

## ON A $\Pi_1^0$ SET OF POSITIVE MEASURE

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*Dedicated to Professor Katuzi Ono for his 60th birthday anniversary*

**Introduction.** Some basis results for arithmetic, hyperarithmetic (*HA*) or  $\Pi_1^1$  sets which have positive measure (or which are not meager, i.e., of the second Baire category) have been obtained by several authors.<sup>1)</sup> For example, every non-meager  $\Sigma_3^0$  set must have a recursive element (Shoenfield-Hinman, Hinman [2]) but there exists a non-meager  $\Pi_3^0$  set (as well as of measure 1) that contains no recursive element (Shoenfield [7]), and every  $\Sigma_n^0$  set (i.e., arithmetic set) of positive measure contains an arithmetic element (Sacks [5], and Tanaka [12]).<sup>2)</sup> In view of these results, Hinman [2] asked whether a  $\Sigma_3^0$  set of positive measure must contain a recursive element. The main aim of this note is to give a negative answer for this question; thus, *there is a  $\Pi_1^0$  set of positive measure with no recursive element* (§1). In §2, we shall mention some remarks on hierarchy problems.

### §1. Answer for the question.

**LEMMA 1.** *For each positive integer  $k$ , the measure of every Baire's interval of order  $k$  is not greater than  $1/k(k+1)$ .*

*Proof.* Let  $\{a_1, \dots, a_k, \dots\}$  be an arbitrary sequence of positive integers. We define  $p_k = [a_1, \dots, a_k]$  as follows:

$$(1) \quad \begin{cases} p_0 = [\phi] = 1, & p_1 = [a_1] = a_1, \\ p_k = [a_1, \dots, a_k] = [a_1, \dots, a_{k-1}]a_k + [a_1, \dots, a_{k-2}] & (k \geq 2), \\ \quad = p_{k-1}a_k + p_{k-2}. \end{cases}$$

Further, let  $q_0 = 0$  and  $q_k = [a_2, \dots, a_k]$  ( $k \geq 1$ ). Then, by (1), we have

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<sup>1)</sup> In the present paper, sets means subsets of Baire's zero-space  $N^N$ . Measure means the Lebesgue measure, and we shall write  $\mu(E)$  for the measure of a measurable set  $E$ .

<sup>2)</sup> An element of Baire's space is regarded as a 1-place number-theoretic function.

$$(2) \quad q_k = q_{k-1}a_k + q_{k-2} \quad (k \geq 2).$$

It is well-known by elementary number theory that the following equations hold:

$$(3) \quad p_k q_{k-1} - p_{k-1} q_k = (-1)^k \quad (k \geq 1),$$

$$(4) \quad \frac{1}{|a_1|} + \frac{1}{|a_2|} + \dots + \frac{1}{|a_k|} = \frac{q_{k-1}a_k + q_{k-2}}{p_{k-1}a_k + p_{k-2}} \text{ if } k \geq 2.$$

Let  $\delta = \langle a_1, \dots, a_k \rangle$  be an arbitrary Baire's interval of order  $k$ . Then by (3) and (4), we have

$$\begin{aligned} \mu(\delta) &= \left| \left( \frac{1}{|a_1|} + \dots + \frac{1}{|a_{k-1}|} + \frac{1}{|a_k|} \right) - \left( \frac{1}{|a_1|} + \dots + \frac{1}{|a_{k-1}|} + \frac{1}{|a_k + 1|} \right) \right| \\ &= 1 / (p_{k-1}a_k + p_{k-2})(p_{k-1}a_k + p_{k-1} + p_{k-2}). \end{aligned}$$

Since  $p_k \geq k$  for all  $k \geq 1$ ,  $\mu(\delta) \leq \frac{1}{(2k-3)(3k-4)}$  if  $k \geq 2$ . Hence we have

$$(5) \quad \mu(\delta) \leq 1/k(k+1),$$

if  $k \geq 3$ . Obviously (5) holds for  $k = 1$  or  $2$ , too. (Q.E.D.)

In the following, a method by which one can evaluate the outer-measure of a countable set is available.

For each numbers  $p$  and  $e$  we shall define a set  $M_{p,e}$  as follows:

$$\alpha \in M_{p,e} \iff (\forall x)_{x < p+e+1} (\exists y) [T_1(e, x, y) \ \& \ \alpha(x) = U(y)],$$

and let

$$M_p = \bigcup_{e=0}^{\infty} M_{p,e}.$$

For each  $p$  and  $e$ ,  $M_{p,e}$  is either the empty set or a Baire's interval of order  $p + e + 1$ , and  $M_p$  is a  $\Sigma_1^0$  set which contains *all* recursive elements. By Lemma 1, we have

$$\mu(M_p) \leq \sum_{e=0}^{\infty} \mu(M_{p,e}) \leq \sum_{e=0}^{\infty} \frac{1}{(p+e+1)(p+e+2)} = \frac{1}{p+1}.$$

Thus we obtain the

**THEOREM 2.** *There exists a  $\Sigma_1^0$  set  $M (\subset N \times N^N)$  such that each  $M_p = \{ \alpha : \langle p, \alpha \rangle \in M \}$  contains all recursive elements and satisfies the following condition:*

$$\mu(M_p) \leq \frac{1}{p+1} \quad ^3)$$

**COROLLARY 3.** *There exists a  $\Pi_1^0$  set of positive measure that contains no recursive element.<sup>3)</sup>*

This gives a negative answer for Hinman’s problem. By a theorem obtained by Sacks [5] and the author [12] (see Introduction), any set obtained in Corollary 3 must contain an *arithmetic* element.

**COROLLARY 4.** *There exists a  $\Sigma_2^0$  set of measure 1 which has no recursive element.*

It follows from Shoenfield-Hinman’s Theorem [2; p. 1] (see Introduction) that such a set as in Corollary 4 is an *example of arithmetic, meager* (first Baire category) *sets having measure 1.*<sup>4),5)</sup>

**§2. Some remarks.** 1°) Evidently, there is a  $\Sigma_1^0$  set  $E$  of measure 1 such that  $E \not\supseteq \mathfrak{R}$ , where  $\mathfrak{R}$  is the set of all 1-place recursive functions.

2°) Contrasting with Corollary 4, *if  $E$  is a  $\Pi_2^0$  set of measure 1 then  $E$  contains a recursive element.* For, since every  $\Sigma_1^0$  set of measure 1 is an open dense set,  $E$  is co-meager (the complement of a meager set) and hence  $E$  is not meager. By the Shoenfield-Hinman Theorem,  $E$  contains a recursive element.

3°) There is a  $\Pi_1^0$  set consisting of a single element that is not arithmetical. (Spector [10; Corollary 2])

4°) It is known as Kripke-Feferman-Harrison’s Theorem (e.g. Mathias [4; T 3200]) that every countable  $\Sigma_1^1$  set contains only *HA* elements. This can be proved, for example, by the fact that a non-empty  $\Sigma_1^1$  set with no *HA* element is dense-in-itself. The elements of a countable  $\Sigma_1^1$  set are not necessarily enumerated by a *HA* function, as is obvious; but, by the following proposition, *the elements of a countable  $\Delta_1^1$  set can be enumerated by a HA function:*

<sup>3)</sup> N. Tsukada has pointed out that this result can be straightforwardly extended in the case of sets of level  $|a|$  for  $a \in O$ .

<sup>4)</sup> The referee called my attention to this fact.

<sup>5)</sup> Theorem 2, Corollaries 3 and 4 hold true for the case of the space  $2^N$  (instead of  $N^N$ ).

PROPOSITION 5. A countable  $\Sigma_1^1$  set  $E$  can not contain  $HA$  elements of arbitrarily high degrees; that is, there is a  $HA$  function  $\varphi$  such that

$$(\forall \alpha)[\alpha \in E \rightarrow \alpha \leq_T \varphi].$$

*Proof.* By the Kripke-Feferman-Harrison Theorem, we have

$$(\forall \beta) (\exists a) [\beta \in E \rightarrow a \in O \ \& \ \beta \leq_T H_a],$$

where  $\beta \leq_T A$  means that  $\beta$  is Turing reducible to  $A$ , namely  $\beta$  is recursive in  $A$ . Since  $E$  is  $\Sigma_1^1$ , the predicate described in the brackets is  $\Pi_1^1$ . Hence, by Kreisel's Lemma [3; Lemma 1], there exists a  $HA$  functional  $\Psi \in N^{N^N}$  such that

$$(\forall \beta) [\beta \in E \rightarrow \Psi \langle \beta \rangle \in O \ \& \ \beta \leq_T H_{\Psi \langle \beta \rangle}].$$

The set  $\{\Psi \langle \beta \rangle : \beta \in E\}$  is a  $\Sigma_1^1$  subset of  $O$ . Therefore, by a fact known as a direct consequence of Spector [9; Theorem 1], there exists a number  $b \in O$  such that

$$|\Psi \langle \beta \rangle| \leq |b| \quad \text{for all } \beta \in E.$$

Thus we obtain the following implication:

$$\beta \in E \rightarrow \beta \leq_T H_b.$$

This completes the proof.

5°) It is a difficult work that one performs any enumeration of a countable  $CA$  (i.e., co-analytic) subset of  $N^N$ . Now one knows Mansfield-Solovay's Theorem [11; Appendix II], [4; T3206] and [6]: Let  $E$  be a  $\Sigma_2^1$ -in- $\alpha$  set ( $\alpha$  is a code of  $E$ ). If  $E$  has a non constructible-from- $\alpha$  element, then  $E$  contains a perfect subset. By the theorem, we shall try to do this work for a countable  $PCA$  set, thus:

For the sake of simplicity, we shall deal with effective case, i.e., with a countable  $\Sigma_2^1$  set  $E$ , instead of a classical  $PCA$  set. By the above theorem we have

$$(1) \quad E \subset L \cap N^N.$$

Since  $E$  is  $\Sigma_2^1$ , by Shoenfield's Theorem [4; T3101] together with (1)  $E$  is a constructible set. Since  $\alpha \in L \cap N^N \rightarrow \alpha \in F'' \aleph_1^L \subset F'' \aleph_1$ , we have

$$E \in L \text{ \& } E \subset F^{cc} \aleph_1 \text{ \& } \text{Card}(E) = \aleph_0.$$

(L and F are Gödel's.) Hence by [8; p. 317] we have

$$E \in F^{cc} \aleph_1; \text{ i.e., } Od(E) < \aleph_1.$$

Thus we obtain

PROPOSITION 6.<sup>6)</sup> Let  $E$  be a countable  $\Sigma_2^1$  set. Then  $E$  itself is constructible and  $Od(E) < \aleph_1$ .

Let  $\sigma = Od(E)$ . Then  $(\forall \beta) [\beta \in E \rightarrow Or(\beta) < \sigma]$ .<sup>7)</sup> Note that  $Or(\beta) \leq Od(\beta)$ . Let  $\varphi$  be a code for the countable ordinal  $\sigma$ . We shall inductively define  $\alpha$  as follows:

$$\begin{cases} \alpha(0) = (\mu i)_{i \in \omega} (\exists \beta) [\omega \times \omega \cdot F(\varphi_i) = \beta \text{ \& } \beta \in E], \\ \alpha(n + 1) = (\mu i)_{i \in \omega} (\exists \beta) [\omega \times \omega \cdot F(\varphi_i) = \beta \text{ \& } \beta \in E \text{ \& } (\forall k)_{k \leq n} (i \neq \alpha(k))]. \end{cases}$$

Then we can see that  $\alpha$  is  $\mathcal{A}_3^1$ -in- $\varphi$ . Let  $\beta_n = \omega \times \omega \cdot F(\varphi_{\alpha(n)})$ . ( $\beta_n$  is a different notation from  $\varphi_i$ .) Then  $E = \{\beta_0, \beta_1, \beta_2, \dots\}$ . Now, since

$$\beta_n(x) = y \leftrightarrow (\exists \varepsilon) (\exists \beta) [M(\varphi, \varepsilon) \text{ \& } A(\varphi, \varepsilon, \beta, \alpha(n)) \text{ \& } \beta(x) = y],$$

it is  $\Sigma_3^1$ -in- $\varphi$  and hence  $\mathcal{A}_3^1$ -in- $\varphi$ . Consequently,  $E$  can be enumerated by a  $\mathcal{A}_3^1$ -in- $\varphi$  function. We do not know, however, what  $\varphi$  is.

If  $\varphi$  is a constructible function (e.g., if  $\aleph_1^L = \aleph_1$  then it is the case), then

$$(\exists \varphi) [\varphi \in L \cap N^N \text{ \& } W(\varphi) \text{ \& } (\forall \beta) [\beta \in E \rightarrow (\exists i) [Or(\beta) < \varphi_i]].]$$

Hence we can choose a  $\mathcal{A}_3^1$  function  $\varphi$  satisfying the bracketed condition. After all,  $E$  can be enumerated by a  $\mathcal{A}_3^1$  function.

### REFERENCES

- [ 1 ] J.W. Addison, Some consequences of the axiom of constructibility, *Fund. Math.*, **46** (1959), 337–357.
- [ 2 ] P.G. Hinman, Some applications of forcing to hierarchy problems in arithmetic, to appear.
- [ 3 ] G. Kreisel, The axiom of choice and the class of hyperarithmetic functions, *Indag. Math.*, **24** (1962), 308–319.
- [ 4 ] A.R.D. Mathias, A survey of recent results in set theory, Mimeographed note, Stanford University, (1968).

<sup>6)</sup> This proposition is due to Y. Sampei.

<sup>7)</sup> We use freely some results and notations in Addison [1].

- [ 5 ] G.E. Sacks, Measure-theoretic uniformity in recursion theory and set theory, to appear. Summary of results in Bull. Amer. Math. Soc., **73** (1967), 169–174.
- [ 6 ] Y. Sampei, A proof of Mansfield's Theorem by forcing method, to appear.
- [ 7 ] J.R. Shoenfield, The class of recursive functions, Proc. Amer. Math. Soc., **9** (1958), 690–692.
- [ 8 ] ———, Mathematical Logic, Addison-Wesley Company (1967).
- [ 9 ] C. Spector, Recursive well-orderings, J. Symbolic Logic, **20** (1955), 151–163.
- [10] ———, Hyperarithmetical quantifiers, Fund. Math., **48** (1960), 313–320.
- [11] R. Solovay, On the cardinality of  $\sum_1^1$  sets of reals, to appear.
- [12] H. Tanaka, Some results in the effective descriptive set theory, Publ. RIMS, Kyoto Univ., Ser. A, **3** (1967), 11–52.

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