4

Wess–Zumino–Witten model and coset models

The two-dimensional Wess–Zumino–Witten (WZW) model was introduced in the seminal paper of Witten [224]. The model makes use of the WZ term that was introduced by Wess and Zumino in [217]. Sometimes the model is referred to as the WZWN model, where the N stands for Novikov, who independently invoked a similar model [170]. Here we follow only [224].

4.1 From free massless scalar theory to the WZW model

Consider the free massless scalar theory that was described in Section 1.2, but now with $\hat{X}(z,\bar{z})$ being an angle variable defined in the interval $[0,2\pi]$. The action of the scalar field can now be re-written in the following form,

$$S = \int \mathrm{d}^2 x \mathcal{L} = \frac{1}{8\pi}\int \mathrm{d}^2 z \partial_\nu \hat{X}\partial^\nu \hat{X}$$
$$= \frac{1}{8\pi}\int \mathrm{d}^2 z \partial_\nu (\mathrm{e}^{i\hat{X}})\partial^\nu(\mathrm{e}^{-i\hat{X}}) = \frac{1}{4\pi}\int \mathrm{d}^2 z \partial u \bar{\partial} u^{-1}, \tag{4.1}$$

where $u = \mathrm{e}^{i\hat{X}(z,\bar{z})}$ is an abelian group element. Recall that the theory is characterized by a Virasoro algebra and an abelian ALA structure. In terms of this variable the currents J and $\bar{J}$ can be written as,

$$J = -iu^{-1}\partial u = iu\partial u^{-1}, \quad \bar{J} = -iu^{-1}\bar{\partial} u = iu\bar{\partial} u^{-1},$$

with

$$\bar{\partial} J = \partial \bar{J} = 0$$

and

$$T =: (u^{-1}\partial u)^2 :=: (u\partial u^{-1})^2 : .$$

It is now tempting to replace the abelian u with a non-abelian group element,

$$u \in G, \quad G = SO(N) \;\; or \;\; SU(N), \tag{4.2}$$

and consider the action,

$$S_{sm} = \frac{1}{4\pi}\int \mathrm{d}^2 z Tr[\partial u \bar{\partial} u^{-1}], \tag{4.3}$$

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

where the trace is taken in fundamental representation so that,

$$Tr[T^a T^b] = \frac{1}{2}\delta^{ab}.$$

The question here is whether this action admits a similar non-abelian affine Lie algebra and Virasoro algebra. Let us analyze the equations of motion, symmetries and the corresponding currents of this action. The variation of the action under $u \to u + \delta u$ is,

$$\begin{aligned}
\delta S_{sm} &= \frac{1}{4\pi}\int \mathrm{d}^2 z Tr[\partial(\delta u)\bar{\partial}u^{-1} - \partial u\bar{\partial}(u^{-1}\delta u u^{-1})] \\
&= \frac{1}{4\pi}\int \mathrm{d}^2 z Tr\left[u^{-1}\delta u[u^{-1}\bar{\partial}\partial(u) - \partial\bar{\partial}(u^{-1})u]\right] \\
&= \frac{1}{4\pi}\int \mathrm{d}^2 z Tr\left[u^{-1}\delta u \partial^\mu(u^{-1}\partial_\mu u)\right] \\
&= \frac{1}{4\pi}\int \mathrm{d}^2 z Tr\left[\delta u u^{-1}\partial^\mu(u\partial_\mu u^{-1})\right],
\end{aligned} \tag{4.4}$$

were we use $\delta u^{-1} = -u^{-1}\delta u u^{-1}$ and $\partial u^{-1} u = -u^{-1}\partial u$, following from $\delta(u^{-1}u) = 0$ and $\partial(u^{-1}u) = 0$. It is easy to realize that for a constant group element g the action is invariant under,

$$u \to gu \quad (u^{-1} \to u^{-1}g^{-1}), \quad u \to uh \quad (u^{-1} \to h^{-1}u^{-1}), \tag{4.5}$$

and the currents corresponding to the left and right multiplications take the form,

$$J^\mu = \frac{1}{4\pi}u^{-1}\partial^\mu u \quad \tilde{J}^\mu = -\frac{1}{4\pi}\partial^\mu u u^{-1}. \tag{4.6}$$

Both currents are conserved. Note that the conservation of one implies the conservation of the other. However, unlike the massless free scalar theory, now we do not have an ALA structure associated with a separate holomorphic and anti-holomorphic conservation. The latter would have taken the form of $J_L(z)$ corresponding to left transformation of the form $u \to g(z)u$ and $J_R(\bar{z})$, corresponding to right transformation of the form $u \to ug(\bar{z})$. In a similar manner one finds that the energy-momentum tensor,

$$T_{\mu\nu} \sim Tr[J_\mu J_\nu] - 1/2\, g_{\mu\nu} Tr[J^\alpha J_\alpha],$$

and there is only the overall conservation law $\partial_\mu T^{\mu\nu} = 0$, not $\bar{\partial}T = \partial\bar{T} = 0$, namely not an external product of two Virasoro algebras.

Can we modify the action (4.3) so that it does have the desired ALA and Virasoro algebraic structure? For that let us consider first the variation of the action we are looking for. If instead of (4.4) one assumes a variation of the form,

$$\delta S = \frac{1}{4\pi}\int \mathrm{d}^2 z Tr\left[u^{-1}\delta u \partial(u^{-1}\bar{\partial}u)\right] = \frac{1}{4\pi}\int \mathrm{d}^2 z Tr\left[\delta u u^{-1}\bar{\partial}(u\partial u^{-1})\right], \tag{4.7}$$

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

then the global transformations of (4.5) are elevated into

$$u \to g(z)u \qquad u \to uh(\bar{z}), \tag{4.8}$$

with the corresponding currents,

$$J_L \equiv J = \frac{k}{4\pi}\partial u u^{-1} \qquad J_R \equiv \bar{J} = -\frac{k}{4\pi}u^{-1}\bar{\partial}u, \tag{4.9}$$

which have the desired ALA property,

$$\partial\bar{J} = \bar{\partial}J = 0. \tag{4.10}$$

Moreover, it can be shown that for an action whose variation takes the form of (4.7) the energy-momentum takes the form,

$$T \sim Tr[JJ], \quad \bar{T} \sim Tr[\bar{J}\bar{J}], \tag{4.11}$$

and hence it also has the appropriate Virasoro behavior.

The next question is obviously what action has a variation of the form (4.7), and in particular can it be built from S_{sm} plus an additional term that has the standard form of $\tilde{S} = \frac{1}{4\pi}\int \mathrm{d}^2 z\mathcal{L}$. To address this question we rewrite the variation (4.7) in the form,

$$\begin{aligned}\delta S &= \frac{1}{4\pi}\int \mathrm{d}^2 z Tr\left[u^{-1}\delta u\partial^\mu(g_{\mu\nu}+\epsilon_{\mu\nu})(u^{-1}\partial^\nu u)\right]\\ &= \frac{1}{4\pi}\int \mathrm{d}^2 z Tr\left[\delta u u^{-1}\partial^\mu(g_{\mu\nu}-\epsilon_{\mu\nu})(u\partial^\nu u^{-1})\right].\end{aligned} \tag{4.12}$$

Clearly the term, proportional to $g_{\mu\nu}$ in both forms, is exactly the variation δS_{sm}, so that we need to find what action $\tilde{S}$ has a variation that takes the form of the $\epsilon^{\mu\nu}$ term. It may seem that the action,

$$\tilde{S} = \frac{1}{4\pi}\int \mathrm{d}^2 z\epsilon^{\mu\nu}Tr[\partial^\mu u\partial_\nu u^{-1}]$$

does the job, but in fact it vanishes.

It was the proposal of Witten to take for $\tilde{S}$ the so-called WZ action, which for the present case takes the form of a three-dimensional integral over a ball whose boundary is an S^2, which is the two-dimensional space-time,

$$S_{WZ} = \frac{1}{12\pi}\int \mathrm{d}^3\sigma\epsilon^{ijk}Tr[(u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u)], \tag{4.13}$$

where σ_i with $i = 1, 2, 3$ are the coordinates of the ball. Using the fact that $\int \mathrm{d}^3\sigma\epsilon^{ijk}\partial^k(\ldots) = \int \mathrm{d}^2\sigma\epsilon^{ij}(\ldots)$, it is straightforward to show that indeed the variation of (4.13) yields the extra term to change (4.4) to (4.7).

The map u, from the Euclidean space-time that we now take to be S^2 to the group manifold (Fig. 4.1) can be extended into a map from the ball to the group manifold. This is based on the fact that the homotopy group associated with

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

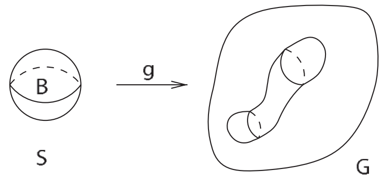

Fig. 4.1. The map between the space-time S^2 and the group manifold.

maps from S^2 to the group space G vanishes, namely, $\pi_2(G) = 0$[1] for any non-abelian group G.

On the other hand since $\pi_3(G) = \mathcal{Z}$, there are topologically inequivalent ways to extend the map u to a map from the ball to the group manifold. This implies that there is an ambiguity in S_{WZ} and it is well defined only modulo $S_{WZ} \to S_{WZ} + 2\pi$. Thus the coefficient of this term must be an integer k, and to have a variation of the form (4.12) it is clear that the sigma term has to have the same coefficient.

Let us now summarize. The classical action of the WZW model is,

$$\begin{aligned} S_{WZW} = &\frac{k}{4\pi}\int \mathrm{d}^2 z Tr[\partial u \bar{\partial} u^{-1}] \\ &+ \frac{k}{12\pi}\int \mathrm{d}^3\sigma \epsilon^{ijk} Tr[(u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u)]. \end{aligned} \tag{4.14}$$

The variation of this action is given by,

$$\delta S = \frac{k}{4\pi}\int \mathrm{d}^2 z Tr\left[u^{-1}\delta u \partial(u^{-1}\bar{\partial} u)\right] = \frac{k}{4\pi}\int \mathrm{d}^2 z Tr\left[\delta u u^{-1}\bar{\partial}(u\partial u^{-1})\right], \tag{4.15}$$

so that the equation of motion takes the form,

$$\partial(u^{-1}\bar{\partial} u) = \bar{\partial}(u\partial u^{-1}) = 0. \tag{4.16}$$

The solutions of these equations of motion take the form,

$$u(z,\bar{z}) = u(z)\bar{u}(\bar{z}), \tag{4.17}$$

where clearly $u \in G, \bar{u} \in G$.

We should state, that the form (4.14), with a term extended to one dimension higher, follows from general properties. Equations of motion that we want, in even space-time dimensions, imply a term in the action with one dimension higher, otherwise the action will involve singular terms, like the introduction of Dirac strings in the case of elementary monopoles.

The symmetries of the action are the ALA transformations,

$$u \to g(z)u \quad u \to uh(\bar{z}), \tag{4.18}$$

[1] $\pi_n(G)$ denotes the group of homotopy classes of maps f: $S^n \to G$

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

and the conformal transformations,

$$z \to f(z) \quad \bar{z} \to \bar{f}(\bar{z}). \tag{4.19}$$

The ALA currents are,

$$J = -\frac{k}{4\pi}\partial u u^{-1} \quad \bar{J} = \frac{k}{4\pi}u^{-1}\bar{\partial}u, \tag{4.20}$$

and the classical energy-momentum tensor takes the form,

$$T = \frac{1}{k}Tr[JJ] \quad \bar{T} = \frac{1}{k}Tr[\bar{J}\bar{J}]. \tag{4.21}$$

4.2 Perturbative conformal invariance

In the following section it will be shown in an exact way, based on algebraic properties, that the WZW model is a CFT. Prior to that we present now a perturbative computation, demonstrating that to a given order indeed the theory has a vanishing β function. Here we restrict ourselves to the one loop order. Of course this only serves as a motivation, as the CFT is at a finite coupling, and so exact demonstration is needed.

The idea is to use the background field method, expanding u around a solution of the equations of motion which we denote by u_0, so $u = u_0 e^{iT^a \pi^a}$. Substituting this ansatz into the action 4.14 one finds,

$$\begin{aligned} S_{WZW} = \frac{k}{4\pi}\int d^2z\{&Tr[\partial u_0 \bar{\partial} u_0^{-1}] + \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a \\ &+ \frac{1}{2}(\eta^{\mu\nu} - \epsilon^{\mu\nu})Tr\{(u_0^{-1}\partial_\mu u_0)[T^a\pi^a, T^b\partial_\nu \pi^b]\} + O(\pi^3)\} \end{aligned} \tag{4.22}$$

The one loop renormalization diagram is shown in Fig. 4.2.

The non vanishing contributions are only when both vertices are proportional to $\eta^{\mu\nu}$ or to $\epsilon^{\mu\nu}$. The two contributions are the same apart from a sign since $\eta^{\rho\mu}\eta_\rho^{\cdot\nu} = \eta^{\mu\nu} = -\epsilon^{\rho\mu}\epsilon_\rho^{\cdot\nu}$, so that the one loop beta function vanishes. Obviously, this result relates to the choice of the coefficient of the sigma model term versus the WZ term, with the latter being fixed by topological arguments. The vanishing of the β function at this stage is an indication that we have chosen the coefficients in a way that is compatible with conformal invariance.

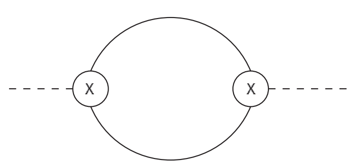

Fig. 4.2. Calculation of the one loop beta function.

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

4.3 ALA, Sugawara construction and the Virasoro algebra

An alternative approach to the quantization methods discussed in the previous section is the ALA and CFT approach. Notice first that under infinitesimal left transformation $\delta u = \epsilon(z)u$ the left current transforms as,

$$\delta_\epsilon J = \frac{k}{4\pi}[\partial(\epsilon u)u^{-1} - \partial u u^{-1}\epsilon] = \frac{k}{4\pi}(\partial\epsilon + [\epsilon, J]), \tag{4.23}$$

which translates into $\delta_\epsilon J^a = \frac{k}{4\pi}[\partial\epsilon^a + i f^a_{bc}\epsilon^b J^c]$. Since the transformation of J is generated by,

$$\delta_\epsilon J^a(w) = \frac{1}{2\pi i}\oint_w \epsilon(z)J(z)J^a(w), \tag{4.24}$$

it is easy to realize that the OPE that is compatible with such a transformation is,

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{i f^{ab}_c J^c}{(z-w)}, \tag{4.25}$$

which is the OPE associated with the ALA discussed in Section 3.2.

Next we want to determine the conformal properties of u, in particular its confomal dimension. The classical form of the currents (4.20) is elevated to the quantum expression via the equations,

$$\kappa\partial u(z,\bar{z}) =: J^a T^a u(z,\bar{z}) :, \quad \kappa\bar{\partial} u(z,\bar{z}) =: \bar{J}^a u(z,\bar{z})T^a :, \tag{4.26}$$

where κ, which is a renormalized level, will be determined shortly, and the normal ordering refers as usual to subtracting the singular parts of the product. Assuming that u is an ALA primary field, the OPE takes the form,

$$J^a(z)T^a u(w,\bar{w}) = \frac{c_2}{z-w}u(w,\bar{w}) + \kappa\partial u(w,\bar{w}) + \sum_{n=2}^{\infty}(z-w)^{n-1}T^a J^a_{-n}u(w,\bar{w}), \tag{4.27}$$

where (4.26) was inserted as the $(z-w)^0$ term, and c_2 is the quadratic Casimir operator in the representation of u, $\sum_a T^a T^a = c_2$. If we assume that u is a Virasoro primary as well, then $L_{-1}u = \partial u$, so that combined with (4.27) one can write down the following null vector,

$$v_{\text{null}} \equiv (J^a_{-1}T^a - \kappa L_{-1})u = 0. \tag{4.28}$$

This is a special case of the degeneracy in the combined ALA–Virasoro algebra discussed in Section 3.3.

The null vectors obey,

$$\begin{aligned} L_0 V_{\text{null}} &= (h+1)V_{\text{null}} \\ J^a_0 V_{\text{null}} &= T^a V_{\text{null}} \\ L_n V_{\text{null}} &= J^a_n V_{\text{null}} = 0 \quad \text{for } n > 0, \end{aligned} \tag{4.29}$$

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

where $L_0 u = hu$. For $n = 1$ the conditions $L_1 V_{\text{null}} = J_1^a V_{\text{null}} = 0$ imply that the renormalized level and the conformal dimension of u take the form,

$$\kappa = \frac{1}{2}(C_2 + k) \quad h = \frac{c_2}{C_2 + k}, \tag{4.30}$$

where C_2 is the quadratic Casimir in the adjoint representation, defined as $f^{acd} f^{bcd} = C_2 \delta^{ab}$.

The use of null vectors and the differential equations that determine correlators of primary fields were introduced in the landmark paper of Knizhnik and Zamolodchikov [143]. An elaboration of the application of these equations appears in [77]. This direction was further developed by Gepner and Witten [108].

The WZW has an in-built Sugawara construction. In fact it is very often taken as the prototype model for this structure. According to the discussion in Section 3.3 the quantum version of the classical energy-momentum tensor (4.21) takes the form of (3.46),

$$T(z) = \frac{1}{2(k + C_2)} : J^a(z) J^a(z) :, \tag{4.31}$$

and the Virasoro anomaly of the model is,

$$c = \frac{k \dim G}{k + C_2}. \tag{4.32}$$

The Sugawara construction is described in [203]. This paper, however, does not have the correct expression that includes the finite renormalization. This was done later in the paper of Dashen and Frishman [73].

4.4 Correlation functions of primary fields

Primary fields of theories invariant under ALA were defined and discussed in Section 3.4. The group element of the WZW theory is an example of a primary field in the fundamental representation of $G \times G$. Indeed the transformation properties of $u(z, \bar{z})$ imply that it has the following OPE with the currents,

$$J^a(z) u(w, \bar{w}) = -\frac{t^a u(w, \bar{w})}{z - w} \quad \bar{J}^a(\bar{z}) u(w, \bar{w}) = -\frac{u(w, \bar{w}) t^a}{\bar{z} - \bar{w}}. \tag{4.33}$$

Next we would like to compute the n-point correlation function of the group element primary field of the WZW model. In Section 3.6 we presented the Knizhnik–Zamolodchikov equation which determines the correlators of theories invariant under ALA. We now demonstrate its use in determining the four-point correlation function of the primary field $u(z, \bar{z})$ of $SU(N)$ WZW model. We denote this correlator as,

$$G_4 = \left\langle u(z_1, \bar{z}_1) u^{-1}(z_2, \bar{z}_2) u^{-1}(z_3, \bar{z}_3) u(z_4, \bar{z}_4) \right\rangle. \tag{4.34}$$

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

Recall from (2.59) that in general due to the conformal Ward identity the four-point function can be written as,

$$G_4 = [z_{14}z_{23}\bar{z}_{14}\bar{z}_{23}]^{-2h}\mathcal{G}(x,\bar{x}), \tag{4.35}$$

where

$$x = \frac{z_{12}z_{34}}{z_{14}z_{32}} \quad \bar{x} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{14}\bar{z}_{32}}, \tag{4.36}$$

with $z_{ij} = z_i - z_j$ and h, the dimension of u, is given by $h = \frac{N^2-1}{2N(N+k)}$.

Now $\mathcal{G}(x,\bar{x})$ can be decomposed into a sum of terms, each one representing a conformal block, the latter having the form of a product of a holomorphic and anti-holomorphic function,

$$\mathcal{G}(x,\bar{x}) = \text{Sum of terms of the form } G(x)\bar{G}(\bar{x}). \tag{4.37}$$

Since $u(z,\bar{z})$ is in the fundamental representation of $SU(N)$, the four-point function is a product of two fundamentals and two anti-fundamentals, so each term in the last equation can be decomposed into,

$$G(x) = I_1G_1 + I_2G_2, \tag{4.38}$$

where I_1 and I_2 are the $SU(N)$ invariant factors,

$$I_1 \equiv \delta_{m_1,m_2}\delta_{m_3,m_4} \quad I_2 \equiv \delta_{m_1,m_3}\delta_{m_2,m_4}. \tag{4.39}$$

If we now substitute this decomposed form of the four-point function into the Knizhnik–Zamolodchikov equation (3.69) we find,

$$\left(\partial_{z_i} + \frac{1}{k+N}\sum_{j\neq i}\frac{t_i^a\otimes t_j^a}{z_i - z_j}\right)[z_{14}z_{23}]^{-4h}(I_1G_1 + I_2G_2) = 0. \tag{4.40}$$

As was discussed in Section 2.9 conformal invariance allows us to fix three out of the four points. Using the standard convenient choice,

$$z_1 = x, \quad z_2 = 0, \quad z_3 = 1, \quad z_4 = \infty, \tag{4.41}$$

and the equation now reads,

$$\left(\partial_x + \frac{1}{k+N}\frac{t_1^a\otimes t_2^a}{x} + \frac{1}{k+N}\frac{t_1^a\otimes t_3^a}{x-1}\right)(I_1G_1 + I_2G_2) = 0, \tag{4.42}$$

After introducing the explicit expressions for the various group theoretical products $t_i^a \otimes t_j^a I_k$ and projecting to the I_1 and I_2 factors we get,

$$\begin{aligned}
\partial_x G_1 &= \frac{-1}{k+N}\left(\frac{(N^2-1)}{N}\frac{G_1}{x} + \frac{G_2}{x} - \frac{1}{N}\frac{G_1}{x-1}\right) \\
\partial_x G_2 &= \frac{-1}{k+N}\left(\frac{(N^2-1)}{N}\frac{G_2}{x-1} + \frac{G_1}{x-1} - \frac{1}{N}\frac{G_2}{x}\right).
\end{aligned} \tag{4.43}$$

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

Extracting G_2 from the first equation and plugging it back into the second equation, the latter translates into a hypergeometric differential equation,

$$\frac{x(1-x)}{N^2}[N^2\kappa^2\partial_x^2 + A(x)\partial_x + B(x)]g_1(x) = 0, \tag{4.44}$$

where $\kappa = k + N$ and with the following two possible values for $A(x)$, $B(x)$ and the relation between g_1 and G_1 as

$$\begin{aligned}
A(x) &= \left(\frac{N(N+\kappa)}{x} - \frac{N^2}{1-x}\right)N\kappa \quad B(x) = -\frac{N^4 - N^2 + 2}{x(1-x)},\\
G_1 &= [x(1-x)]^{\frac{1}{\kappa N}} g_1^+\\
&\quad \textit{or}\\
A(x) &= \left(\frac{-N(N-\kappa)}{x} - \frac{N^2}{1-x}\right)N\kappa \quad B(x) = -\frac{2(N^2-1)}{x(1-x)},\\
G_1 &= x^{-\frac{N^2-1}{\kappa N}}(1-x)^{\frac{1}{\kappa N}} g_1^-
\end{aligned} \tag{4.45}$$

The solutions of the differential equations are the following hypergeometric functions,

$$g_1^- = F\left(\frac{1}{\kappa}, -\frac{1}{\kappa}, 1 - \frac{N}{\kappa}; x\right) \quad g_1^+ = F\left(\frac{N-1}{\kappa}, \frac{N+1}{\kappa}, 1 + \frac{N}{\kappa}; x\right). \tag{4.46}$$

In a similar way the solutions for G_2 are found, defining an appropriate g_2.

To fully determine the correlator we still have to fix the linear combination of the solutions. This is done using crossing symmetry, as discussed in Section 2.10.[2] The latter implies that,

$$\mathcal{G}(x,\bar{x}) = \mathcal{G}(1-x, 1-\bar{x}). \tag{4.47}$$

With parametrization,

$$\mathcal{G}(x,\bar{x}) = \sum_{i,j=1,2} I_i \bar{I}_j \mathcal{G}_{i,j}(x,\bar{x}) \quad \mathcal{G}_{i,j} = \sum_{n,m=+,-} \xi_{mn} G_i^{(m)} G_j^{(n)}. \tag{4.48}$$

Crossing symmetry implies that,

$$\mathcal{G}_{i,j}(x,\bar{x}) = \mathcal{G}_{3-i,3-j}(1-x, 1-\bar{x}), \tag{4.49}$$

which follows from the fact that under crossing symmetry $I_1 \leftrightarrow I_2$. Single valuedness implies that $\xi_{+-} = \xi_{-+} = 0$. To obey the crossing symmetry requirement

[2] Crossing symmetry to determine correlators of fermions in the fundamental representation of $SU(N)$ was used by Dashen and Frishman in [73].

we make use of the following property of hypergeometric functions:

$$F(a,b,c;x) = A_1 F(a,b,a+b-c;1-x) \\ + A_2(1-x)^{c-a-b}F(c-a,c-b,c-a-b+1;1-x), \qquad (4.50)$$

where

$$A_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad A_2 = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \qquad (4.51)$$

Finally we find that

$$\mathcal{G}_{ij} = G_i^{(-)}(x)G_j^{(-)}(\bar{x}) + \frac{c_{--}^2 - 1}{c_{+-}^2} G_i^{(+)}(x)G_j^{(+)}(\bar{x}), \qquad (4.52)$$

where

$$c_{--} = N\frac{\Gamma(\frac{N}{\kappa})\Gamma(-\frac{N}{\kappa})}{\Gamma(\frac{1}{\kappa})\Gamma(-\frac{1}{\kappa})} \quad c_{+-} = -N\frac{[\Gamma(\frac{N}{\kappa})]^2}{\Gamma(\frac{N+1}{\kappa})\Gamma(\frac{N-1}{\kappa})}. \qquad (4.53)$$

For $k=1$ we have $c_{--}=-1$, and hence the second term in (4.52) vanishes. Using $F\left(\frac{1}{N+1}, -\frac{1}{N+1}, \frac{1}{N+1}; x\right) = (1-x)^{\frac{1}{N+1}}$ the four-point function takes the form,

$$\mathcal{G}(x,\bar{x}) = [x\bar{x}(1-x)(1-\bar{x})]^{\frac{1}{N}} \left[I_1\frac{1}{x} + I_2\frac{1}{1-x}\right]\left[\bar{I}_1\frac{1}{x} + \bar{I}_2\frac{1}{1-x}\right]. \qquad (4.54)$$

In Section 6.3 it will be shown that the WZW theory of $U(N)$ at level $k=1$ is a bosonized equivalent to that of N Dirac fermions controlled by the action,

$$S_f = \int d^2z\,[\psi_\alpha^\dagger\bar{\partial}\psi_\alpha + \tilde{\psi}_\beta^\dagger\partial\tilde{\psi}_\beta]. \qquad (4.55)$$

In particular the fermion bilinear is equivalent to the group element as,

$$M\tilde{u}_\alpha^\beta(z,\bar{z}) =: \psi_\alpha(\tilde{\psi}^\dagger)^\beta : \quad M(\tilde{u}^{-1})_\beta^\alpha(z,\bar{z}) =: \tilde{\psi}_\beta{\psi^\dagger}^\alpha :, \qquad (4.56)$$

where M is a mass scale and where $\tilde{u}$ denotes the $U(N)$ group element. In the theory of free Dirac fermions the four-point function of the fermion bilinears takes the form,

$$\mathcal{G}(x,\bar{x}) = \left[I_1\frac{1}{x} + I_2\frac{1}{1-x}\right]\left[\bar{I}_1\frac{1}{x} + \bar{I}_2\frac{1}{1-x}\right]. \qquad (4.57)$$

It is easy to see that by converting (4.54) to a similar correlator of $U(N)$ we find exactly the same answer. This is done as follows: define the $U(N)$ group element to be,

$$\tilde{u}(z,\bar{z}) = e^{i\sqrt{4\pi/N}\gamma\varphi(z,\bar{z})}u(z,\bar{z}). \qquad (4.58)$$

Then the four-point function of group elements of $U(N)$ is,

$$\tilde{\mathcal{G}}(x,\bar{x}) = M^{-2\frac{\gamma^2}{N}}[x\bar{x}(1-x)(1-\bar{x})]^{-\frac{\gamma^2}{N}}\mathcal{G}(x,\bar{x}). \qquad (4.59)$$

For $\gamma = 1$ we observe that indeed the correlator is identical to that of the fermion bilinears. For arbitrary γ the correlator corresponds to that of fermion bilinears of the Thirring model defined by,

$$S = S_f + \frac{\gamma^2 - 1}{2\gamma^2} \int \mathrm{d}^2 z J(z) \bar{J}(\bar{z}). \tag{4.60}$$

This generalized bosonization will also be addressed in Section 6.2.

4.5 WZW models with boundaries – D branes

The WZW model described in Section 4.1 was shown to be based on a map from Σ, a compact two-dimensional manifold, in paticular an S^2, into a group manifold G. Let us now study the case where Σ has boundaries. For concreteness we take it to be the upper half-plane. In the bulk the theory is invariant under the holomorphic and anti-holomorphic ALA (4.18) and there is corresponding holomorphic and anti-holomorphic conservation of the associated currents (4.9). On the boundary the two types of modes mix, the symmetry is reduced to,

$$u \to g(\tau) u g(\tau)^{-1}, \tag{4.61}$$

where τ denotes the coordinate on the boundary, and accordingly there is a relation between J_L and J_R,

$$J(z) = \Omega_{\text{aut}} \bar{J}(\bar{z}) \quad \text{at } z = \bar{z}, \tag{4.62}$$

where Ω_{aut} is an automorphism of the ALA.

The notion of boundary conformal field theory was introduced in [58]. The gluing conditions used for D branes in the WZW model were introduced in [135]. From the many papers that have been written on the subject we have chosen to describe it following [11] and [85].

Let us first address the simplest case of a level k $S\hat{U}(2)$ WZW, for which $\Omega = -1$, and then later discuss the general case. In terms of ∂_t, ∂_x and the adjoint action of G on its Lie algebra,

$$Ad(g)u = gug^{-1}, \tag{4.63}$$

the gluing condition reads,

$$(1 - Ad(u))u^{-1}\partial_t u = (1 + Ad(u))u^{-1}\partial_x u. \tag{4.64}$$

The tangent space to the group G at the point u can be split into $T_u G = T_u^{\perp} G \oplus T_u^{||} G$, where $T_u^{||} G$ consists of vectors tangential to the orbit of Ad through u. On $T_u^{\perp} G$, $(1 - Ad(u)) = 0$ and $(1 + Ad(u)) = 2$, so that $(u^{-1}\partial_x u)^{\perp} = 0$ and the corresponding D branes, namely the submanifolds where the condition (4.62) is obeyed, coincide with the conjugacy classes. In the case that $(1 - Ad(u))$ is

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

invertible, (4.64) can be written as,

$$u^{-1}\partial_t u = \frac{1+Ad(u)}{1-Ad(u)} u^{-1}\partial_x u. \tag{4.65}$$

We can now define a two form on the conjugacy class as,

$$\omega = \frac{k}{8\pi}\left(u^{-1}\mathrm{d}u \frac{1+Ad(u)}{1-Ad(u)} u^{-1}\mathrm{d}u\right). \tag{4.66}$$

Applying an exterior derivative to this form we find,

$$\mathrm{d}\omega = \frac{k}{12\pi} Tr(\mathrm{d}uu^{-1})^3, \tag{4.67}$$

namely, it is not closed. The submanifold $D \subset G$ on which the WZ term is exact, $(WZ) = \mathrm{d}\omega$ defines a D brane in G. There is a further restriction which follows from reasoning similar to that discussed in Section 4.1.

Consider the wave functional $\Psi(u(x))$ on the space of closed loops $u(x)$ in some conjugacy class C. The latter, for the group manifold $SU(2)$, are typically two-spheres so that C can be constructed in two different ways, and hence there is an ambiguity in the phase of the wave functional. It can be shown that the phase can take the values $2\pi j$ with $j = 1, \ldots, k-1$. Thus there are $k-1$ two-dimensional conjugacy classes or D_2 branes for the k level $S\hat{U}(2)$ WZW model. In addition there are two D_0 branes associated with the two points $\pm e$, where e is the identity on the group space.

To address the issue of the conjugacy classes in other groups it is convenient to rewrite the two form (4.66) as,

$$\omega = \frac{k}{8\pi} Tr[\tilde{k}^{-1}\mathrm{d}\tilde{k}h\tilde{k}^{-1}\mathrm{d}\tilde{k}h^{-1}], \tag{4.68}$$

for u that belongs to the conjugacy class,

$$C_h^G = \{u \in G | u = \tilde{k}h\tilde{k}^{-1}\}. \tag{4.69}$$

The WZW action that corresponds to the map from the two manifold with a boundary to the group space can be constructed as follows. Instead of considering the map from Σ that has a boundary, we take it from $\Sigma \cup D$ where D is an auxilliary disc that closes the hole in Σ, having a common boundary with it. The disc is mapped into the conjugacy class allowing for its boundary (4.69). The WZW action is now written as,

$$S = S_{WZW} + \int_D w, \tag{4.70}$$

where S_{WZW} is the ordinary WZW action with a three-dimensional WZ term defined now on a ball whose boundary is $\Sigma \cup D$. It can be explicitly checked that the new WZW action is invariant under (4.18). Similar to the topologically distinct D_2 branes of the k level $S\hat{U}(2)$ WZW model, there are different embeddings of the disc in a conjugacy class in a general group manifold G. This is related to the second homotopy group of the conjugacy class, which in general

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

is non-trivial. The group element k of (4.69) is defined modulo a right multiplication with any element that commutes with h, and the group of such elements is isomorphic to the Cartan torus $\mathcal{T}_G$ generated by the generators in the Cartan subalgebra. Thus the conjugacy classes can be described as $\frac{G}{\mathcal{T}_G}$ and the second homotopy group reads,

$$\Pi_2(C_h^G) = \Pi_1(\mathcal{T}_G) \tag{4.71}$$

For a rank r algebra of G, the D_2 will be characterized by an r-dimensional vector in the coroot lattice of G. Namely, if two embeddings given by khk^{-1} and $k'hk'^{-1}$ then on the boundary of the world sheet they have to be related as,

$$k(\tau)k'(\tau)^{-1} = t(\tau), \tag{4.72}$$

where $t(\tau)$ is an element of the subgroup isomorphic to $\mathcal{T}_G$ which commutes with h. This relation determines a mapping from the boundary to $\mathcal{T}_G$. Since the latter is R^r modulo $2\pi \times$ (coroot lattice), every such mapping belongs to the topological sector parameterized by a vector in the coroot lattice describing the winding of the boundary circle on the torus $\mathcal{T}_G$. This lattice vector determines via (4.71) the element of $\Pi^2(C_h^G)$ corresponding to the union of the two embeddings. For the group element $h \in \mathcal{T}_G$ of the form $h = \mathrm{e}^{i\theta\cdot H}$, where H are the Cartan subalgebra generators, the change in the action resulting from the topological change of the embedding is $\Delta S = k(\theta \cdot s)$, where s is a coroot lattice vector. Consistency of the model then implies the condition,

$$\theta \cdot \alpha \in \frac{2\pi\mathcal{Z}}{k}. \tag{4.73}$$

This generalizes the condition that led to the set of $k-1$ D_2 branes for level k $\hat{SU}(2)$. It implies that θ should be $\frac{2\pi}{k} \times$ (weight lattice vector). Since a point in $\mathcal{T}_G$ is defined modulo $2\pi \times$ (coroot lattice), the allowed conjugacy classes correspond to points in the weight lattice divided by k modulo the coroot lattice. This is also the characterization of the primary fields of the corresponding WZW model.

4.6 G/H coset models

The concept of coset models dates back to [110] or in fact even to as early as [27]. A Lagrangian formulation in terms of a gauged WZW model was introduced for instance in [12]. Here we follow the review of [111].

The WZW models constitute a large class of conformal field theories which are invariant under ALA. An even larger class of CFTs can be constructed by taking the quotient of two WZW models. Consider an ALA $\hat{g}$ at level k and a subalgebra of it $\hat{h}$ at level k_h. We denote the currents associated with the former as J^a and with the latter $J_h^{a_h}$ where $a = 1, \ldots, \dim G$ and $a_h = 1, \ldots, \dim H$. The currents

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

associated with the subalgebra $\hat{h}$ can be expressed as a linear combination of $\hat{g}$ as,

$$J_h^{a_h} = \sum_a m_{a_h\, a} J^a. \tag{4.74}$$

Using the commutator of J^a and the corresponding generator of the Virasoro algebra constructed via the Sugawara construction,

$$[L_m, J_n^a] = -mJ^a_{m+n}, \tag{4.75}$$

it follows that,

$$[L_m, J_h^{a_h}{}_n] = -mJ_h^{a_h}{}_{m+n}. \tag{4.76}$$

It is also obvious that a similar relation holds with L_m^h which is the Virasoro generator built by a Sugawara construction from the currents of $\hat{h}$, namely,

$$[L_m^h, J_h^{a_h}{}_n] = -mJ_h^{a_h}{}_{m+n}. \tag{4.77}$$

Thus it follows that,

$$[L_m - L_m^h, J_h^{a_h}] = 0. \tag{4.78}$$

Since L_m^h is a bilinear of $J_h^{a_h}$ it follows that,

$$[L_m - L_m^h, L_n^h] = 0 \quad \rightarrow \quad [L_m, L_n^h] = [L_m^h, L_n^h]. \tag{4.79}$$

We can now define,

$$L_m^{(g/h)} \equiv L_m - L_m^h. \tag{4.80}$$

The algebra of these coset generators is a Virasoro algebra, as follows from,

$$\begin{aligned}[L_m^{(g/h)}, L_n^{(g/h)}] &= [L_m, L_n] - [L_m^h, L_n^h] \\ &= (m-n)L_{m+n}^{(g/h)} + [c(\hat{g}_k) - c(\hat{h}_{k_h})]\frac{(m^3-m)}{12}\delta_{m+n,0}.\end{aligned} \tag{4.81}$$

Thus we have just found that the Virasoro generators of the coset $L_m^{(g/h)}$ obey a Virasoro algebra with a central charge of

$$c = \frac{k dim g}{k + C_2(g)} - \frac{k_h dim h}{k_h + C_2(h)}. \tag{4.82}$$

A special class of coset models are the *diagonal coset models* $\frac{\hat{g}\oplus\hat{g}}{\hat{g}}$. The generators of the coset in this case are given by $J_h^a = J_{(1)}^a + J_{(2)}^a$, namely the sum of the generators of each copy. It thus follows that the level of the coset must be the sum of the two levels since $[J_{(1)}^a, J_{(2)}^a] = 0$. The coset therefore takes the form,

$$\frac{\hat{g}_{k_1} \oplus \hat{g}_{k_2}}{\hat{g}_{k_1+k_2}},$$

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

and its corresponding central charge is given by,

$$c = \dim g \left(\frac{k_1}{k_1 + C_2(g)} + \frac{k_2}{k_2 + C_2(g)} - \frac{k_1 + k_2}{k_1 + k_2 + C_2(g)} \right). \tag{4.83}$$

Consider the case where $g = SU(2)$ and the coset is,

$$\frac{S\hat{U}(2)_k \oplus S\hat{U}(2)_1}{S\hat{U}(2)_{k+1}}, \tag{4.84}$$

with the central charge,

$$c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)} = 1 - \frac{6}{p(p+1)}, \tag{4.85}$$

where in the last step we introduced $p = k + 2 \geq 3$. This is exactly the central term of the unitary minimal models discussed in Section 2.7. In fact one can show that this is indeed an equivalence in the sense that the characters of the minimal models are the same as those of the coset model.

4.7 G/G coset models

The concept of the G/G model was introduced in [200] and [227]. Our description of the G/G model and in particular its BRST analysis follows [9] and [8].

A special class of the G/H models is the case where $H = G$, namely where we gauge the maximal anomaly-free diagonal group. Using the gauging procedure that will be discussed in Section 9.3.1 the classical action takes the form,

$$S_k(g, A^\mu) = S_k(g) - \frac{k}{2\pi} \int d^2 z Tr(g^{-1}\partial g \bar{A}_{\bar{z}} + g\bar{\partial} g^{-1} A_z - \bar{A}_{\bar{z}} g^{-1} A_z g + A_z \bar{A}_{\bar{z}}). \tag{4.86}$$

Next we introduce the following parameterization of the gauge fields, $A_z = ih^{-1}\partial_z h, \bar{A}_{\bar{z}} = ih^* \partial_z h^{*-1}$ where $h(z) \in G^c$. In Section 15 we will elaborate more about this formulation. The action then reads,

$$S_k(g, A) = S_k(g) - S_k(hh^*). \tag{4.87}$$

The Jacobian of the change of variables introduces a dimension $(1, 0)$ system of anticommuting ghosts χ and ρ in the adjoint representation of the group. The quantum action thus takes the form of,

$$S_k(g, h, \rho, \chi) = S_k(g) - S_{k+2C_G}(hh^*) - i \int d^2 z Tr[\rho\bar{\partial}\bar{\chi} + c.c], \tag{4.88}$$

where c_G is the second Casimir of the adjoint representation.[3] The second term can be viewed as $S_{-(k+2C_G)}(h)$. Since the Hilbert space of the model decomposes into holomorphic and anti-holomorphic sectors we restrict our discussion only to the former.

[3] This was $C_2(g)$ in the previous section; notation has changed according to the literature.

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

There are three sets of holomorphic G transformations which leave (4.88) invariant,

$$\begin{aligned} \delta_J g = i[\epsilon(z), g] \quad \delta_I h = i[\epsilon(z), h] \\ \delta_{J^{(gh)}} \chi^a = i f^a_{bc} \epsilon^b \chi^c \quad \delta_{J^{(gh)}} \rho^a = -i f^a_{bc} \epsilon^b \rho^c, \end{aligned} \tag{4.89}$$

with ϵ in the algebra of G. The corresponding currents J^a, I^a and ${J^{(gh)}}^a = i f^a_{bc} \chi_b \rho_c$ satisfy the G ALA with the levels k,$-(k + 2c_G)$ and $2c_G$, respectively. We now define ${J^{(\text{tot})}}^a$

$${J^{(\text{tot})}}^a = J^a + I^a + {J^{(gh)}}^a = J^a + I^a + i f^a_{bc} \chi_b \rho_c, \tag{4.90}$$

which obeys an affine Lie algebra of level,

$$k^{(\text{tot})} = k - (k + 2c_G) + 2c_G = 0. \tag{4.91}$$

The energy-momentum tensor $T(z)$ is a sum of Sugawara terms of the J^a and I^a currents and the usual contribution of a $(1,0)$ ghost system, namely,

$$T(z) = \frac{1}{k + c_G} : J^a J^a : - \frac{1}{k + c_G} : I^a I^a : + \rho^a \partial \chi^a. \tag{4.92}$$

The corresponding Virasoro central charge vanishes,[4]

$$c^{(\text{tot})} = \frac{k d_G}{k + c_G} - \frac{(k + 2c_G) d_G}{-(k + 2c_G) + c_G} - 2 d_G = 0 \tag{4.93}$$

The symmetry structure of the model is in fact richer. It is easy to realize that there are also two odd conserved currents, a dimension one current which is the BRST current $J^{(\text{BRST})}$ and a dimension two operator G. These holomorphic symmetry generators are given by

$$\begin{aligned} J^{(\text{BRST})} &= \chi_a \left[J^a + I^a + \frac{1}{2} {J^{(gh)}}^a \right], \\ G &= \frac{1}{k + c_G} \rho_a [J^a - I^a]. \end{aligned} \tag{4.94}$$

The reason we denote the dimension one current as a BRST current is that one can express both $T(z)$, ${J^{(\text{tot})}}_a$ and $J^{(\text{BRST})}$ itself in terms of its corresponding charge $Q = \int J^{(\text{BRST})}(z)$ as follows,

$$\begin{aligned} T(z) &= \{Q, G(z)\}, \\ {J^{(\text{tot})}}^a(z) &= \{Q, \rho^a\}, \\ J^{(\text{BRST})} &= \{Q, j^{\#}(z)\}, \end{aligned} \tag{4.95}$$

where $j^{\#}$ is the ghost number current.

The fact that $T(z)$ is BRST exact namely $T(z) = \{Q, G(z)\}$ and that the total Virasoro anomaly vanishes, are indications that the $\frac{G}{G}$ model is a topological quantum field theory. These theories which were found to be very useful in dealing with various issues in physics and mathematics are beyond the scope

[4] d_G is what we called $\dim g$ in the previous section; changed according to the literature.

of this book. We thus do not discuss here the topological quantum field theory aspects of the $\frac{G}{G}$ models.

By construction of the BRST procedure the space of physical states of a $\frac{G}{G}$ model is given by the cohomology of Q. That is to say, a physical state $|\text{phys}\rangle$ has to be closed under Q, namely $Q|\text{phys}\rangle = 0$ and not exact, namely $|\text{phys}\rangle \neq Q|\text{state}\rangle$ where $|\text{state}\rangle$ is any other state. It can be shown that taking the trace over those states one finds the torus partition function of the $\frac{G}{G}$ model which is based on the decomposition into WZW characters, discussed in Section 3.5. The torus partition function can be expressed as

$$\mathcal{Z}_{\frac{G}{G}} = c\tau_2^{-r} \int \mathrm{d}u \mathcal{Z}^g(\tau,u)\mathcal{Z}^{hh^*}(\tau,u)\mathcal{Z}^{gh}(\tau,u), \tag{4.96}$$

where $\mathrm{d}u$ is the measure over the flat gauge connections on the torus and r is the rank of G; $\mathcal{Z}^g(\tau,u)$ is the torus partition function of the G_k WZW model,

$$\mathcal{Z}^g(\tau,u) = (q\bar{q})^{\frac{-c}{24}} \sum_{\lambda_L,\lambda_R} \chi_{k,\lambda_L}(\tau,u)\bar{\chi}_{k,\lambda_R}(\tau,u)N_{\lambda_R,\lambda_L}, \tag{4.97}$$

where $q = \mathrm{e}^{2i\pi\tau}$, λ_R, λ_L denote the G_k highest weights, and for the diagonal invariant $N_{\lambda_R,\lambda_L} = \delta_{\lambda_R,\lambda_L}$. The character can be written as,

$$\chi_{k,\lambda}(\tau,u) = \frac{M_{k,\lambda}(\tau,u)}{M_{0,0}(\tau,u)}, \tag{4.98}$$

with $M_{k,\lambda}$ defined explicitly for the $SU(2)$ case below. $\mathcal{Z}^{hh^*}(\tau,u)$ in (4.96) is the contribution of $h \in \frac{G^c}{G}$ at level $k + 2C_G$ or equivalently $h \in G$ at level $-(k + 2c_G)$. This takes the form $\mathcal{Z}^{hh^*}(\tau,u) \sim |M_{0,0}(\tau,u)|^{-2}$ indicating that $\frac{G^c}{G}$ contains just one conformal block. It is straightforward to calculate $\mathcal{Z}^{gh}(\tau,u)$, the ghost contribution to the partiton function $\mathcal{Z}^{gh}(\tau,u) \sim |M_{0,0}(\tau,u)|^4$. Thus there is a cancellation of the $|M_{0,0}(\tau,u)|$ factors and the resulting character is given by the numerator of the character of the "matter" sector. In the $\frac{G}{G}$ model it is $M_{k,\lambda}$. This cancellation property leads to an index interpretation for $M_{k,\lambda}$. For $G = SU(2)$ it was found that one can express,

$$M_{k,j}(\tau,\theta) = \sum_{l=-\infty}^{\infty} q^{(k+2)(l+\frac{j+\frac{1}{2}}{(k+2)})^2} \sin\left\{\pi\theta\left[(k+2)l + j + \frac{1}{2}\right]\right\}, \tag{4.99}$$

as

$$M_{k,j}(\tau,\theta) = \frac{1}{2i} q^{\frac{(j+\frac{1}{2})^2}{(k+2)}} \mathrm{e}^{i\pi\theta(j+\frac{1}{2})} Tr[(-)^{\hat{G}} q^{\hat{L}_0} \mathrm{e}^{i\pi\theta \hat{J}^0_{(\mathrm{tot})}}], \tag{4.100}$$

where $\theta = Reu$, $\hat{G}$ is the ghost number, $\hat{L}_0$ is the excitation level and $\hat{J}^0_{(\mathrm{tot})}$ is the $J^0_{(\mathrm{tot})}$ eigenvalue of the excitation. Note that $M_{k,j}(\tau,\theta)$ is obtained from the torus $M_{k,j}(\tau,u)$ by restricting to just one angle. This amounts to considering the propagation along a cylinder rather than around the torus. As long as we are interested in the spectrum it is sufficient to consider $M_{k,j}(\tau,\theta)$. This index interpretation enables us to read important information about the physical spectrum

https://doi.org/10.1017/9781009401654.005 Published online by Cambridge University Press

from (4.99). For a positive integer k, $2j = 0, \ldots, k$. Hence there are $k+1$ zero ghost number primary states which correspond to the first term in the q expansion of the different $M_{k,j}$s, i.e. the term corresponding to $l = 0$ with $\hat{L}_0 = \frac{j(j+1)}{k+2}$. On each of these states there is a whole tower of descendant states correponding to the higher terms in the q expansion. For further discussion of the G/G model the reader is referred to [9].