A special linear associative algebra

By J. H. M. WEDDERBURN.

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The following algebra possesses certain points of interest and is, I think, worth putting on record; it includes the algebra of matrices as a special case. Consider the algebra H over a ring F defined by

(1) $$h_{pq} h_{rs} = k_{qr} h_{ps}, \qquad k_{qr} < F$$

where $h_{pq} (p, q = 1, 2, \ldots, n)$ are linearly independent over F. If $a = \Sigma a_{ij} h_{ij}$, $b = \Sigma b_{ij} h_{ij}$ are any elements of H, then from (1)

(2) $$ab = \sum_{p,q,r,s} a_{pq} b_{rs} k_{qr} h_{ps} = \Sigma c_{ij} h_{ij}$$
$$c_{ij} = \sum_{qr} a_{iq} k_{qr} b_{rj}.$$

Hence, if we set $A = \| a_{ij} \|$, $B = \| b_{ij} \|$, $K = \| k_{ij} \|$ and consider the isomorphism $a \sim A$, we have $a + b \sim A + B$, $ab \sim AKB$; this shows, as is otherwise obvious, that H is associative.

If K has an inverse in F, we may define a set of elements e_{pq} in H by

(3) $$e_{pq} \sim K^{-1} E_{pq}, \qquad E_{pq} = \| \delta_{ij}^{pq} \|$$

which gives

$$e_{pq} e_{rs} \sim K^{-1} E_{pq} K K^{-1} E_{rs} = K^{-1} E_{pq} E_{rs} = \delta_{qr} K^{-1} E_{ps} \sim \delta_{qr} e_{ps},$$

so that e_{pq} have the law of combination of ordinary matric units. Further the e_{pq} are linearly independent since the E_{pq} are and K is non-singular; hence in this case H is equivalent to the algebra of matrices.

Suppose now that K^{-1} does not exist in F but that the latter is restricted to be a Euclidean domain of integrity, that is, one in which the Euclidean algorism is valid. If we define a new basis for H by

(4) $$h'_{1q} = h_{1q} + \theta h_{2q}, \quad (q = 1, 2, \ldots, n), \quad \theta \text{ in } F$$
$$h'_{pq} = h_{pq}, \quad (p \neq 1),$$

then these elements are linearly independent, and so in fact form a basis, and a short calculation shows that their law of combination is

(5) $$h'_{pq} h'_{rs} = \begin{cases} (k_{q1} + \theta k_{q2}) h'_{ps} & (r = 1) \\ k_{qr} h'_{ps} & (r \neq 1). \end{cases}$$

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Hence the transformation (4) gives rise to the corresponding elementary transformation on the columns of K. Interchanging the rôles of the subscripts clearly gives the same transformation on the rows of K; and similar results apply to permutations of one set of subscripts. Hence, since F is Euclidean, we may assume K to be in its normal form, that is, such that if its rank is r, the first r coefficients in the main diagonal are $k_1, k_2, \ldots, k_r$ with $k_i \mid k_{i+1}$ $(i = 1, 2, \ldots, r-1)$ and all other coefficients are 0. If the rank is n, this form differs from a matric algebra only in this that k_{ss} is not necessarily unity so that the algebra is the matric algebra when F is a field. If $r < n$, the subalgebra H_1: $(h_{pq},\ p, q = 1, 2, \ldots, r)$ is simple and when F is a field is a simple matric algebra; and

$$N: (h_{pq},\ p = r+1, \ldots, n;\ q = 1, 2, \ldots, n \\ \text{or } p = 1, 2, \ldots, r;\ q = r+1, \ldots, n)$$

is the radical since in $h_{pq} h_{ij} = k_{qi} h_{pj}$ we have $k_{qi} = 0$ if $p, q \leqq r$ and $i > r$ while, if $j > r$, then $h_{pj} < N$. The difference algebra $(H - N)$ is, of course, isomorphic with H_1.

It is of some interest to observe that when F is a field and no k_{pq} is zero, which can always be secured by elementary transformations, then the basis defined by

$$f_{pq} = h_{pq}/k_{qp}$$

is composed entirely of idempotent elements. An example of this has already been given.[1]

[1] Question 3700, *Amer. Math. Monthly*, 41 (1934), p. 521 and 43 (1936), p. 378.

PRINCETON UNIVERSITY,
PRINCETON, N.J.