the graph defined in Section 5, we obtain a graph for this simplification π. More blowing ups of $\widetilde{X}$ induce operations of graphs described by the following: Replace (something)$\to\overset{a}{\bigcirc}$—$\overset{b}{\bigcirc}\leftarrow$(something) by (something)$\to\overset{a}{\bigcirc}$—$\overset{a+b}{\bigcirc}$—$\overset{b}{\bigcirc}\leftarrow$(something), and (something)$\to\overset{a}{\bigcirc}$ by (something)$\to\overset{a}{\bigcirc}$—$\overset{a}{\bigcirc}$. These operations generate an equivalence relation on the set of all such graphs. For $f_\alpha:(\mathbf{R}^2,0)\to(\mathbf{R},0)$ $(\alpha=1,2)$ $f_1\overset{\text{b.a.}-\text{V}}{\sim}f_2$ implies that the equivalence classes of graphs of f_1 and f_2 coincide. We obtain that the graphs in that equivalence classes with possible minimal numbers of vertices are same, by a discussion similar to (5.1). For nondegenerate-real analytic germ $f:(\mathbf{R}^n,0)\to(\mathbf{R},0)$, the graph of f is obtained by a procedure similar to (5.2), and we omit the details.

(8.6) Forgetting about the sign morphism σ of p.o.sets, we can also discuss them analogously to Section 6. We omit the details, because it is almost same.

9. Conjectures

To end the paper, we formulate several conjectures in this direction.

CONJECTURE 9.1 Let $f_\alpha:(\mathbf{R}^2,0)\to(\mathbf{R},0)$ $(\alpha=1,2)$ be real analytic function-germs. Then $f_1\overset{\text{b.a.}-\text{V}}{\sim}f_2$ implies $\Gamma_+(f_1)=\Gamma_+(f_2\circ h)$ for a suitably chosen coordinate change h of $(\mathbf{R}^2,0)$.

For function-germs in 3 variables, the conjecture analogous to (9.1) cannot be expected. In fact, set $f_t(x_1,x_2,x_3)=x_3^5+tx_2^6x_3+x_1x_2^7+x_1^{15}$ ([2]). By [6], $f_0\overset{\text{b.a.}}{\sim}f_1$, but there are no coordinate changes h with $\Gamma_+(f_0)=\Gamma_+(f_1\circ h)$.

CONJECTURE 9.2 Let $f_\alpha : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ $(\alpha = 1, 2)$ be weighted homogeneous polynomial-germs with isolated singularities at the origin. Then, $f_1 \overset{\text{b.a.}-V}{\sim} f_2$ implies that f_1 and f_2 have same weights in suitably chosen coordinate systems of $(\mathbf{R}^n, 0)$.

Acknowledgements

I would like to take this opportunity to thank Satoshi Koike, Tzee-Char Kuo and Masahiro Shiota for valuable communications.

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