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INEQUALITIES FOR SUPERADDITIVE FUNCTIONALS WITH APPLICATIONS

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Abstract

Some inequalities for superadditive functionals defined on convex cones in linear spaces are given, with applications for various mappings associated with the Jensen, Hölder, Minkowski and Schwarz inequalities.

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1. Introduction

Let X be a linear space. A subset $C \subseteq X$ is called a *convex cone* in X provided the following conditions hold:

(i) $x, y \in C$ imply $x + y \in C$;
(ii) $x \in C$, $\alpha \geq 0$ imply $\alpha x \in C$.

A functional $\nu : C \to \mathbb{R}$ is called *superadditive* on C if

(iii) $\nu(x + y) \geq \nu(x) + \nu(y)$ for any $x, y \in C$,

and *nonnegative* on C if

(iv) $\nu(x) \geq 0$ for each $x \in C$.

The functional ν is *s-positive homogeneous* on C, for a given $s > 0$, if

(v) $\nu(\alpha x) = \alpha^s \nu(x)$ for any $\alpha \geq 0$ and $x \in C$.

The main aim of the present paper is to provide some fundamental inequalities for the values $\nu(x)$ and $\nu(y)$ of a superadditive functional ν defined on a convex cone C provided that there exist constants $M \geq m > 0$ for which $My - x$ and $x - my$ remain in C. Natural applications in refining some fundamental inequalities such as the Jensen, Hölder, Minkowski and Schwarz inequalities are also provided.

2. The results

The following fundamental result holds.

THEOREM 1. *Let $x, y \in C$, and let $\nu : C \to \mathbb{R}$ be a nonnegative, superadditive and s-positive homogeneous functional on C. If $M \geq m \geq 0$ are such that $x - my$ and $My - x \in C$, then*

$$M^s \nu(y) \geq \nu(x) \geq m^s \nu(y). \tag{1}$$

PROOF. We have successively

$$\begin{aligned}\nu(x) &= \nu(x - my + my) \geq \nu(x - my) + \nu(my) \\ &= m^s \nu(y) + \nu(x - my) \geq m^s \nu(y),\end{aligned}$$

giving the second inequality in (1).

For $M = 0$, (1) is obviously true. Suppose that $M > 0$. We have successively

$$\begin{aligned}\nu(y) &= \nu\left(\frac{1}{M} \cdot My\right) = \frac{1}{M^s}\nu(My - x + x) \\ &\geq \frac{1}{M^s}[\nu(My - x) + \nu(x)] \geq \frac{1}{M^s}\nu(x),\end{aligned}$$

giving the first inequality in (1). □

Now, let $\ell : C \to \mathbb{R}$ be an *additive* and *strictly positive* functional on C that is also *positive homogeneous* on C, that is,

(vi) $\ell(\alpha x) = \alpha \ell(x)$ for any $\alpha > 0$ and $x \in C$.

We have the following result concerning other bounds for a composite functional.

THEOREM 2. *Let $x, y \in C$, let ν be strictly positive, superadditive and positive homogeneous on C, and let ℓ be an additive, strictly positive and positive homogeneous functional on C. If $M \geq m > 0$ are such that $x - my$ and $My - x \in C$, then*

$$\left[\frac{\nu(y)}{\ell(y)}\right]^{M\ell(y)} \geq \left[\frac{\nu(x)}{\ell(x)}\right]^{\ell(x)} \geq \left[\frac{\nu(y)}{\ell(y)}\right]^{m\ell(y)}. \tag{2}$$

PROOF. Consider the new functional $\mu : C \to \mathbb{R}$ defined by

$$\mu(x) := \ell(x) \ln\left[\frac{\nu(x)}{\ell(x)}\right].$$

Observe that, for $\alpha > 0$ and $x \in C$,

$$\mu(\alpha x) = \ell(\alpha x) \ln\left[\frac{\nu(\alpha x)}{\ell(\alpha x)}\right] = \alpha \ell(x) \ln\left[\frac{\nu(x)}{\ell(x)}\right] = \alpha \mu(x),$$

showing that μ is positive homogeneous on C.

Using the arithmetic mean–geometric mean inequality,

$$\frac{\alpha a+\beta b}{\alpha+\beta}\geq a^{(\alpha/(\alpha+\beta))}\cdot b^{(\beta/(\alpha+\beta))},$$

when $a, b>0$, $\alpha, \beta\geq 0$ with $\alpha+\beta>0$,

$$\begin{aligned}\mu(x+y) &= \ell(x+y)\ln\left[\frac{\nu(x+y)}{\ell(x+y)}\right]\geq \ell(x+y)\ln\left[\frac{\nu(x)+\nu(y)}{\ell(x)+\ell(y)}\right]\\ &= \ell(x+y)\ln\left[\frac{\ell(x)\cdot(\nu(x)/\ell(x))+\ell(y)\cdot(\nu(y)/\ell(y))}{\ell(x)+\ell(y)}\right]\\ &\geq [\ell(x)+\ell(y)]\ln\left[\left(\frac{\nu(x)}{\ell(x)}\right)^{\ell(x)/(\ell(x)+\ell(y))}\cdot\left(\frac{\nu(y)}{\ell(y)}\right)^{\ell(y)/(\ell(x)+\ell(y))}\right]\\ &= [\ell(x)+\ell(y)]\left\{\frac{\ell(x)}{\ell(x)+\ell(y)}\cdot\ln\left[\frac{\nu(x)}{\ell(x)}\right]+\frac{\ell(y)}{\ell(x)+\ell(y)}\ln\left[\frac{\nu(y)}{\ell(y)}\right]\right\}\\ &= \mu(x)+\mu(y),\end{aligned}$$

showing that μ is superadditive on C.

Now, if we apply Theorem 1 for $s=1$ and μ, we get

$$M\ell(y)\ln\left[\frac{\nu(y)}{\ell(y)}\right]\geq \ell(x)\ln\left[\frac{\nu(x)}{\ell(x)}\right]\geq m\ell(y)\ln\left[\frac{\nu(y)}{\ell(y)}\right],$$

which is clearly equivalent to (2). □

3. Applications for Jensen's inequality

Let K be a convex subset of the real linear space X and let $f:K\to\mathbb{R}$ be a convex mapping. Here we consider the following well-known form of Jensen's discrete inequality:

$$f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\leq\frac{1}{P_I}\sum_{i\in I}p_if(x_i), \tag{3}$$

where I denotes a finite subset of the set $\mathbb{N}$ of natural numbers, $x_i\in K$, $p_i\geq 0$ for $i\in I$ and $P_I:=\sum_{i\in I}p_i>0$.

Let us fix $I\in\mathcal{P}_f(\mathbb{N})$ (the class of finite parts of $\mathbb{N}$) and $x_i\in K$ $(i\in I)$. Now consider the functional $J:S_+(I)\to\mathbb{R}$ given by

$$J_I(p):=\sum_{i\in I}p_if(x_i)-P_If\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\geq 0, \tag{4}$$

where $S_+(I):=\{p=(p_i)_{i\in I}\mid p_i\geq 0, i\in I \text{ and } P_I>0\}$ and f is convex on K.

We observe that $S_+(I)$ is a cone and the functional J_I is nonnegative, superadditive [3] and positive homogeneous on $S_+(I)$.

Using Theorem 1 we can state the following proposition.

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PROPOSITION 3. *If $p, q \in S_+(I)$ and $M \geq m \geq 0$ are such that $Mp \geq q \geq mp$, then*

$$\begin{aligned} M\left[\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right] &\geq \sum_{i\in I} q_i f(x_i) - Q_I f\left(\frac{1}{Q_I}\sum_{i\in I} q_i x_i\right) \\ &\geq m\left[\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right] \quad (\geq 0). \end{aligned} \tag{5}$$

Now, on choosing $\nu(p) := J_I(p)$ and $\ell(p) := P_I$ and applying Theorem 2, we can state the following result as well.

PROPOSITION 4. *With the assumptions of Proposition 3,*

$$\begin{aligned} \left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^{MP_I} &\geq \left[\frac{1}{Q_I}\sum_{i\in I} q_i f(x_i) - f\left(\frac{1}{Q_I}\sum_{i\in I} q_i x_i\right)\right]^{Q_I} \\ &\geq \left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^{mP_I}. \end{aligned} \tag{6}$$

The above results can be used to obtain various inequalities generated by the appropriate choices of the convex function f.

(1) If $f : X \to \mathbb{R}$, $f(x) = \|x\|^r$, $r \geq 1$, where $(X, \|\cdot\|)$ is a normed linear space, then

$$\begin{aligned} M\left[\sum_{i\in I} p_i \|x_i\|^r - P_I^{1-r}\left\|\sum_{i\in I} p_i x_i\right\|^r\right] &\geq \sum_{i\in I} q_i \|x_i\|^r - Q_I^{1-r}\left\|\sum_{i\in I} q_i x_i\right\|^r \\ &\geq m\left[\sum_{i\in I} p_i \|x_i\|^r - P_I^{1-r}\left\|\sum_{i\in I} p_i x_i\right\|^r\right], \end{aligned} \tag{7}$$

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and

$$\frac{Q_I^{rQ_I}}{P_I^{rMP_I}}\left[P_I^{r-1}\sum_{i\in I}p_i\|x_i\|^r-\left\|\sum_{i\in I}p_ix_i\right\|^r\right]^{MP_I}$$
$$\geq\left[Q_I^{r-1}\sum_{i\in I}q_i\|x_i\|^r-\left\|\sum_{i\in I}q_ix_i\right\|^r\right]^{Q_I}$$
$$\geq\frac{Q_I^{rQ_I}}{P_I^{rmP_I}}\left[P_I^{r-1}\sum_{i\in I}p_i\|x_i\|^r-\left\|\sum_{i\in I}p_ix_i\right\|^r\right]^{mP_I}, \tag{8}$$

for $I\in\mathcal{P}_{\mathrm{f}}(\mathbb{N})$ and $p, q\in S_+(I)$ with $Mp\geq q\geq mp$ and $M\geq m>0$ and for any vectors $x_i\in X$, $i\in I$.

(2) For $x_i>0$ and $p_i\geq 0$ ($i\in\mathbb{N}$) such that $P_I>0$, let

$$A(I,p,x):=\frac{1}{P_I}\sum_{i\in I}p_ix_i,\quad G(I,p,x):=\left(\prod_{i\in I}(x_i)^{p_i}\right)^{1/P_I}$$

denote the weighted arithmetic and geometric means respectively.

Applying the above two propositions for the convex function $f(x)=-\ln x$, $x\in(0,\infty)$, we can state the following inequalities:

$$\left[\frac{A(I,p,x)}{G(I,p,x)}\right]^{MP_I}\geq\left[\frac{A(I,p,x)}{G(I,p,x)}\right]^{Q_I}\geq\left[\frac{A(I,p,x)}{G(I,p,x)}\right]^{mP_I} \tag{9}$$

and

$$\left\{\ln\left[\frac{A(I,p,x)}{G(I,p,x)}\right]\right\}^{MP_I}\geq\left\{\ln\left[\frac{A(I,p,x)}{G(I,p,x)}\right]\right\}^{Q_I}\geq\left\{\ln\left[\frac{A(I,p,x)}{G(I,p,x)}\right]\right\}^{mP_I}, \tag{10}$$

for $I\in\mathcal{P}_{\mathrm{f}}(\mathbb{N})$ and $p, q\in S_+(I)$ with $Mp\geq q\geq mp$ and $M\geq m>0$ and for any $x_i>0$, $i\in I$.

4. Applications for Hölder's inequality in normed spaces

Let $(X,\|\cdot\|)$ be a normed space and $I\in\mathcal{P}_{\mathrm{f}}(\mathbb{N})$. We define

$$E(I):=\{x=(x_j)_{j\in I}\mid x_j\in X,\ j\in I\},$$
$$\mathbb{K}(I):=\{\lambda=(\lambda_j)_{j\in I}\mid \lambda_j\in\mathbb{K},\ j\in I\}.$$

We consider for $\alpha,\beta>1$, $(1/\alpha)+(1/\beta)=1$ the functional

$$H_I(p,\lambda,x;\alpha,\beta):=\left(\sum_{j\in I}p_j|\lambda_j|^\alpha\right)^{1/\alpha}\left(\sum_{j\in I}p_j\|x_j\|^\beta\right)^{1/\beta}-\left\|\sum_{j\in I}p_j\lambda_jx_j\right\|.$$

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LEMMA 5. *For any $p, q \in S_+(I)$,*

$$H_I(p+q, \lambda, x; \alpha, \beta) \geq H_I(p, \lambda, x; \alpha, \beta) + H_I(q, \lambda, x; \alpha, \beta), \tag{11}$$

where $x \in E(I)$, $\lambda \in \mathbb{K}(I)$ and $\alpha, \beta > 1$ with $(1/\alpha) + (1/\beta) = 1$.

PROOF. Using the elementary inequality

$$(a^\alpha + b^\alpha)^{1/\alpha}(c^\beta + d^\beta)^{1/\beta} \geq ac + bd,$$

with $\alpha, \beta > 1$, $(1/\alpha) + (1/\beta) = 1$ and $a, b, c, d \geq 0$, and the triangle inequality,

$$\begin{aligned}
&H_I(p+q, \lambda, x; \alpha, \beta)\\
&\quad = \left(\sum_{i\in I} p_i|\lambda_i|^\alpha + \sum_{i\in I} q_i|\lambda_i|^\alpha\right)^{1/\alpha}\left(\sum_{i\in I} p_i\|x_i\|^\beta + \sum_{i\in I} q_i\|x_i\|^\beta\right)^{1/\beta}\\
&\qquad - \left\|\sum_{i\in I} p_i\lambda_i x_i + \sum_{i\in I} q_i\lambda_i x_i\right\|\\
&\quad = \left\{\left[\left(\sum_{i\in I} p_i|\lambda_i|^\alpha\right)^{1/\alpha}\right]^\alpha + \left[\left(\sum_{i\in I} q_i|\lambda_i|^\alpha\right)^{1/\alpha}\right]^\alpha\right\}\\
&\qquad \times\left\{\left[\left(\sum_{i\in I} p_i\|x_i\|^\beta\right)^{1/\beta}\right]^\beta + \left[\left(\sum_{i\in I} q_i\|x_i\|^\beta\right)^{1/\beta}\right]^\beta\right\}\\
&\qquad - \left\|\sum_{i\in I} p_i\lambda_i x_i + \sum_{i\in I} q_i\lambda_i x_i\right\|\\
&\quad \geq \left(\sum_{i\in I} p_i|\lambda_i|^\alpha\right)^{1/\alpha}\left(\sum_{i\in I} p_i\|x_i\|^\beta\right)^{1/\beta}\\
&\qquad + \left(\sum_{i\in I} q_i|\lambda_i|^\alpha\right)^{1/\alpha}\left(\sum_{i\in I} q_i\|x_i\|^\beta\right)^{1/\beta} - \left\|\sum_{i\in I} p_i\lambda_i x_i\right\| - \left\|\sum_{i\in I} q_i\lambda_i x_i\right\|\\
&\quad = H_I(p, \lambda, x; \alpha, \beta) + H_I(q, \lambda, x; \alpha, \beta),
\end{aligned}$$

and the superadditivity of H is proved. □

REMARK 1. The same result can be stated if $(B, \|\cdot\|)$ is a normed algebra and the functional H is defined by

$$H_I(p, \lambda, x; \alpha, \beta) := \left(\sum_{i\in I} p_i\|x_i\|^\alpha\right)^{1/\alpha}\left(\sum_{i\in I} p_i\|y_i\|^\beta\right)^{1/\beta} - \left\|\sum_{i\in I} p_i x_i y_i\right\|,$$

where $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I} \subset B$, $p \in S_+(I)$ and $\alpha, \beta > 1$ with $(1/\alpha) + (1/\beta) = 1$.

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Since, obviously, $H(\cdot, \lambda, x; \alpha, \beta)$ is positive homogeneous, on using Theorems 1 and 2, we can state the following propositions.

PROPOSITION 6. *If $p, q \in S_+(I)$ and $M \geq m \geq 0$ with $Mp \geq q \geq mp$, then*

$$\begin{aligned}
M & \left[\left(\sum_{i \in I} p_i |\lambda_i|^\alpha \right)^{1/\alpha} \left(\sum_{i \in I} p_i \|x_i\|^\beta \right)^{1/\beta} - \left\| \sum_{i \in I} p_i \lambda_i x_i \right\| \right] \\
& \geq \left(\sum_{i \in I} q_i |\lambda_i|^\alpha \right)^{1/\alpha} \left(\sum_{i \in I} q_i \|x_i\|^\beta \right)^{1/\beta} - \left\| \sum_{i \in I} q_i \lambda_i x_i \right\| \\
& \geq m \left[\left(\sum_{i \in I} p_i |\lambda_i|^\alpha \right)^{1/\alpha} \left(\sum_{i \in I} p_i \|x_i\|^\beta \right)^{1/\beta} - \left\| \sum_{i \in I} p_i \lambda_i x_i \right\| \right] \quad (\geq 0).
\end{aligned}$$

Now, for $\ell(p) := P_I$ and $\nu(p) = H_I(p, \lambda, x; \alpha, \beta)$, on applying Theorem 2, we have the following.

PROPOSITION 7. *With the assumptions in Proposition 6,*

$$\begin{aligned}
& \left[\left(\frac{1}{P_I} \sum_{i \in I} p_i |\lambda_i|^\alpha \right)^{1/\alpha} \left(\frac{1}{P_I} \sum_{i \in I} p_i \|x_i\|^\beta \right)^{1/\beta} - \left\| \frac{1}{P_I} \sum_{i \in I} p_i \lambda_i x_i \right\| \right]^{MP_I} \\
& \quad \geq \left[\left(\frac{1}{Q_I} \sum_{i \in I} q_i |\lambda_i|^\alpha \right)^{1/\alpha} \left(\frac{1}{Q_I} \sum_{i \in I} q_i \|x_i\|^\beta \right)^{1/\beta} - \left\| \frac{1}{Q_I} \sum_{i \in I} q_i \lambda_i x_i \right\| \right]^{Q_I} \\
& \quad \geq \left[\left(\frac{1}{P_I} \sum_{i \in I} p_i |\lambda_i|^\alpha \right)^{1/\alpha} \left(\frac{1}{P_I} \sum_{i \in I} p_i \|x_i\|^\beta \right)^{1/\beta} - \left\| \frac{1}{P_I} \sum_{i \in I} p_i \lambda_i x_i \right\| \right]^{mP_I},
\end{aligned}$$

for $x \in E(I)$, $\lambda \in \mathbb{K}(I)$ and $\alpha, \beta > 1$, $(1/\alpha) + (1/\beta) = 1$.

Similar results may be stated for normed algebras. However, the details are omitted.

5. Applications for Minkowski's inequality

Let $(X, \|\cdot\|)$ be a normed space and $I \in \mathcal{P}_f(\mathbb{N})$. We define the functional

$$M_I(p, x, y; \alpha) = \left[\left(\sum_{i \in I} p_i \|x_i\|^\alpha \right)^{1/\alpha} + \left(\sum_{i \in I} p_i \|y_i\|^\alpha \right)^{1/\alpha} \right]^\alpha - \sum_{i \in I} p_i \|x_i + y_i\|^\alpha, \tag{12}$$

where $p \in S_+(I)$, $\alpha \geq 1$ and $x, y \in E(I)$.

LEMMA 8. *For any $p, q \in S_+(I)$,*

$$M_I(p + q, x, y; \alpha) \geq M_I(p, x, y; \alpha) + M_I(q, x, y; \alpha),$$

where $x, y \in E(I)$ and $\alpha \geq 1$.

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PROOF. Using the elementary inequality

$$(a^\alpha + b^\alpha)^{1/\alpha} + (c^\alpha + d^\alpha)^{1/\alpha} \geq [(a+c)^\alpha + (b+d)^\alpha]^{1/\alpha},$$

for $a, b, c, d \geq 0$ and $\alpha \geq 1$,

$$\begin{aligned}
M_I(p+q, x, y; \alpha) &= \Bigg[\Bigg\{\Bigg[\Bigg(\sum_{i\in I} p_i \|x_i\|^\alpha\Bigg)^{1/\alpha}\Bigg]^\alpha + \Bigg[\Bigg(\sum_{i\in I} q_i \|x_i\|^\alpha\Bigg)^{1/\alpha}\Bigg]^\alpha\Bigg\}^{1/\alpha} \\
&\quad + \Bigg\{\Bigg[\Bigg(\sum_{i\in I} p_i \|y_i\|^\alpha\Bigg)^{1/\alpha}\Bigg]^\alpha + \Bigg[\Bigg(\sum_{i\in I} q_i \|y_i\|^\alpha\Bigg)^{1/\alpha}\Bigg]^\alpha\Bigg\}^{1/\alpha}\Bigg]^\alpha \\
&\quad - \sum_{i\in I} p_i \|x_i + y_i\|^\alpha - \sum_{i\in I} q_i \|x_i + y_i\|^\alpha \\
&\geq \Bigg[\Bigg(\sum_{i\in I} p_i \|x_i\|^\alpha\Bigg)^{1/\alpha} + \Bigg(\sum_{i\in I} p_i \|y_i\|^\alpha\Bigg)^{1/\alpha}\Bigg]^\alpha \\
&\quad + \Bigg[\Bigg(\sum_{i\in I} q_i \|x_i\|^\alpha\Bigg)^{1/\alpha} + \Bigg(\sum_{i\in I} q_i \|y_i\|^\alpha\Bigg)^{1/\alpha}\Bigg]^\alpha \\
&\quad - \sum_{i\in I} p_i \|x_i + y_i\|^\alpha - \sum_{i\in I} q_i \|x_i + y_i\|^\alpha \\
&= M_I(p, x, y; \alpha) + M_I(q, x, y; \alpha),
\end{aligned}$$

which proves the superadditivity property of the functional M. □

Since the functional $M_I(\cdot, x, y; \alpha)$ is positive homogeneous on $S_+(I)$, on using Theorem 1, we can state the following proposition.

PROPOSITION 9. *If $p, q \in S_+(I)$ and $M \geq m \geq 0$ with $Mp \geq q \geq mp$, then*

$$\begin{aligned}
M &\Bigg\{\Bigg[\Bigg(\sum_{i\in I} p_i \|x_i\|^\alpha\Bigg)^{1/\alpha} + \Bigg(\sum_{i\in I} p_i \|y_i\|^\alpha\Bigg)^{1/\alpha}\Bigg]^\alpha - \sum_{i\in I} p_i \|x_i + y_i\|^\alpha\Bigg\} \\
&\geq \Bigg[\Bigg(\sum_{i\in I} q_i \|x_i\|^\alpha\Bigg)^{1/\alpha} + \Bigg(\sum_{i\in I} q_i \|y_i\|^\alpha\Bigg)^{1/\alpha}\Bigg]^\alpha - \sum_{i\in I} q_i \|x_i + y_i\|^\alpha \\
&\geq m \Bigg\{\Bigg[\Bigg(\sum_{i\in I} p_i \|x_i\|^\alpha\Bigg)^{1/\alpha} + \Bigg(\sum_{i\in I} p_i \|y_i\|^\alpha\Bigg)^{1/\alpha}\Bigg]^\alpha - \sum_{i\in I} p_i \|x_i + y_i\|^\alpha\Bigg\}.
\end{aligned}$$

Now, since $\ell(p) = P_I$ is additive and positive homogeneous on $S_+(I)$, on using Theorem 2 we can state the following result as well.

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PROPOSITION 10. *With the assumptions in Proposition 9,*

$$\begin{aligned}
&\left\{\left[\left(\frac{1}{P_I}\sum_{i\in I}p_i\|x_i\|^\alpha\right)^{1/\alpha}+\left(\frac{1}{P_I}\sum_{i\in I}p_i\|y_i\|^\alpha\right)^{1/\alpha}\right]^\alpha-\frac{1}{P_I}\sum_{i\in I}p_i\|x_i+y_i\|^\alpha\right\}^{MP_I}\\
&\quad\geq\left\{\left[\left(\frac{1}{Q_I}\sum_{i\in I}q_i\|x_i\|^\alpha\right)^{1/\alpha}+\left(\frac{1}{Q_I}\sum_{i\in I}q_i\|y_i\|^\alpha\right)^{1/\alpha}\right]^\alpha\right.\\
&\qquad\left.-\frac{1}{Q_I}\sum_{i\in I}q_i\|x_i+y_i\|^\alpha\right\}^{Q_I}\\
&\quad\geq\left\{\left[\left(\frac{1}{P_I}\sum_{i\in I}p_i\|x_i\|^\alpha\right)^{1/\alpha}+\left(\frac{1}{P_I}\sum_{i\in I}p_i\|y_i\|^\alpha\right)^{1/\alpha}\right]^\alpha\right.\\
&\qquad\left.-\frac{1}{P_I}\sum_{i\in I}p_i\|x_i+y_i\|^\alpha\right\}^{mP_I}.
\end{aligned}$$

6. Applications for the Schwarz inequality

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive-semi-definite Hermitian forms on X, or, for simplicity, nonnegative forms on X: that is, the mapping $\langle\cdot,\cdot\rangle : X\times X\to\mathbb{K}$ belongs to $\mathcal{H}(X)$ if it satisfies the conditions:

(i) $\langle x, x\rangle\geq 0$ for all $x\in X$;
(ii) $\langle\alpha x+\beta y, z\rangle=\alpha\langle x, z\rangle+\beta\langle y, z\rangle$ for all $x, y\in X$ and $\alpha, \beta\in\mathbb{K}$;
(iii) $\langle y, x\rangle=\overline{\langle x, y\rangle}$ for all $x, y\in X$.

If $\langle\cdot,\cdot\rangle\in\mathcal{H}(X)$, then the functional $\|\cdot\|=\langle\cdot,\cdot\rangle^{1/2}$ is a semi-norm on X and the following version of Schwarz' inequality holds:

$$\|x\|\,\|y\|\geq|\langle x, y\rangle|, \tag{13}$$

for each $x, y\in H$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$. Also, we can introduce on $\mathcal{H}(X)$ the following *binary relation* [2]

$$\langle\cdot,\cdot\rangle_2\geq\langle\cdot,\cdot\rangle_1\quad\text{if and only if}\quad\|x\|_2\geq\|x\|_1\quad\text{for any}\quad x\in H. \tag{14}$$

This is an *order relation* on $\mathcal{H}(X)$.

Consider the functional [2]

$$\sigma:\mathcal{H}(X)\times X^2\to\mathbb{R}_+,\qquad\sigma(\langle\cdot,\cdot\rangle; x, y):=\|x\|\|y\|-|\langle x, y\rangle|,$$

which is closely related to the second version of the Schwarz inequality in (13).

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LEMMA 11 (Dragomir–Mond [2]). *The functional* $\sigma(\cdot; x, y)$ *is nonnegative, superadditive and positive homogeneous on* $\mathcal{H}(X)$.

PROPOSITION 12. *Let* $M \geq m > 0$, *and let* $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ *be two nonnegative Hermitian forms on* X *such that* $M\|x\|_1 \geq \|x\|_2 \geq m\|x\|_1$ *for each* $x \in X$. *Then*

$$M^2(\|x\|_1\|y\|_1 - |\langle x, y\rangle_1|) \geq \|x\|_2\|y\|_2 - |\langle x, y\rangle_2| \geq m^2(\|x\|_1\|y\|_1 - |\langle x, y\rangle_1|), \tag{15}$$

for any $x, y \in H$.

PROOF. From the hypothesis, $M^2\langle\cdot,\cdot\rangle_2 - \langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2 - m^2\langle\cdot,\cdot\rangle_1$ are nonnegative Hermitian forms. Then applying Theorem 1 for the functional

$$\sigma(\langle\cdot,\cdot\rangle; x, y) := \|x\|\|y\| - |\langle x, y\rangle|,$$

for x, y fixed in X, we deduce the desired result. □

REMARK 2. If we assume that $A : H \to H$ is a self-adjoint linear operator on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$ satisfying the property that there exist $P \geq p > 0$ such that $PI \geq A \geq pI$ in the operation order (that is, $P\|x\|^2 \geq \langle Ax, x\rangle \geq p\|x\|^2$ for any $x \in H$), then we have the inequality

$$P(\|x\|\|y\| - |\langle x, y\rangle|) \geq \langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2} - |\langle Ax, y\rangle| \geq p(\|x\|\|y\| - |\langle x, y\rangle|), \tag{16}$$

for any $x, y \in H$.

For $e \in X$, $e \neq 0$ we can define the functional

$$\ell(\langle\cdot,\cdot\rangle; e) := \|e\|^2 = \langle e, e\rangle.$$

For fixed $e \in H$, the functional $\ell(\cdot; e)$ is additive and positive homogeneous on $\mathcal{H}(X)$.

Using Theorem 2, we can state the following result as well.

PROPOSITION 13. *Let* $M \geq m > 0$, *and let* $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ *be two inner products on* X *such that* $M\|x\|_1 \geq \|x\|_2 \geq m\|x\|_1$ *for each* $x \in H$. *Then for any* $e \in X$, $e \neq 0$,

$$\left[\frac{\|x\|_1\|y\|_1 - |\langle x, y\rangle_1|}{\|e\|_1^2}\right]^{M\|e\|_1^2} \geq \left[\frac{\|x\|_2\|y\|_2 - |\langle x, y\rangle_2|}{\|e\|_2^2}\right]^{\|e\|_2^2} \geq \left[\frac{\|x\|_1\|y\|_1 - |\langle x, y\rangle_1|}{\|e\|_1^2}\right]^{m\|e\|_1^2}.$$

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REMARK 3. Similar results can be stated if one uses the following nonnegative, superadditive and s-positive homogeneous functionals on $\mathcal{H}(X)$ (see [1, pp. 8–15]):

$$\sigma_{\mathrm{r}}(\langle\cdot,\cdot\rangle; x, y) := \|x\|\|y\| - \mathrm{Re}\langle x, y\rangle;$$
$$\delta(\langle\cdot,\cdot\rangle; x, y) := \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2;$$
$$\delta_{\mathrm{r}}(\langle\cdot,\cdot\rangle; x, y) := \|x\|^2\|y\|^2 - (\mathrm{Re}\langle x, y\rangle)^2;$$
$$\gamma(\langle\cdot,\cdot\rangle; x, y) := \frac{\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2}{\|y\|^2};$$

where in the definition of γ, $\langle\cdot,\cdot\rangle$ is an inner product and y is not zero, and

$$\beta(\langle\cdot,\cdot\rangle; x, y) := (\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2)^{1/2},$$

for each $x, y \in X$.

The details are left to the interested reader.

References

[1] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces* (Nova Science Publishers, New York, 2007).

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[3] S. S. Dragomir, J. Pečarić and L. E. Persson, 'Properties of some functionals related to Jensen's inequality', *Acta Math. Hungar.* **70** (1996), 129–143.

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