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The Chowla–Selberg Formula and The Colmez Conjecture

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Abstract. In this paper, we reinterpret the Colmez conjecture on the Faltings height of CM abelian varieties in terms of Hilbert (and Siegel) modular forms. We construct an elliptic modular form involving the Faltings height of a CM abelian surface and arithmetic intersection numbers, and prove that the Colmez conjecture for CM abelian surfaces is equivalent to the cuspidality of this modular form.

1 Introduction

The celebrated Chowla–Selberg formula [SC] asserts

$$\prod_{[\mathfrak{a}]\in \mathcal{CL}(K)} |\Delta(\tau_{\mathfrak{a}})|\,\mathrm{Im}(\tau_{\mathfrak{a}})^6 = \Big(\frac{1}{2\pi\sqrt{l}}\Big)^{6h} \prod_{0<c<l} \Gamma\Big(\frac{c}{l}\Big)^{6\epsilon(c)}. \tag{1.1}$$

Here $K = \mathbb{Q}(\sqrt{-l})$ is an imaginary field of prime discriminant $-l$, h is the ideal class number of K, and $\epsilon(c) = (\frac{-l}{c})$. Moreover, Δ is the well-known cusp form of weight 12, and $\Gamma(x)$ is the usual Gamma function. Gross re-interpreted this formula (up to a constant multiple in $\overline{\mathbb{Q}}$) as a period relation for a CM elliptic curve in his thesis [Gr2]. Later, he generalized this period relation to a CM abelian variety with CM by a CM abelian extension of $\mathbb{Q}$ [Gr1]. Anderson reformulated the right-hand side of Gross's formula in terms of a log-derivative of Dirichlet L-series [An]. In 1993, Colmez [Co] defined p-adic periods of a CM abelian variety (using an integral model) and conjectured that there should be a product formula for periods. Using that, he derived a conjecture which gives a very precise identity between the Faltings height of a CM abelian variety and the logarithmic derivative of certain virtual Artin L-functions at $s = 0$. It can be roughly stated as follows. Let K be a CM number field and let Φ be a CM type of K. Let A be a CM abelian variety of CM type $(\mathcal{O}_K, \Phi)$ defined over a number field L such that A has good reduction everywhere, and let $\alpha \in \Lambda^g\Omega_A$ be a Neron differential of A over $\mathcal{O}_L$, non-vanishing everywhere. Then the Faltings height of A is defined as (our normalization is slightly different from that of [Co])

$$h_{\mathrm{Fal}}(A) = -\frac{1}{2[L:\mathbb{Q}]}\sum_{\sigma:L\hookrightarrow\mathbb{C}} \log\Big|\Big(\frac{1}{2\pi i}\Big)^g \int_{\sigma(A)(\mathbb{C})} \sigma(\alpha)\wedge\overline{\sigma(\alpha)}\Big| + \log \#\Lambda^g\Omega_A/\mathcal{O}_L\alpha.$$

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Here $g = \dim A$. It is independent of the choice of L. In fact, Colmez proved that $h_{\mathrm{Fal}}(\Phi) = \frac{1}{[K:\mathbb{Q}]} h_{\mathrm{Fal}}(A)$ depends only on the CM type Φ, not on A or K [Co, Theorem 0.3]. On the other hand, Colmez constructed a class function A^0_Φ on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ from the CM type Φ (see §3 for details), which can be viewed as a linear combination of characters of Artin representations, say $\sum a_\chi \chi$. The Colmez conjecture asserts

$$h_{\mathrm{Fal}}(\Phi) = -\sum a_\chi \frac{L'(0,\chi)}{L(0,\chi)} - \frac{1}{2}\sum a_\chi \log f_{\mathrm{Art}}(\chi) + \frac{1}{4}\log 2\pi,$$

where $f_{\mathrm{Art}}(\chi)$ is the analytic Artin conductor of χ.

When the CM abelian variety is an elliptic curve, it is a reformulation of the Chowla–Selberg formula. In the same paper, Colmez proved the conjecture for an abelian CM number field, by combining Gross's work with his computation of the p-adic period of the Jacobian of the Fermat curves. Recently a less precise version of the conjecture and the result have been generalized to CM motives by V. Maillot and Roessler [MR] and Köhler and Roessler [KR] using the Lefschetz fixed point theorem in Arakelov geometry. Yoshida independently developed conjectures about absolute CM periods that are very close to the Colmez conjecture and provided some nontrivial numerical evidence as well as partial results [Yo]. We should also mention that van der Poorten and Williams [VW] gave another proof of the Chowla–Selberg formula by computing the CM values of the η-function.

Nothing is known about the Colmez conjecture besides what he has proved. It remains a mystery in the non-abelian case. The goal of this note is to try to understand the conjecture in terms of modular forms and arithmetic intersection. In Section 2, we interpret the Faltings height from the moduli point of view as in Faltings' original definition and relate it to Siegel modular forms and Hilbert modular forms, and arithmetic intersections. For example, we have the following (Corollary 2.4).

Proposition *Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field with prime discriminant $D \equiv 1$* mod 4. *Let K be a quartic CM number field with real quadratic subfield F, and Φ a CM type of K. Let $\mathcal{X}$ be the moduli stack of abelian surfaces with real multiplication by $\mathcal{O}_F$, and let $\hat{\mathcal{M}}_k$ be the line bundle on $\mathcal{X}$ of Hilbert modular forms of weight k with the Petersson metric. Then*

$$\frac{k\#\,\mathrm{CM}(K,\Phi)}{W_K} h_{\mathrm{Fal}}(A) = h_{\hat{\mathcal{M}}_k}(\mathcal{CM}(K,\Phi)).$$

Here W_K is the number of roots of unity in K, $\mathrm{CM}(K,\Phi)$ *is the* 0*-cycle of CM abelian surfaces of CM type* $(\mathcal{O}_K,\Phi)$ *in* $\mathcal{X}(\bar{\mathbb{Q}})$*, and* $\mathcal{CM}(K,\Phi)$ *is the flat closure of* $\mathrm{CM}(K,\Phi)$ *in $\mathcal{X}$. Let Ψ be a normalized meromorphic Hilbert modular form for* $\mathrm{SL}_2(\mathcal{O}_F)$ *of weight k such that* $\operatorname{div}\Psi$ *and* $\mathcal{CM}(K,\Phi)$ *intersect properly. Then*

$$\frac{k\#\,\mathrm{CM}(K,\Phi)}{W_K} h_{\mathrm{Fal}}(A) = \operatorname{div}\Psi.\mathcal{CM}(K,\Phi) - \frac{1}{W_K}\sum_{z\in\mathrm{CM}(K,\Phi)} \|\Psi(z)\|_{\mathrm{Pet}}$$

for an abelian surface of the CM type (K,Φ).

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In Section 3, we review the Colmez conjecture and unravel his definition of class function A^0_Φ associated with a CM type Φ, and prove the following proposition.

Proposition 1.1 *Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of prime discriminant $D \equiv 1 \bmod 4$. Let K be a non-biquadratic CM quartic field with real quadratic field F with a CM type Φ. Then the Colmez conjecture for the CM type Φ is the same as*

$$h_{\mathrm{Fal}}(A) = \tfrac{1}{2}\beta(K/F).$$

Here

$$\beta(K/F) = -\frac{\Lambda'(0, \chi_{K/F})}{\Lambda(0, \chi_{K/F})} + \Gamma'(1) - \log 4\pi$$

and $\Lambda(s, \chi_{K/F})$ is the complete L-function of the quadratic Hecke character $\chi_{K/F}$ associated with K/F as defined in (3.4). *In particular, the Faltings height is independent of the choice of CM types of K.*

Finally, let $\mathcal{X}$ be the moduli stack over $\mathbb{Z}$ of abelian varieties (A, ι, λ) with real multiplications (see Section 2 for a precise definition). Let $\mathcal{CM}(K)$ be the moduli stack of (A, ι, λ) where $\iota : \mathcal{O}_K \subset \mathrm{End}(A)$ is an $\mathcal{O}_K$-action on A such that $(A, \iota|_{\mathcal{O}_F}, \lambda) \in \mathcal{X}$, and the Rosati involution associated with the polarizations λ gives the complex conjugation on K. The map $(A, \iota, \lambda) \mapsto (A, \iota|_{\mathcal{O}_F}, \lambda)$ is a finite proper map from $\mathcal{CM}(K)$ into $\mathcal{M}$, and we denote its direct image in $\mathcal{M}$ still by $\mathcal{CM}(K)$ by abuse of notation. Finally, let $\mathcal{T}_m$ be the flat closure of the well-known Hirzebruch–Zagier divisors T_m in $\mathcal{X}$; see [BBK] for more information. Then $\mathcal{T}_m$ and $\mathcal{CM}(K)$ are arithmetic two- and one-cycles in the arithmetic three-fold $\mathcal{X}$ and they intersect properly. In [BY, (1.10)] (a minor mistake in the conjectured formula), it is conjectured that

$$\mathcal{T}_m.\mathcal{CM}(K) = \tfrac{1}{2}b_m.$$

Here $b_m = \sum_p b_m(p) \log p$ is defined as follows. Let $\tilde{K}$ be the reflex field of (K, Φ) with real quadratic field $\tilde{F} = \mathbb{Q}(\sqrt{\tilde{D}})$. Then

$$b_m(p) \log p = \sum_{\mathfrak{p}|p} \sum_{\substack{t = \frac{n+m\sqrt{\tilde{D}}}{2D} \in d^{-1}_{\tilde{K}/\tilde{F}} \\ |n|<m\sqrt{\tilde{D}}}} B_t(\mathfrak{p}),$$

where

$$B_t(\mathfrak{p}) = \begin{cases} 0 & \text{if } \mathfrak{p} \text{ is split in } \tilde{K}, \\ (\mathrm{ord}_{\mathfrak{p}} t + 1)\rho(t d_{\tilde{K}/\tilde{F}} \mathfrak{p}^{-1}) \log |\mathfrak{p}| & \text{if } \mathfrak{p} \text{ is not split in } \tilde{K}, \end{cases}$$

and

$$\rho(\mathfrak{a}) = \#\{\mathfrak{A} \subset \mathcal{O}_{\tilde{K}} : N_{\tilde{K}/\tilde{F}} \mathfrak{A} = \mathfrak{a}\}.$$

The main result of this paper is the following.

Theorem 1.2 *Let the notation be as above, and assume that* $d_K = D^2\tilde{D}$ *with* $\tilde{D} \equiv 1 \bmod 4$ *being prime. Then*

$$g(\tau) = \frac{\#\,\mathrm{CM}(K)}{2}\Big(-h_{\mathrm{Fal}}(A) + \frac{1}{2}\beta(K/F)\Big) + \sum_{m>0}\Big(\mathcal{T}_m.\mathcal{CM}(K) - \frac{1}{2}b_m\Big)q^m$$

is a modular form of weight 2*, level* D*, and character* $\epsilon_D = (\frac{\cdot}{D})$*. Moreover, the Colmez conjecture holds for* K *if and only if* $g(\tau)$ *is a cusp form.*

We will prove this theorem in Section 4, Here is the rough idea. Bruinier, Burgos Gil, and Kühn defined an arithmetic of Hirzebruch–Zagier divisors $\hat{\mathcal{T}}_m$ in $\mathcal{X}$ and proved that

$$\hat{\phi}(\tau) = \hat{\phi}(\tau) = \mathcal{M}^{\vee}_{\frac{1}{2}} + \sum_{M\geq 1}\hat{\mathcal{T}}_m e(m\tau)$$

is a modular form of weight 2, level D, and Nybentypus character $(\frac{D}{\cdot})$ with values in $\widehat{\mathrm{CH}}^1(\mathcal{X})$. Doing height pairing with $\mathcal{CM}(K)$ gives rise to the following modular form (see 4.1)

$$\phi(\tau) = -\frac{\#\,\mathrm{CM}(K)}{W_K}h_{\mathrm{Fal}}(A) + \sum_{m>0}\big(\mathcal{T}_m.\mathcal{CM}(K) + \frac{2}{W_K}G_m(\mathrm{CM}(K))\big)q^m.$$

On the other hand, [BY, Theorems 5.1, 8.1] (see also Theorem 4.1) asserts that

$$f(\tau) = -\frac{\#\,\mathrm{CM}(K)}{2W_K}\beta(K/F) + \sum_{m>0}\Big(\frac{1}{2}b_m + \frac{2}{W_K}G_m(\mathrm{CM}(K))\Big)q^m$$

is a modular form of weight 2, level D, Nybentypus character $(\frac{D}{\cdot})$. Since $g(\tau) = \phi(\tau) - f(\tau)$, one obtains the theorem.

We proved that $\mathcal{T}_1.\mathcal{CM}(K) = \frac{1}{2}b_1$ if furthermore $\mathcal{O}_K$ is a free $\mathcal{O}_F$-module [Ya1]. In particular, for $D = 5, 13, 17$, this, together with Theorem 1.2, implies that $g(\tau)$ is cuspidal, and so the Colmez conjecture holds in these cases. We proved that $\mathcal{T}_m.\mathcal{CM}(K) = \frac{1}{2}b_m$ for all $m \geq 1$, assuming further that $\mathcal{O}_K$ is a free $\mathcal{O}_F$-module [Ya2]. It gives the first non-abelian Chowla–Selberg formula.

2 The Faltings Height

Let $g \geq 1$ be an integer, and let $\mathbb{H}_g$ be the Siegel upper plane of genus g, *i.e.,* the set of symmetric matrices $z = x + iy \in \mathrm{Sym}_g(\mathbb{C})$ such that $y > 0$ is totally positive. Let $A_g = \mathrm{Sp}_g(\mathbb{Z})\backslash\mathbb{H}_g$ be the open Siegel modular variety of genus g over $\mathbb{C}$. Let $\mathcal{A}_g$ be the moduli stack over $\mathbb{Z}$ of principally polarized abelian varieties (A, λ), then $\mathcal{A}_g(\mathbb{C}) = [A_g]$ as orbifolds. Let $\tilde{\mathcal{A}}_g$ be a toroidal compactification, and let ω be the Hodge bundle on $\tilde{\mathcal{A}}_g$. It has a natural metric defined as follows. Let α be a section of ω and let $z = (A_z, \lambda_z) \in \mathcal{A}_g(\mathbb{C})$, The value α_z of α at z has metric

$$\|\alpha_z\|^2_{\mathrm{nat}} = \Big|\Big(\frac{1}{2\pi i}\Big)^g\int_{A_z(\mathbb{Z})}\alpha\wedge\bar{\alpha}\Big|.$$

We write $\widehat{\omega} = (\omega, \|\cdot\|_{\mathrm{nat}})$ for this *naturally* metrized Hodge bundle. We remark that different authors use different normalizing factors (we use $(\frac{1}{2\pi i})^g$ here). For a primitive arithmetic one-cycle $\mathcal{Z} = (A, \lambda) \in \mathcal{A}_g(\mathcal{O}_L)$ where L is a number field and $\mathcal{O}_L$ is the ring of integers of L, we define its Faltings height with respect to $\widehat{\omega}$ as

$$(2.1) \qquad h_{\hat{\omega}}(\mathcal{Z}) = \frac{1}{[L:\mathbb{Q}]} \operatorname{div} s.\mathcal{Z} - \sum_{\sigma\colon L \hookrightarrow \mathbb{C}} \frac{1}{\#\operatorname{Aut}(\sigma(A), \lambda))} \log \|s(\sigma(A), \lambda)\|_{\mathrm{nat}}.$$

Here $\operatorname{Aut}(\sigma(A), \lambda)$ is the automorphism group of $(\sigma(A), \lambda)$ over $\mathbb{C}$. It does not change when we replace L by its finite extensions. We define the Faltings height of an arithmetic 1-cycle $\mathcal{Z}$ by linearity. Let $\omega_{A/L} = \wedge^g \Omega_{A/\mathcal{O}_L}$ which is an invertible $\mathcal{O}_L$-module (since A has good reduction everywhere). Let L' be the Hilbert class field of L. Then $\omega_{A/L'} = \omega_{A/L} \otimes \mathcal{O}_{L'}$ is a principal $\mathcal{O}_{L'}$-module. Without loss of generality, we may thus assume that $\omega_{A/L} = \mathcal{O}_L \alpha$ is already principal. In this case (2.1) gives

$$h_{\hat{\omega}}((A, \lambda)) = -\frac{1}{2[L:\mathbb{Q}]} \sum_{\sigma\colon L \hookrightarrow \mathbb{C}} \frac{1}{\operatorname{Aut}(\sigma(A), \lambda))} \log \Big| \Big(\frac{1}{2\pi i}\Big)^g \int_{\sigma(A)(\mathbb{C})} \sigma(\alpha) \wedge \overline{\sigma(\alpha)} \Big|.$$

Let (A, ι, λ) be a CM abelian variety over $\mathbb{C}$ of CM type $(\mathcal{O}_K, \Phi)$, *i.e.,*

$$\iota\colon \mathcal{O}_K \hookrightarrow \operatorname{End}(A)$$

such that the induced action of $\mathcal{O}_K$ on Ω_A is given by the CM type Φ. Then (A, ι, λ) descends to an abelian variety (A_L, ι, λ) where A_L is an abelian variety over $\mathcal{O}_L$ with good reduction everywhere, and ι and λ are also defined over $\mathcal{O}_L$. In such a case,

$$\operatorname{Aut}((\sigma(A_L), \lambda)) = \mu_K$$

is the group of unity in K, and is independent of the choice of L or $\sigma\colon L \hookrightarrow \mathbb{C}$. So it is natural to define

$$h_{\mathrm{Fal}}(A) = W_K h_{\hat{\omega}}((A_L, \lambda)) = -\frac{1}{2[L:\mathbb{Q}]} \sum_{\sigma\colon L \hookrightarrow \mathbb{C}} \log \Big| \Big(\frac{1}{2\pi i}\Big)^g \int_{\sigma(A_L)(\mathbb{C})} \sigma(\alpha) \wedge \overline{\sigma(\alpha)} \Big|.$$

Here $W_K = \#\mu_K$. Notice that this normalization differs from Colmez's normalization by $\frac{g}{2} \log 2\pi$ [Co]. It is not independent of L. In fact, Colmez proved that it is only dependent of (K, Φ) [Co, Theorem 0.3].

By [FC, p. 141], if $f(\tau)$ is a Siegel modular form for $\operatorname{Sp}_g(\mathbb{Z})$ of weight k, then

$$\alpha(f) = f(\tau)(2\pi i dw_1 \wedge 2\pi i dw_2 \wedge \cdots \wedge 2\pi i dw_g)^k$$

is a section of $\omega_{\mathbb{C}}^k$, when pulling back to $\mathbb{H}_g$, where $dw_1 \wedge dw_2 \wedge \cdots \wedge dw_g$ is a trivialization of $\omega_{\mathbb{C}}$ over $\mathbb{H}_g$. Moreover, $\alpha(f)$ gives a section of ω^k over a subring R if and only if the Fourier coefficients of f are defined over R. Conversely, every section of ω^k can be identified this way. Let $\widehat{\mathcal{M}}_k = (\mathcal{M}_k, \|\cdot\|_{\mathrm{Pet}})$ be the line bundle of Siegel modular forms of weight k with the following Petersson metric

$$\|f(\tau)\|_{\mathrm{Pet}} = |f(\tau)|(4\pi)^g \det \operatorname{Im}(\tau))^{\frac{k}{2}}.$$

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Then it is easy to check that $f \mapsto \alpha(f)$ gives an isomorphism between $\widehat{\mathcal{M}}_k$ and $\widehat{\omega}^k_{\mathrm{nat}}$. Indeed,

$$\begin{aligned}\|\alpha(f)\|^2_{\mathrm{nat}} &= |f(z)|^2 \Big| (2\pi i)^g \int_{A_z} dw_1 \wedge d\bar{w}_1 \wedge dw_2 \wedge d\bar{w}_2 \wedge \cdots \wedge dw_g \wedge d\bar{w}_g \Big|^k \\ &= |f(z)|^2((4\pi)^g \mathrm{Im}(z))^k.\end{aligned}$$

Let K be a CM number field of degree $2g$ with a CM type Φ, let $\mathrm{CM}(K, \Phi)$ be the set of CM abelian varieties with CM type $(\mathcal{O}_K, \Phi)$. We extend it to an arithmetic 1-cycle in $\mathcal{A}_g$ over $\mathcal{O}_L$ for some number field L, and denote it by $\mathcal{CM}(K, \Phi)$. Then the following lemma is now obvious.

Lemma 2.1 *Let f be a normalized meromorphic Siegel modular form defined over $\mathcal{O}_L$,* i.e., *its Fourier coefficients are all defined over $\mathcal{O}_L$ and generate $\mathcal{O}_L$. Assume that* $\operatorname{div} f$ *and $\mathcal{CM}(K, \Phi)$ intersect properly. Then*

$$\begin{aligned}\frac{k\#\,\mathrm{CM}(K, \Phi)}{W_K} h_{\mathrm{Fal}}(A) &= h_{\widehat{\mathcal{M}}_k}(\mathcal{CM}(K, \Phi)) \\ &= \operatorname{div} f . \mathcal{CM}(K, \Phi) - \frac{1}{W_K} \sum_{A_\tau \in \mathrm{CM}(K,\Phi)} \log \|f(\tau)\|_{\mathrm{Pet}}\end{aligned}$$

for any CM abelian variety $A \in \mathrm{CM}(K, \Phi)$. Here for $\tau \in \mathbb{H}_g$, $A_\tau = \mathbb{C}^g / L_\tau$ is its associated principally polarized abelian variety where $L_\tau = \tau\mathbb{Z}^g \oplus \mathbb{Z}^g$.

Next, let F be a totally real number field of degree g, and let ∂ be its different. Let

$$\Gamma(\mathfrak{f}) = \mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{f}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(F) : a, d \in \mathcal{O}_F, c \in \mathfrak{f}^{-1}, b \in \mathfrak{f} \right\},$$

and let $X(\mathfrak{f}) = \Gamma(\mathfrak{f}) \backslash \mathbb{H}^g$ be a Hilbert modular variety. Let $\mathcal{X}(\mathfrak{f})$ be the moduli stack of the triples (A, ι, λ) defined over some number field where A is an abelian variety of dimension g with real multiplication

$$\iota \colon \mathcal{O}_F \subset \mathrm{End}(A) \quad \text{and} \quad \lambda \colon \mathfrak{f}^{-1}\partial^{-1} \mapsto \mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)^{\mathrm{Sym}}$$

is a polarization module map satisfying the Deligne–Pappa condition (see [Go]):

$$\mathfrak{f}^{-1}\partial^{-1} \otimes A \to A^\vee, \quad (r, a) \mapsto \lambda(r)a$$

is an isomorphism. Then $X(\mathfrak{f})$ is the coarse moduli scheme of $\mathcal{X}_{\mathbb{C}}$ and the map $(A, \iota, \lambda) \mapsto (A, \lambda(1))$ gives a natural map from $\mathcal{X}(\partial^{-1})$ to $\mathcal{A}_g$ which extends to a map ϕ from a toroidal compactification $\tilde{\mathcal{X}}(\partial^{-1})$ to some $\tilde{\mathcal{A}}_g$. Over $X(\partial^{-1} = \Gamma(\partial^{-1}) \backslash \mathbb{H}^g$ the map is given as follows. Let $e = \{e_1, \dots e_g\}$ be an ordered $\mathbb{Z}$-basis of $\mathcal{O}_F$ and let $f = \{f_1, \dots, f_g\}$ be a basis of ∂^{-1} such that

$$\mathrm{tr}_{F/\mathbb{Q}}\, e_i f_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_g\}$ be the (ordered) set of real embeddings of F, and set

$$R = \sigma(e) = (\sigma_i(e_j)) \in M_g(\mathbb{R}).$$

Then it is easy to check ${}^tR^{-1} = \sigma(f) = (\sigma_i(f_j))$. Finally for $a \in \mathcal{O}_F$ and $z = (z_1, \dots z_g) \in \mathbb{C}^g$, we set

$$a^* = \operatorname{diag}(\sigma_1(a), \dots, \sigma_g(a)), \quad z^* = \operatorname{diag}(z_1, \dots, z_g).$$

Lemma 2.2 *Let the notation be as above, then the map*

$$\phi\colon \Gamma(\partial^{-1})\backslash\mathbb{H}^g \longrightarrow \mathrm{Sp}_g(\mathbb{Z})\backslash\mathbb{H}_g$$

is given by $\phi(z) = {}^tRz^*R$. The associated map $\Gamma(\partial^{-1}) \to \mathrm{Sp}_g(\mathbb{Z})$ is given by

$$\phi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \operatorname{diag}(R^{-1}, {}^tR)\left(\begin{smallmatrix} a^* & b^* \\ c^* & d^* \end{smallmatrix}\right)\operatorname{diag}(R, {}^tR^{-1}).$$

Proof Let $\Lambda = \mathcal{O}_F \oplus \partial^{-1}$ be with the symplectic form

$$\langle {}^t(x_1, x_2), {}^t(y_1, y_2)\rangle = \operatorname{tr}_{F/\mathbb{Q}}(x_1y_2 - x_2y_1).$$

We embed F into $\mathbb{R}^g$ via σ and then embed Λ into $\mathbb{R}^{2g} = \mathbb{R}^g \oplus \mathbb{R}^g$. Then $\Lambda = \operatorname{diag}(R, {}^tR^{-1})L$, with $L = \mathbb{Z}^g \oplus \mathbb{Z}^g$ being the standard lattice of $\mathbb{R}^{2g}$ with the standard symplectic form. Since $\Gamma(\partial^{-1})$ acts on Λ linearly and preserves the symplectic form, so it acts on L and preserves its symplectic form, this gives the map $\phi\colon \Gamma(\partial^{-1}) \to \mathrm{Sp}_g(\mathbb{Z})$ in the lemma. Indeed, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(\partial^{-1})$, one has

$$\begin{aligned} \gamma L &= \operatorname{diag}(R^{-1}, {}^tR)\gamma\Lambda = \operatorname{diag}(R^{-1}, R)\left(\begin{smallmatrix} a^* & b^* \\ c^* & d^* \end{smallmatrix}\right)\Lambda \\ &= \operatorname{diag}(R^{-1}, R)\left(\begin{smallmatrix} a^* & b^* \\ c^* & d^* \end{smallmatrix}\right)\operatorname{diag}(R, {}^tR^{-1})L. \end{aligned}$$

For $z \in \mathbb{H}^g$, its associated abelian variety is $A_z = \mathbb{C}^g/\Lambda_z$, where

$$\begin{aligned} \Lambda_z &= \left\{{}^t\big(\sigma_1(a)z_1 + \sigma_1(b), \dots, \sigma_g(a)z_g + \sigma_g(b)\big) : a \in \mathcal{O}_F, b \in \partial^{-1}\right\} \\ &= z^*R\mathbb{Z}^g + {}^tR^{-1}\mathbb{Z}^g = {}^tR^{-1}L_\tau. \end{aligned}$$

Here $\tau = {}^tRz^*R \in \mathbb{H}_g$, and $L_\tau = \tau\mathbb{Z}^g + \mathbb{Z}^g = \{\tau a + b : a, b \in \mathbb{Z}^g\}$. So A_z is isomorphic to A_τ, where A_τ is the abelian variety associated with $\tau \in \mathbb{H}_g$. ■

Notice that $|\det R| = \sqrt{d_F}$ where d_F is the absolute discriminant of F. So for a Siegel modular form f of weight k,

$$\|f(\phi(z))\|^2_{\mathrm{Pet}} = |f(\phi(z))|^2((4\pi)^g \operatorname{Im}({}^tRz^*R))^k = |f(\phi(z))|^2((4\pi)^g d_F \prod \operatorname{Im}(z_i))^k.$$

Let $\mathcal{M}_k(\partial^{-1})$ be the line bundle of Hilbert modular forms of weight k on $\mathcal{X}(\partial^{-1})$, and let $\widehat{\mathcal{M}}_k(\partial^{-1}) = (\mathcal{M}_k(\partial^{-1}), \|\cdot\|_{\mathrm{Pet}})$ be the metrized line bundle of Hilbert modular forms of weight k with the following Petersson metric

$$\|\Psi(z)\|^2_{\mathrm{Pet}} = |\Psi(z)|^2((4\pi)^g d_F \prod \operatorname{Im}(z_i))^k.$$

It can be extended to a metrized line bundle on $\tilde{\mathcal{X}}(\partial^{-1})$, which we still denote by $\widehat{\mathcal{M}}_k(\partial^{-1})$. Notice that for a CM number field K with maximal totally real subfield F, $\mathcal{CM}(K, \Phi)$ can be viewed as an arithmetic 1-cycle in $\mathcal{X}(\partial^{-1})$. So we have the following.

Corollary 2.3 *Let Ψ be a normalized meromorphic Hilbert modular form for $\Gamma(\partial^{-1})$ of weight k such that* $\operatorname{div}\Psi$ *intersect with $\mathcal{CM}(K,\Phi)$ properly. Then*

$$\frac{k\#\,\mathrm{CM}(K,\Phi)}{W_K}h_{\mathrm{Fal}}(A) = h_{\widehat{\mathcal{M}}_k(\partial^{-1})}(\mathcal{CM}(K,\Phi))$$
$$= \operatorname{div}\Psi.\mathcal{CM}(K,\Phi) - \frac{1}{W_K}\sum_{z\in\mathcal{CM}(K,\Phi)}\log\|\Psi(z)\|_{\mathrm{Pet}}.$$

Now we consider a special case which is in the main interest of this paper. Let $F=\mathbb{Q}(\sqrt{D})$ be a real quadratic field with discriminant $D\equiv 1 \bmod 4$ being a prime number. In this case,

$$\Gamma := \Gamma(\mathcal{O}_F)\cong\Gamma(\partial^{-1}),\quad \gamma\mapsto\tilde{\gamma}=\operatorname{diag}\Big(1,\frac{\sqrt{D}}{\epsilon}\Big)\,\gamma\,\operatorname{diag}\Big(1,\frac{\epsilon}{\sqrt{D}}\Big),$$

where $\epsilon>1$ is a fundamental unit of F so that $\epsilon\epsilon'=-1$. This induces an isomorphism

$$\Gamma\backslash H^2\to\Gamma(\partial^{-1})\backslash H^2,(z_1,z_2)\mapsto\Big(\frac{\epsilon}{\sqrt{D}}z_1,\frac{-\epsilon'}{\sqrt{D}}z_2\Big).$$

Let $\widehat{\mathcal{M}}_k=(\mathcal{M}_k,\|\cdot\|_{\mathrm{Pet}})$ be the metrized line bundle on $\tilde{\mathcal{X}}$ of Hilbert modular forms for $\Gamma=\mathrm{SL}_2(\mathcal{O}_F)$ with the following Petersson metric:

$$\|\Psi(z)\|_{\mathrm{Pet}}=|\Psi(z)|(16\pi^2y_1y_2)^{k/2}$$

Then the above remark and Corollary 2.3 give the following.

Corollary 2.4 *Let Ψ be a normalized meromorphic Hilbert modular form for $\mathrm{SL}_2(\mathcal{O}_F)$ of weight k such that* $\operatorname{div}\Psi$ *and $\mathcal{CM}(K,\Phi)$ intersect properly. Then*

$$\frac{k\#\,\mathrm{CM}(K,\Phi)}{W_K}h_{\mathrm{Fal}}(A)=h_{\widehat{\mathcal{M}}_k}(\mathcal{CM}(K,\Phi))$$
$$=\operatorname{div}\Psi.\mathcal{CM}(K,\Phi)-\frac{1}{W_K}\sum_{z\in\mathrm{CM}(K,\Phi)}\|\Psi(z)\|_{\mathrm{Pet}}$$

for an abelian surface of the CM type (K,Φ).

3 The Colmez Conjecture

In this section, we review the Colmez conjecture [Co] and pay special attention in the end for the case K is a quartic CM number field.

We fix an embedding $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$, and view all number fields as subfields of $\overline{\mathbb{Q}}$. Let $\mathbb{Q}^{\mathcal{CM}}$ be the composite of all CM number fields in $\overline{\mathbb{Q}}$. It has a unique complex conjugation ρ. For a CM number field L, we denote $G_L^{\mathcal{CM}}=\mathrm{Gal}(\mathbb{Q}^{\mathcal{CM}}/L)$ and simply $G^{\mathcal{CM}}=G_{\mathbb{Q}}^{\mathcal{CM}}$. We define the Haar measure on $G^{\mathcal{CM}}$ with total volume 1, *i.e.*,

$$\int_{G^{\mathcal{CM}}}d\mu=\mathrm{Vol}(G^{\mathcal{CM}})=1.$$

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So $\mathrm{Vol}(G_L^{\mathcal{CM}}) = [L:\mathbb{Q}]^{-1}$.

For a field R of characteristic 0, let $H(G^{\mathcal{CM}}, R)$ be the Hecke algebra of $G^{\mathcal{CM}}$, *i.e.*, the ring (without identity) of locally constant functions Φ on $G^{\mathcal{CM}}$ with values in R with the convolution as the multiplication:

$$\Phi_1 * \Phi_2(g) = \int_{G^{\mathcal{CM}}} \Phi_1(h)\Phi_2(h^{-1}g)\, dh.$$

When $R = \mathbb{C}$ (or a subfield invariant under the complex conjugation), we define the reflex function $\Phi^\vee$ via $\Phi^\vee(g) = \overline{\Phi(g^{-1})}$, and define a positive definite Hermitian form

$$\langle \Phi_1, \Phi_2 \rangle = \int_{G^{\mathcal{CM}}} \Phi_1(h)\overline{\Phi_2(h)}\, dh = (\Phi_1 * \Phi_2^\vee)(1).$$

Let $H^0(G^{\mathcal{CM}}, R)$ be the subring of locally constant class functions on $G^{\mathcal{CM}}$ with values in R, *i.e.*, $\Phi \in H(G^{\mathcal{CM}}, R)$ such that $\Phi(hgh^{-1}) = \Phi(g)$ for all $g, h \in G^{\mathcal{CM}}$. By Brauer's theorem, $H^0(G^{\mathcal{CM}}, \bar{\mathbb{Q}})$ has a basis given by all Artin characters $\chi = \chi_\pi$ of $G^{\mathcal{CM}}$, where π runs over all irreducible representations of $G^{\mathcal{CM}}$. For an Artin character χ of $G^{\mathcal{CM}}$, we denote by $f_{\mathrm{Art}}(\chi)$ the analytic Artin conductor (*i.e.*, the one used for the functional equation) $L(s, \chi)$, the Artin L-function, and define

$$Z(s, \chi) = \frac{L'(s, \chi)}{L(s, \chi)}, \quad \mu_{\mathrm{Art}}(\chi) = \log f_{\mathrm{Art}}(\chi).$$

We extend the definition linearly to all functions $\Phi \in H^0(G^{\mathcal{CM}}, \bar{\mathbb{Q}})$.

Notice that there is a projection map $\Phi \mapsto \Phi^0$ from $H(G^{\mathcal{CM}}, \bar{\mathbb{Q}})$ to $H^0(G^{\mathcal{CM}}, \bar{\mathbb{Q}})$, given by

$$\Phi^0(g) = \int_{G^{\mathcal{CM}}} \Phi(hgh^{-1})\, dh = \sum_\chi \langle \Phi, \chi \rangle \chi.$$

A CM type is a function $\Phi \in H(G^{\mathcal{CM}}, \mathbb{Z})$ such that $\Phi(g) \in \{0, 1\}$ and

$$\Phi(g) + \Phi(\rho g) = 1 \quad \text{for every } g \in G^{\mathcal{CM}}.$$

This is consistent with the usual definition of a CM type. Indeed, let K be a subfield of finite degree over $\mathbb{Q}$ such that

$$\Phi(gh) = \Phi(g) \quad \text{for all } h \in G_K^{\mathcal{CM}}, g \in G^{\mathcal{CM}}. \tag{3.1}$$

Then Φ can be viewed as a formal sum

$$\Phi = \sum_{\sigma : K \hookrightarrow \bar{\mathbb{Q}}} a_\sigma(\Phi)\sigma, \tag{3.2}$$

where $a_\sigma(\Phi) = \Phi(g)$ for any $g \in G^{\mathcal{CM}}$ with $g|K = \sigma$. The two conditions on a CM type function Φ are exactly what is needed to make the formal sum Φ a CM type of

K in the usual sense. Conversely, a formal sum as (3.2) gives rise to a function Φ on $G^{\mathcal{CM}}$. We will use the same notation Φ for the two meanings of a CM type. If we take K to be the smallest subfield of $\mathbb{Q}^{\mathcal{CM}}$ such that (3.1) holds, then (K, Φ) is a primitive CM type. When K is Galois over $\mathbb{Q}$, the reflex type $\tilde{\Phi}$ in the usual sense corresponds to the reflex function $\Phi^\vee$.

For a CM type Φ, we define $A_\Phi = \Phi * \Phi^\vee$ and let A^0_Φ be the projection of A_Φ to $H^0(G^{\mathcal{CM}}, \bar{\mathbb{Q}})$. Concretely, let (K, Φ) be a CM type of a CM number field K in the usual sense, and let M be a CM Galois extension of $\mathbb{Q}$ containing K, and let $\Phi_M = \sum_{\sigma|_K \in \Phi} \sigma$ be the extension of Φ. Then

$$A_\Phi = \frac{1}{[M:\mathbb{Q}]} \Phi_M \tilde{\Phi}_M.$$

Here we recall that $\tilde{\Phi}_M = \sum a_\sigma \sigma^{-1}$ if $\Phi_M = \sum a_\sigma \sigma$. Moreover, if

$$A_\Phi = \sum_{\sigma \in \mathrm{Gal}(M/\mathbb{Q})} c(\sigma)\sigma,$$

then

$$A^0_\Phi = \sum_\sigma c^0(\sigma)\sigma, \qquad \text{with } c^0(\sigma) = \frac{1}{\#[\sigma]} \sum_{\tau \in [\sigma]} c(\tau).$$

Here $[\sigma]$ is the conjugacy class of σ in $\mathrm{Gal}(M/\mathbb{Q})$.

Let (K, Φ) be a CM type, and let A be a CM abelian variety of CM type $(\mathcal{O}_K, \Phi)$. We may assume that A is defined over a number field L with good reduction everywhere. Let $h_{\mathrm{Fal}}(A)$ be the Faltings height of A. It can be proved that

$$h_{\mathrm{Fal}}(\Phi) = \frac{1}{[K:\mathbb{Q}]} h_{\mathrm{Fal}}(A)$$

is independent of the choices of A and is even independent of the choice of K if we view Φ as a function of $G^{\mathcal{CM}}$. We call it the Faltings height of Φ. Colmez [Co, Theorem 0.3] asserts that there is a unique $\mathbb{Q}$-linear function ht from $H^0(G^{\mathcal{CM}}, \mathbb{R})$, the height function, satisfying a specific condition and

$$h_{\mathrm{Fal}}(\Phi) = -\,\mathrm{ht}(A^0_\Phi) - \tfrac{1}{2}\mu_{\mathrm{Art}}(A^0_\Phi) + \tfrac{1}{4}\log 2\pi.$$

Here the extra term $\frac{1}{4}\log 2\pi$ is due to the different normalization of the Faltings height between our definition and Colmez's. Furthermore, he conjectured [Co, Conjecture 0.4] that for any $\Phi \in H^0(G^{\mathcal{CM}}, \bar{\mathbb{Q}})$, one has $\mathrm{ht}(\Phi) = Z(0, \Phi^\vee)$. In terms of the Faltings height, it means the following.

Conjecture 3.1 **(Colmez)** $h_{\mathrm{Fal}}(\Phi) = -Z(0, A^0_\Phi) - \frac{1}{2}\mu_{\mathrm{Art}}(A^0_\Phi) + \frac{1}{4}\log 2\pi.$

Two CM types Φ_1 and Φ_2 are called *equivalent* if there is $\tau \in G^{\mathcal{CM}}$ such that $\Phi_1(\sigma) = \Phi_2(\tau\sigma)$ for every $\sigma \in G^{\mathcal{CM}}$. Clearly, two equivalent CM types have the same Faltings height.

Now we consider some simple examples. First let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field with the CM type $\Phi = \sigma_0$, where σ_0 is the identity map. In this case, $\tilde{\Phi} = \Phi$ and $A^0_\Phi = A_\Phi = \frac{1}{2}\sigma_0 = \frac{1}{4}(\chi_0+\chi_{-d})$, where χ_0 is the trivial character and χ_{-d} is the Dirichlet quadratic character associated with $K/\mathbb{Q}$. So the Colmez conjecture is simply

$$
\begin{aligned}
(3.3)\qquad 2h_{\mathrm{Fal}}(E) &= -\frac{\zeta'(0)}{\zeta(0)} - \frac{L'(0,\chi_{-d})}{L(0,\chi_{-d})} - \frac{1}{2}\log d + \log 2\pi \\
&= -\frac{L'(0,\chi_{-d})}{L(0,\chi_{-d})} - \frac{1}{2}\log d
\end{aligned}
$$

for a CM elliptic curve with CM by $\mathcal{O}_{-d}$. This is a reformulation of the Chowla–Selberg formula (1.1) [Co].

Next let $K = \mathbb{Q}(\sqrt{D}, \sqrt{-d})$ be a bi-quadratic CM number field with real quadratic subfield $F = \mathbb{Q}(\sqrt{D})$ and two imaginary quadratic field $F_1 = \mathbb{Q}(\sqrt{-d})$ and $F_2(\sqrt{-Dd})$. Let $\Phi = 1 + \sigma$ be a CM type of K with 1 being the identity map and $\sigma(\sqrt{D}) = -\sqrt{D}$, $\sigma(\sqrt{-d}) = \sqrt{-d}$, *i.e.*, σ fixes F_1. Then $\tilde{\Phi} = \Phi$, and

$$
A^0_\Phi = A_\Phi = \tfrac{1}{4}(1+\sigma)^4 = \tfrac{1}{2}(1+\sigma) = \tfrac{1}{4}(\chi_0 + \chi_{-d}),
$$

where χ_0 is the trivial character of $\mathrm{Gal}(K/\mathbb{Q})$ and χ_{-d} is the nontrivial character of $\mathrm{Gal}(F_1/\mathbb{Q})$ viewed as a character of $\mathrm{Gal}(K/\mathbb{Q})$. So the Colmez conjecture implies

$$
h_{\mathrm{Fal}}(A) = -\frac{\zeta'(0)}{\zeta(0)} - \frac{L'(0,\chi_{-d})}{L(0,\chi_{-d})} - \frac{1}{2}\log d + \log 2\pi = -\frac{L'(0,\chi_{-d})}{L(0,\chi_{-d})} - \frac{1}{2}\log d
$$

for a CM abelian surface of CM type $(\mathcal{O}_K, \Phi)$. That is the same as $h_{\mathrm{Fal}}(A) = 2h_{\mathrm{Fal}}(E)$ for a CM elliptic curve with CM by $\mathcal{O}_{-d} = \mathbb{Z}[\frac{-d+\sqrt{-d}}{2}]$, by the Chowla–Selberg formula (3.3). Indeed, let E be a CM elliptic curve with CM by $\mathcal{O}_{-d}$. Then $A = E \otimes \mathcal{O}_D \cong E \times E$ is of CM type (K, Φ) with CM by $\mathcal{O}_K$. So $h_{\mathrm{Fal}}(A) = 2h_{\mathrm{Fal}}(E)$. We summarize the two examples in the next proposition.

Proposition 3.2 *Let $K = \mathbb{Q}(\sqrt{D}, \sqrt{-d})$ be a bi-quadratic CM number field. Let E be a CM elliptic curve with CM by $\mathcal{O}_{-d}$ and let A be a CM abelian surface of CM type $(\mathcal{O}_K, \Phi)$, where $\Phi = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{-d}))$. Then*

$$
h_{\mathrm{Fal}}(A) = 2h_{\mathrm{Fal}}(E) = \frac{1}{2}\Gamma'(1) - \frac{1}{2}\log 4\pi + \frac{\Lambda'(0,\chi_{-d})}{\Lambda(0,\chi_{-d})}.
$$

Here

$$
\Lambda(s,\chi_{-d}) = \Big(\frac{d}{\pi}\Big)^{(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)L(s,\chi_{-d}).
$$

Now let $K = F(\sqrt{\Delta})$ be a non-biquadratic quartic CM number field with maximal totally real subfield $F = \mathbb{Q}(\sqrt{D})$, the case of special interest in this paper. We first assume that K is not Galois over $\mathbb{Q}$, and let M be the smallest Galois extension of $\mathbb{Q}$ containing K. Then $\mathrm{Gal}(M/\mathbb{Q}) = \langle \sigma, \tau \rangle$ is a dihedral group D_4 with

$$\sigma(\sqrt{\Delta}) = \sqrt{\Delta'}, \quad \sigma(\sqrt{\Delta'}) = -\sqrt{\Delta},$$
$$\tau(\sqrt{\Delta}) = \sqrt{\Delta'}, \quad \tau(\sqrt{\Delta'}) = \sqrt{\Delta}.$$

Let $\Phi = 1 + \sigma$ be a CM type of K, and let $\tilde{K}$ be its reflex field with maximal real quadratic field $\tilde{F}$. Then $\Phi_M = 1 + \sigma + \tau\sigma + \tau$ and $\tilde{\Phi}_M = 1 + \sigma^{-1} + \tau\sigma + \tau$. So

$$A_\Phi = \frac{1}{8}\Phi_M \tilde{\Phi}_M = \frac{1}{4}(2 + 2\tau + \sigma + \sigma^{-1} + \tau\sigma + \sigma\tau),$$

and

$$A_\Phi^0 = \frac{1}{4}\Big(1 - \rho + \sum_{\alpha \in \mathrm{Gal}(M/Q)} \alpha\Big) = \frac{1}{4}\Big(\frac{1}{2}\chi_\pi + \chi_1\Big).$$

Here χ_0 is the trivial character of $G_{\mathbb{Q}}$, and π is the unique two-dimensional representation of $\mathrm{Gal}(M/Q)$, and can be realized as $\mathrm{Ind}_{G_{\tilde{F}}}^{G_{\mathbb{Q}}} \chi_{\tilde{K}/\tilde{F}}$, where $\chi_{\tilde{K}/\tilde{F}}$ is the nontrivial quadratic character of $G_{\tilde{F}}$ factoring through $\mathrm{Gal}(\tilde{K}/\tilde{F})$. Notice that π is also $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \chi_{K/F}$.

When K is a cyclic quartic CM field with a CM type Φ, the same calculation (slightly simpler) shows

$$A_\Phi^0 = \frac{1}{4}\Big(\frac{1}{2}\chi_\pi + \chi_0\Big)$$

as above, with $\pi = \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \chi_{K/F}$. Notice that π is not irreducible in this case.

Let $\chi = \chi_{\tilde{K}/\tilde{F}}$ be the quadratic Hecke character of $\tilde{F}$ associated with $\tilde{K}/\tilde{F}$, and let

$$\Lambda(s, \chi) = (f_\chi)^{\frac{s}{2}} \pi^{-s-1} \Gamma\Big(\frac{s+1}{2}\Big)^2 L(s, \chi) \tag{3.4}$$

be the complete L-function of χ as defined in [BY, §6], where $f_\chi = N_{\tilde{F}/Q} d_{\tilde{K}/\tilde{F}} d_{\tilde{F}}$ is the Artin conductor of χ. Then we have the following.

Proposition 3.3 *Let K be a non-biquadratic quartic CM number field with real quadratic subfield F. Let Φ be a CM type of K and let $\tilde{K}$ be its reflex field with real quadratic field $\tilde{F}$. Let $\chi = \chi_{\tilde{K}/\tilde{F}}$ be as above. Then the Colmez conjecture for Φ is the same as*

$$8h_{\mathrm{Fal}}(\Phi) = -\frac{\Lambda'(0, \chi)}{\Lambda(0, \chi)} + \Gamma'(1) - \log 4\pi.$$

That is, $h_{\mathrm{Fal}}(A) = \frac{1}{2}\beta(\tilde{K}/\tilde{F}) = \frac{1}{2}\beta(K/F)$ for any CM abelian surface with CM by $\mathcal{O}_K$.

Proof The above calculation gives $A_\Phi^0 = \frac{1}{8}\chi_\pi + \frac{1}{4}\chi_0$, where χ_π is the character of the two-dimensional representation $\pi = \operatorname{Ind}_{G_{\tilde F}}^{G_\mathbb{Q}} \chi_{\tilde K/\tilde F}$, and χ_0 is the trivial character. So

$$Z(s,\chi_\pi) = \frac{L'(s,\chi_{\tilde K/\tilde F})}{L(s,\chi_{\tilde K/\tilde F})}, \quad \mu_{\mathrm{Art}}(\chi_\pi) = \log f_{\chi_{\tilde K/\tilde F}}.$$

(we trust the reader will distinguish the representation π from the number π). For $\chi = \chi_{\tilde K/\tilde F}$, one has

$$\begin{aligned} 8\Big(Z(0,A_\Phi^0) + \frac{1}{2}\mu_{\mathrm{Art}}(A_\Phi^0)\Big) &= \frac{L'(0,\chi)}{L(0,\chi)} + 2\frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\log f_\chi \\ &= \frac{\Lambda'(0,\chi)}{\Lambda(0,\chi)} + \log\pi - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} + 2\frac{\zeta'(0)}{\zeta(0)}. \end{aligned}$$

Now recall that

$$\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = -\gamma - 2\log 2, \quad \frac{\zeta'(0)}{\zeta(0)} = \log 2\pi, \quad \Gamma'(1) = -\gamma,$$

where γ is the Euler constant. So

$$8\Big(Z(0,A_\Phi^0) + \frac{1}{2}\mu_{\mathrm{Art}}(A_\Phi^0)\Big) = \frac{\Lambda'(0,\chi)}{\Lambda(0,\chi)} + \gamma + \log 4\pi + 2\log(2\pi).$$

Now the proposition is clear. Notice that $\beta(\tilde K/\tilde F) = \beta(K/F)$, and $L(s,\chi_{\tilde K/\tilde F}) = L(s,\chi_{K/F})$, since $\operatorname{Ind}_{G_{\tilde F}}^{G_\mathbb{Q}} \chi_{\tilde K/\tilde F} = \operatorname{Ind}_{G_F}^{G_\mathbb{Q}} \chi_{K/F} = \pi$ as explained above . ■

This proposition implies that $h_{\mathrm{Fal}}(A)$ for a CM abelian surface with CM by $\mathcal{O}_K$ is independent of choice of the CM abelian surface or the CM type when K is non-biquadratic. This is different from the bi-quadratic case discussed above. It might be interesting to note that

$$h_{\mathrm{Fal}}(A_d) + h_{\mathrm{Fal}}(A_{Dd}) = -\frac{\Lambda'(0,\chi_{K/F})}{\Lambda(0,\chi_{K/F})} + \Gamma'(1) - \log 4\pi$$

for the bi-quadratic case $K = \mathbb{Q}(\sqrt{D},\sqrt{-d})$, which is very much like Proposition 3.3. Here A_d (resp. A_{Dd}) is a CM abelian surface of the CM type $\Phi_d = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{-d}))$ (resp. $\Phi_{Dd} = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{-Dd}))$, and $F = \mathbb{Q}(\sqrt{D})$.

4 Proof of the Main Theorem

The purpose of this section is to prove Theorem 1.2. First we recall a modularity result of Bruinier, Burgos Gil and Kühn on arithmetic Hirzebruch–Zagier divisors.

Let $\tilde{\mathcal{X}}$ be a toroidal compactification of the arithmetic Hilbert modular surface $\mathcal{X}$ and let $\tilde{\mathcal{T}}_m$ be the corresponding compactification of $\mathcal{T}_m$ in $\tilde{\mathcal{X}}$. Bruinier, Burgos Gil

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and Kühn [BBK] defined an Arakelov divisor $\hat{\mathcal{T}}_m = (\tilde{\mathcal{T}}_m, 2G_m) \in \widehat{CH}^1(\tilde{\mathcal{X}})$ (we prefer a slightly different renormalization so that $(\Psi, -\log\|\cdot\|^2)$ is a principal divisor for a rational function on $\tilde{\mathcal{X}}$). They proved in the same paper [BBK, Theorem A] that

$$\hat{\phi}(\tau) = \mathcal{M}^\vee_{\frac{1}{2}} + \sum_{M\geq 1} \hat{\mathcal{T}}_m e(m\tau) \in M_2^+\Big(D, \Big(\frac{D}{\cdot}\Big)\Big) \otimes \widehat{CH}^1(\tilde{\mathcal{X}})$$

is a modular form valued in $\widehat{CH}^1(\tilde{\mathcal{X}})$ for $\Gamma_0(D)$ of weight 2 with Nebentypus character $(\frac{D}{\cdot})$. Here $M_2^+(D, (\frac{D}{\cdot}))$ is the subspace of modular forms of weight 2, level D, and Nebentypus character $(\frac{D}{\cdot})$ such that its Fourier expansion $f(\tau) = \sum_{m\geq 0} a_m e(m\tau)$ satisfies $a_m = 0$ if $(\frac{D}{m}) = -1$.

Recall that there is a bilinear form, the Faltings height pairing

$$\widehat{CH}^1(\tilde{\mathcal{X}}) \times \mathcal{Z}^2(\tilde{\mathcal{X}}) \to \mathbb{C},$$

given by

$$h_{(\mathcal{T},G)}(\mathcal{Z}) = \mathcal{T}.\mathcal{Z} + \frac{1}{2} \sum_{z\in\mathcal{Z}(\mathbb{C})} \frac{1}{\#\operatorname{Aut}(z)} G(z),$$

when $\mathcal{T}$ and $\mathcal{Z}$ intersect properly.

Let $\mathcal{CM}(K)$ be the moduli stack over $\mathbb{Z}$ representing the moduli problem which assigns a base scheme S to the set of the triples (A, ι, λ) where $\iota\colon \mathcal{O}_K \hookrightarrow \operatorname{End}_S(A)$ is a CM action of $\mathcal{O}_K$ on A and $(A, \iota|_{\mathcal{O}_F}, \lambda) \in \mathcal{M}(S)$ such that the Rosati involution associated with λ reduces to the complex conjugation of $\mathcal{O}_K$. The map $(A, \iota, \lambda) \mapsto (A, \iota|_{\mathcal{O}_F}, \lambda)$ is a finite proper map from $\mathcal{CM}(K)$ into $\mathcal{M}$, and we still denote its direct image in $\mathcal{M}$ by $\mathcal{CM}(K)$ by abuse of notation. It was proved in [Ya1, Lemma 2.1] that

$$\mathcal{CM}(K)(\mathbb{C}) = 2\,\mathrm{CM}(K) := 2(\mathrm{CM}(K,\Phi) + \mathrm{CM}(K,\Phi')),$$

where $\Phi = \{1, \sigma\}$ and $\Phi' = \{1, \sigma^{-1}\}$ are CM types of K given in Section 3. As mentioned in Section 2, $h_{\mathrm{Fal}}(A)$ depends only on its CM type. Since $A \mapsto \sigma^{-1}(A)$ is a bijection between $\mathrm{CM}(K,\Phi)$ and $\mathrm{CM}(K,\Phi')$, and $h_{\mathrm{Fal}}(A) = h_{\mathrm{Fal}}(\sigma^{-1}(A))$, we have

$$h_{\mathrm{Fal}}(\mathcal{CM}(K)) = 2\#\,\mathrm{CM}(K) h_{\mathrm{Fal}}(A)$$

for a CM abelian surface A with CM by $\mathcal{O}_K$. By Corollary 2.4, one sees that

$$h_{\mathcal{M}^\vee_{1/2}}(\mathcal{CM}(K)) = -\frac{1}{W_K}\#\,\mathrm{CM}(K) h_{\mathrm{Fal}}(A).$$

Now applying the height paring function to $\hat{\phi}(\tau)$ and $\mathcal{CM}(K)$, one obtains the following modular form in $M_2(D, (\frac{D}{\cdot}))$:

$$\phi(\tau) = -\frac{\#\,\mathrm{CM}(K)}{W_K} h_{\mathrm{Fal}}(A) + \sum_{m>0} \Big(\mathcal{T}_m.\mathcal{CM}(K) + \frac{2}{W_K} G_m(\mathrm{CM}(K))\Big) q^m. \tag{4.1}$$

Here $G_m(\mathrm{CM}(K)) = \sum_{z\in \mathrm{CM}(K)} G_m(z)$. This is the first main step in proving Theorem 1.2. To continue, we need a result of Bruinier and Yang [BY] on modularity of CM values of automorphic Green functions G_m, which we state as the following.

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Theorem 4.1 **(Bruinier and Yang)** *The function*

$$f(\tau) = -\frac{\#\,\mathrm{CM}(K)}{2W_K}\beta(K/F) + \sum_{m>0}\Big(\frac{1}{2}b_m + \frac{2}{W_K}G_m(\mathrm{CM}(K))\Big)q^m$$

is a modular form belonging to $M_2^+(D, (\frac{D}{\cdot}))$. Here b_m are the constants defined in the introduction.

Sketch of Proof Let

$$E_2^+(\tau, 0) = 1 + \sum_{m>0} C(m, 0)q^m \in M_2^+\Big(D, \Big(\frac{D}{\cdot}\Big)\Big)$$

be the (unique) normalized Eisenstein series in $M_2(D, (\frac{D}{\cdot}))$ defined in [BY, Corollary 2.3]. Using a derivative of the incoherent Hilbert Eisenstein series, diagonal restriction (to elliptic modular forms), and holomorphic projection, Bruinier and the author proved [BY, Theorem 8.1] that

$$F(\tau) = \sum_{m>0}\Big(\frac{1}{2}b_m + \frac{1}{2}c_m\Big)q^m + \frac{1}{4}\Lambda(0, \chi_{\tilde{K}/\tilde{F}})\beta(\tilde{K}/\tilde{F})(E_2^+(\tau, 0) - 1) \tag{4.2}$$

is a cusp form belonging to $S_2^+(D, (\frac{D}{\cdot}))$, where

$$c_m = \lim_{s\to 1}\Big\{2\sum_{t=\frac{n\pm m\sqrt{D}}{2p}\in d_{\tilde{K}/\tilde{F}}^{-1,+}} \rho(td_{\tilde{K}/\tilde{F}})Q_{s-1}\Big(\frac{n}{m\sqrt{D}}\Big) + \Lambda(0, \chi_{\tilde{K}/\tilde{F}})\Big(\frac{C(m, 0)}{2(s-1)} - \mathcal{L}_m\Big)\Big\}.$$

Here $d_{\tilde{K}/\tilde{F}}$ is the relative discriminant of $\tilde{K}/\tilde{F}$, the subscript + means totally positive, $\rho(\mathfrak{a}) = \#\{\mathfrak{A} \subset \mathcal{O}_{\tilde{K}} : N_{\tilde{K}/\tilde{F}}(\mathfrak{A}) = \mathfrak{a}\}$ is the norm counting function, $\mathcal{L}_m$ is some normalizing constant depending on m, and $Q_{s-1}(t)$ is the so-called Legendre function of the second kind.

On the other hand, the CM value of G_m is given by [BY, Theorem 5.1] (together with normalization in [BY, (2.24), (2.25)]) that

$$\frac{2}{W_{\tilde{K}}}G_m(\mathrm{CM}(K,\Phi)) = \lim_{s\to 1}\Big[\sum_{\mu=\frac{n-m\sqrt{q}}{2p}\in d_{\tilde{K}/\tilde{F}}^{-1,+}} Q_{s-1}\Big(\frac{n}{m\sqrt{q}}\Big)\rho_{\tilde{K}/\tilde{F}}(\mu d_{\tilde{K}/\tilde{F}}) + \Big(\frac{C(m, 0)}{2(s-1)} - \mathcal{L}_m\Big)\frac{\#\,\mathrm{CM}(K,\Phi)}{W_{\tilde{K}}}\Big].$$

One has the same formula for $G_m(\mathrm{CM}(K, \Phi'))$ with $\frac{n-m\sqrt{D}}{2D}$ replaced by $\frac{n+m\sqrt{D}}{2D}$. So

$$\frac{4}{W_{\tilde{K}}}G_m(\mathrm{CM}(K)) = c_m + \lim_{s\to 1}\Big(\frac{C(m, 0)}{2(s-1)} - \mathcal{L}_m\Big)\Big(\frac{2\#\,\mathrm{CM}(K)}{W_{\tilde{K}}} - \Lambda(0, \chi_{\tilde{K}/\tilde{F}})\Big).$$

This implies

$$\frac{4}{W_{\tilde{K}}} G_m(\mathrm{CM}(K)) = c_m, \quad \Lambda(0, \chi_{\tilde{K}/\tilde{F}}) = \frac{2\#\,\mathrm{CM}(K)}{W_{\tilde{K}}}.$$

Plugging this into (4.2) and using the facts (see the proof of Proposition 3.3)

$$\Lambda(s, \chi_{\tilde{K}/\tilde{F}}) = \Lambda(s, \chi_{K/F}), \quad \beta(\tilde{K}/\tilde{F}) = \beta(K/F)$$

and

$$W_K = W_{\tilde{K}} = \begin{cases} 10 & \text{if } K = \mathbb{Q}(\zeta_5), \\ 2 & \text{otherwise,} \end{cases}$$

one obtains the theorem. ■

Proof of Theorem 1.2 Now the proof of Theorem 1.2 is clear. Indeed, one has by (4.1) and Theorem 4.1 $g(\tau) = \phi(\tau) - f(\tau)$. ■

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