On Compact Riemannian Manifolds with Zero Ricci Curvature

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1. In this note I prove the following result:—

Theorem. *A compact, orientable, Riemannian manifold M_n, with positive definite metric and zero Ricci curvature, is flat if the first Betti number R_1 exceeds $n - 4$.*

In this statement of the theorem it is assumed that the dimensions of M_n are not less than four. If this is not the case, the result is still valid but appears as a purely local result and is true for a metric of arbitrary signature.

To prove the theorem I require two lemmas, the first of which is local in character.

Lemma 1. *A Riemannian space V_n which is a flat extension of a V_3 and has zero Ricci curvature is necessarily flat.*

By a flat extension of a V_r we mean a space whose metric can be written in the form

$$ds^2 = \sum_{\alpha, \beta = 1}^{r} g_{\alpha\beta}\, dx^\alpha\, dx^\beta + \sum_{\lambda = r+1}^{n} (dx^\lambda)^2. \tag{1}$$

To prove the lemma we first show that any V_3 of zero Ricci curvature is necessarily flat. This follows from the well-known result [1] that in any V_3 Weyl's conformal curvature tensor C^h_{ijk} vanishes identically, i.e.

$$C^h_{ijk} \equiv R^h{}_{ijk} + \frac{1}{(n-2)}(\delta^h_j R_{ik} - \delta^h_k R_{ij} + g_{ik}R^h_j - g_{ij}R^h_k) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) = 0. \tag{2}$$

It follows immediately that for any V_3, $R_{ij} = 0$ implies $R^h{}_{ijk} = 0$. Now the only non-zero components of the curvature tensor of a space whose metric is of the form (1) arise from the curvature tensor of V_r. It follows that when $r = 3$, the relation $R_{ij} = 0$ implies $R^h{}_{ijk} = 0$ in any space which is a flat extension of a V_3. This proves Lemma 1.

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[1] L. P. Eisenhart, *Riemannian Geometry* (1926), p. 91.

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LEMMA 2. *A compact, orientable, Riemannian manifold M_n with zero Ricci curvature has its first Betti number R_1* equal *to the number of linearly independent parallel vector fields defined over M.*

To prove this lemma we use a result discovered independently by Bochner[1] and Lichnerowicz,[2] which in the notation of the latter author states that if $T_{\alpha_1\alpha_2\cdots\alpha_p}$ denotes the skew-symmetric tensor associated with a harmonic form of degree p, then

$$\frac{1}{2p}\Delta T^2 = \frac{1}{p} g^{\alpha\beta} T^{\alpha_1\alpha_2\cdots\alpha_p,}{}_{\alpha} T_{\alpha_1\alpha_2\cdots\alpha_p\cdot\beta} - R_{\alpha_1\beta_1} T^{\alpha_1}{}_{\beta_2\cdots\beta_p} T^{\beta_1\beta_2\cdots\beta_p} + \tfrac{1}{2}(p-1).\, R_{\alpha_1\beta_1\alpha_2\beta_2}\,.\, T^{\alpha_1\beta_1}{}_{\beta_3\cdots\beta_p} T^{\alpha_2\beta_2\beta_3\cdots\beta_p},$$

where $\quad T^2 = T^{\alpha_1\alpha_2\cdots\alpha_p} T_{\alpha_1\alpha_2\cdots\alpha_p},$

and Δ denotes the generalised Laplacian operator. If we write $p = 1$ and $R_{\alpha_1\beta_1} = 0$ this assumes the simpler form

$$\tfrac{1}{2}\,\Delta\, T^2 = g^{\alpha\beta}\, T^{\alpha_1,}{}_{\alpha} T_{\alpha_1\cdot\beta} \tag{3}$$

where the member on the right is certainly non-negative. We now appeal to Bochner's Lemma[1]

If on a compact orientable manifold we have for a given scalar ψ the relation $\Delta\psi \geqq 0$ everywhere, then $\Delta\psi = 0$ everywhere.

From this it follows that harmonic forms of degree one arise only from parallel vector fields defined over the manifold. This completes the proof of Lemma 2.

The proof of the theorem now readily follows; for the relation $R_1 > n - 4$ implies, by Lemma 2, that the space admits at least $n - 3$ parallel vector fields and is thus a flat extension of a V_3. Appeal to Lemma 1 now completes the proof of the theorem.

2. Another result analogous to Lemma 1 is the following, the proof of which is trivial.

LEMMA 3. *A Riemannian space V_n which is a flat extension of a V_2 and has zero* scalar *curvature is necessarily flat.*

In a recent paper[3] Lichnerowicz has proved several interesting results including the following:—

[1] S. Bochner, *Annals of Math.*, 49 (1948), 379-390.

[2] A. Lichnerowicz, *C. R. Acad. Sci.*, 226 (1948), 1678-80.

[3] A. Lichnerowicz, *C. R. Acad. Sci.*, 230 (1950), 2146-8.

Let K be a compact, orientable manifold with positive definite recurrent metric. Then if the space has zero scalar curvature, it is necessarily symmetric; and if it has zero Ricci curvature it is flat.

Now A. G. Walker has shown[1] that every real recurrent space with positive definite metric is either a V_2, or the flat extension of a V_2. Using this result with Lemma 3 we find not only that these results of Lichnerowicz are purely local in character but also that the weaker assumption of zero scalar curvature implies the stronger conclusion that the space is flat.

3. The question arises to what extent the restrictions in the hypothesis of our theorem can be relaxed. In particular, are the conditions of compactness and positive definite metric sufficient to ensure that a metric with zero Ricci curvature is flat? To disprove this result it would be sufficient to exhibit a compact manifold with the metric

$$ds^2 = x_1^4\,(dx_1^2 + dx_2^2 + dx_3^2) + x_1^{-2}\,dx_4^2,$$

since this metric is positive definite and satisfies the conditions $R_{ij} = 0$, $R^h{}_{ijk} \neq 0$. It is hoped to deal with this question in a later note.

[1] A. G. Walker, *Proc. London Math. Soc.*, 52 (1950), 36-64.

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