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The gauge technique

The gauge technique goes back a long way [1,2], having been introduced to deal with the Schwinger–Dyson equations of scalar electrodynamics. Its fundamental idea is to find an approximate electron-photon proper vertex expressed in terms of the electron propagator in such a way that the Ward identity is exactly satisfied. These methods can be extended to NAGTs for constructing approximate three- and four-point vertices that are gauge invariant and exactly obey the ghost-free Ward identities of the pinch technique or background-field method. In particular, we will give here several examples of the three-point proper PT vertex approximately but gauge invariantly expressed in terms of the gauge-invariant PT proper self-energy. We do not discuss the four-point vertex, which has been studied elsewhere [3].

Implementing gauge-invariant studies of NAGTs absolutely requires the gauge technique or something like it because, unless the Ward identities are satisfied, gauge invariance is an impossible goal. Outside of perturbation theory, which is not important for us, or exactly solving the Schwinger–Dyson equations, which is not possible for us, there is no other method known for systematically and usefully constructing Green's functions obeying the right Ward identities.

The gauge technique has two related potential drawbacks. The first is that there is no such thing as a unique gauge technique vertex. A gauge technique threepoint vertex Γ_{μ}^{GT} is one that depends only on two-point functions Δ and identically satisfies a Ward identity of the form $k^{\mu}\Gamma_{\mu}^{\text{GT}}(k, p) = \Delta^{-1}(p) - \Delta^{-1}(p-k)$. To any proposed gauge technique vertex, we can always add a term $\widetilde{\Gamma}_{\mu}$ obeying $k^{\mu}\widetilde{\Gamma}_{\mu} \equiv 0$. The second drawback is that the gauge technique vertex is quantitatively but not qualitatively wrong for ultraviolet momenta because it is just the omitted, exactly conserved terms that become as important as the gauge technique terms at large momentum. How then can the gauge technique be justified? The answer is that in a theory with a mass gap,¹ these identically conserved vertex forms, such as $\tilde{\Gamma}_{\mu}$, vanish more rapidly by at least one power of *k* near zero momentum compared to the gauge technique vertex. Both these problems suggest that in gauge theories with a mass gap, which include QED and all the NAGTs of interest to us, the identically conserved terms are unimportant in the infrared and can be dropped. So the gauge technique is meant to be used strictly for small momenta. Fortunately, this is just the region of interest to us, where infrared slavery needs to be cured by nonperturbative phenomena. In practice, we are forced to use it out to momenta that are not small compared to a mass scale but comparable to it, so some quantitative error is inevitable. But we are more interested in the *qualitative* behavior of QCD with its infrared slavery, and in particular, we want to know qualitatively how it is forced to generate a dynamical mass.

Usually, the failure in the ultraviolet can be described as a (partial) failure of the gauge technique to satisfy the renormalization group; for example, the gauge technique vertex may have a renormalization group of standard form, but the beta function coefficients may be wrong. It is possible, in principle at least, to correct these gauge technique errors systematically both in QED [4] and in asymptotically free theories [5], but we will not go into such matters here. For a critique of the gauge technique, see the review [6] by one of the early workers.

The simple QED gauge technique is given first, as a warmup to more complex problems involving spontaneous symmetry breaking and to the ultimate challenge of NAGTs. For both Abelian and non-Abelian gauge theories, there are two basic approaches: via dispersion relations for the propagator, which express the vertex as a spectral integral involving the spectral weight for the propagator, and via purely algebraic methods, expressing the vertex directly in terms of the propagators.

5.1 The original gauge technique for QED

5.1.1 Scalar QED

Consider first the proper vertex $\Gamma_{\mu}(p_1, p_2)$ coupling a photon of momentum q to a charged scalar field. We take all momenta as coming into the vertex and normalize so that the bare vertex is $\Gamma_{\mu}(p_1, p_2) = i(p_1 - p_2)_{\mu}$, with $p_1 + p_2 + q = 0$. This vertex obeys the Ward identity

$$q^{\mu}\Gamma_{\mu}(p_1, p_2) = \Delta^{-1}(p_1) - \Delta^{-1}(p_2).$$
(5.1)

By a mass gap, we mean that there are no massless particles carrying gauge-symmetry charge that appear in the *S*-matrix. For NAGTs, as we already know, generation of a dynamical gluon mass requires longitudinally coupled massless particles akin to Goldstone particles, and such particles are indeed present in the gauge technique vertex. But these particles are absent from the *S*-matrix because they are eaten by the gauge bosons.

Although throughout this book, *proper* Green's functions have been at the forefront for the pinch technique, for the gauge technique, it is sometimes useful to emphasize the improper vertex. Multiplying Γ_{μ} by the two charged-scalar propagators gives the improper vertex or form factor

$$F_{\mu}(p_1, p_2) = \Delta(p_1)\Gamma_{\mu}(p_1, p_2)\Delta(p_2), \qquad (5.2)$$

with a corresponding change in the Ward identity:

$$q^{\mu}F_{\mu}(p, -q - p) = \Delta(p_2) - \Delta(p_1).$$
(5.3)

At tree level, this is satisfied with the usual expressions

$$iF_{\mu}(p, -p-q) = \frac{1}{p_1^2 - m^2}(p_1 - p_2)_{\mu}\frac{1}{p_2^2 - m^2} \qquad \Delta(p) = \frac{i}{p^2 - m^2}.$$
 (5.4)

The Ward identity is true no matter what the charged-particle mass m is, provided that it is the same for both charged lines in the form factor. This is the basis for one useful form of the gauge technique.

The charged-field propagators have a Källen–Lehmann representation:

$$-i\Delta(p) = \int d\sigma \, \frac{\rho(\sigma)}{p^2 - \sigma}.$$
(5.5)

If, in Eq. (5.4), we replace m^2 by σ and integrate with weight function $\rho(\sigma)$, the Ward identity is still satisfied. So we define the *gauge technique improper vertex* as

$$iF_{\mu}^{GT}(p_1, p_2) = \int d\sigma \ \rho(\sigma) \frac{1}{p_1^2 - \sigma} (p_1 - p_2)^{\mu} \frac{1}{p_2^2 - \sigma}.$$
 (5.6)

Clearly, this is not a unique solution because we can add any identically conserved function $[G_{\mu}(p,q), q^{\mu}G_{\mu} \equiv 0]$ to the gauge technique vertex and still solve the Ward identity. Nevertheless, the gauge technique form factor F_{μ}^{GT} is still useful in the region of infrared photon momentum $q_{\mu} \sim 0$, provided that there are no massless charged particles in the *S*-matrix.² The reason, for scalar charged particles, is a simple kinematic one: an identically conserved function without massless poles must vanish at least quadratically in q_{μ} for small q_{μ} . This is proved by exhaustion of a finite number of cases, of which we give only one example:

$$G_{\mu} = (q^2 p_{\mu} - q_{\mu} p \cdot q) H(p, q).$$
(5.7)

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² The only massless charged particles that gauge theories can tolerate are Goldstone-like bosons that get eaten by the gauge particles.

Without some special condition on the theory, there is no reason that H should vanish at $q_{\mu} = 0$. So in the case at hand, corrections to the gauge technique form factor are $\mathcal{O}(q^2)$ at small q.

Equation (5.6) for the gauge technique vertex can easily be transcribed with simple algebra into a form that does not use the Källen–Lehmann representation. We give the result for the proper vertex:

$$\Gamma^{\rm GT}_{\mu}(p, -p-q) = \frac{(2p+q)_{\mu}}{2p \cdot q + q^2} [\Delta^{-1}(p+q) - \Delta^{-1}(p)].$$
(5.8)

Note that there are no singularities at $q_{\mu} = 0$.

The great virtue of the gauge technique, and the reason for its existence, is that one can express an otherwise very complicated three-particle vertex entirely in terms of a propagator, always maintaining exact local gauge invariance. In this way, the Schwinger–Dyson equation for this propagator becomes self-contained.

5.1.2 Fermionic QED

The principles are exactly the same; most of the difference is in notation. In particular, to conform to the usual conventions, we take one momentum to be ingoing and one to be outgoing. There is a proper and an improper vertex, related by fermionic propagators:

$$F_{\mu}(p, p+q) = S(p)\Gamma_{\mu}(p, p+q)S(p+q),$$
(5.9)

and the fermion propagator obeys the Källen-Lehmann representation:

$$S(p) = \int_{-\infty}^{\infty} \mathrm{d}W \, \frac{\rho(W)}{\not p - W}.$$
(5.10)

The Ward identity

$$q^{\mu}F_{\mu}(p, p+q) = S(p) - S(p+q)$$
(5.11)

is solved with the gauge technique form

$$F_{\mu}^{\rm GT}(p, p+q) = \int_{-\infty}^{\infty} \mathrm{d}W \,\rho(W) \frac{1}{\not p - W} \gamma_{\mu} \frac{1}{\not p + \not q - W}.$$
 (5.12)

Remarkably, using this gauge-technique vertex in the Schwinger–Dyson equation linearizes it, as King [4] shows.

In this simple case, it is also straightforward to construct the algebraic version of the gauge technique. Define the electron proper self-energy Σ by

$$S^{-1}(p) = p - M + \Sigma(p).$$
 (5.13)

Then [4] the *proper* gauge technique vertex $\widetilde{\Gamma}_{\mu}^{GT}$ that follows from the spectral form of Eq. (5.12) is

$$\widetilde{\Gamma}_{\mu}^{\rm GT} = \gamma_{\mu} + \frac{1}{p^2 - p'^2} \left[\Sigma(p) (\not p \gamma_{\mu} + \gamma_{\mu} \not p') - (\not p \gamma_{\mu} + \gamma_{\mu} \not p') \Sigma(p') \right].$$
(5.14)

There is no singularity at p = p'. As before, one could add an identically conserved term, such as $i\sigma_{\mu\nu}q^{\nu}/M$, but it is one power of q higher, not two, as in the scalar case, at small momentum compared to the mass.

5.2 Massless longitudinal poles

QED has an exact U(1) gauge symmetry, but it is certainly possible to find gauge technique vertices for gauge theories with dynamically broken gauge symmetry, as a simple $O(2) \times U(1)$ gauge model illustrates [7] (Jackiw and Johnson [8] give an entirely equivalent illustration). There are no scalar fields of any sort in the model, just the fermions and gauge potentials, so the conventional Higgs mechanism cannot apply. Nonetheless, a gauge symmetry can be broken dynamically, with Higgs and associated Goldstone bosons arising as elements of the solution of the Schwinger–Dyson equations of the original model. Only the O(2) symmetry is relevant for us (the U(1) gauge field furnishes a critical attractive force that permits nontrivial symmetry-breaking solutions of the Schwinger–Dyson equations). The fermions form a two-vector in the O(2) space of the form

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{5.15}$$

and they interact with a gauge potential B_{μ} through the interaction $\bar{\psi}\gamma^{\mu}B_{\mu}\tau_{2}\psi$, where τ_{i} are the usual Pauli matrices. The idea is to look for symmetry-breaking solutions of the Schwinger–Dyson equations where the fermion proper self-energy has the form

$$\Sigma(p) = \Sigma_s(p) + \tau_3 \Sigma_v(p). \tag{5.16}$$

Thus, Σ_s preserves the gauge symmetry, and Σ_v violates it. In particular, the symmetry-violating self-energy can split the fermion masses.

The Ward identity for the proper fermion B_{μ} vertex is

$$(p - p')^{\mu} \Gamma_{\mu}(p, p') = S^{-1}(p)\tau_2 - \tau_2 S^{-1}(p').$$
(5.17)

Without the τ_3 term in the fermion self-energy, the vertex, just like the bare vertex, would point solely in the τ_2 direction, and the model would be like two copies of QED. But with the τ_3 term, there must be a term in the vertex behaving like τ_1 . Moreover, this part must be singular at small $q \equiv p - p'$ because at $q_{\mu} = 0$, the

right-hand side of the preceding equation does not vanish. This singularity turns out to be a massless, longitudinally coupled scalar – in other words, a dynamical Goldstone boson. This pole part of the proper vertex must have the following singularity for $q \simeq 0$:

$$\Gamma_{\mu}(p, p') = \frac{2i\tau_1 \Sigma_{\nu}(p)q_{\mu}}{q^2} + \cdots,$$
 (5.18)

as one can readily check; the omitted terms are regular at q = 0. It is this Goldstone boson, now appearing in the completely dressed vertex and not classically, that gives the *B*-field a mass because its pole in the vertex, behaving like $1/q^2$, cancels the usual kinematical factor of q^2 in the *B*-field proper self-energy that would otherwise prevent mass generation in a gauge theory. This cancellation of the pole is what we mean when we say that a Goldstone particle is eaten by a gauge boson.

We will not pursue this model further, except to say that it is essential that the gauge-boson forces acting on the fermions be attractive (which is why there is a second gauge potential). If they are, the Schwinger–Dyson equations indeed have a symmetry-violating fermion self-energy that vanishes at large momentum, fermionic mass splitting, a dynamical Higgs boson, and a nonzero *B*-boson mass, all of whose properties are calculable in terms of the parameters of the original model, whose Lagrangian had none of these effects. However, if the forces are not attractive, the Schwinger–Dyson equations are inconsistent and nonrenormalizable and can only be made consistent³ by introducing *bare* fermion and *B*-boson masses.

For us, the point of considering this model is that much the same properties will turn up in non-Abelian gauge theories (with no matter fields of any sort): gauge-boson mass generation necessarily accompanied by longitudinally coupled massless scalars (really long-range, pure-gauge parts of the gauge potential). And the gauge-boson mass will vanish roughly as $1/q^2$ at large momentum, making the Schwinger–Dyson equations renormalizable and self-consistent.

5.3 The gauge technique for NAGTs

It took many years after the QED gauge technique to develop similar tools for non-Abelian gauge theories. The first construction [9] used a spectral form analogous to the original QED gauge technique, where the spectral integral is the Lehmann representation of the electron propagator. Later, a very general nonspectral construction was given [10] that expressed the gauge technique three-gluon vertex algebraically in terms of the PT proper self-energy. This construction was general

³ In actuality, no Abelian gauge theory is really consistent at asymptotically high energies because of the Landau singularity induced by a positive beta function, but this is not of interest to us.

enough to use at finite temperature or for situations involving dynamical symmetry breaking (which requires other fields in appropriate representations; QCD cannot undergo dynamical symmetry breaking because the quarks are in the fundamental representation [11]).

We repeat here the notation and structure used in Chapter 1 for the PT propagator, both in a covariant R_{ξ} gauge and in the light-cone gauge. In both cases, $\hat{d}(q)$ is the gauge-invariant scalar part of the PT propagator; these two propagators differ only in gauge terms that receive no corrections and play no essential role. The Ward identity for the (inverse) propagator is simply that it is transverse, aside from the irrelevant gauge-fixing terms:

• Covariant gauge

$$i\widehat{\Delta}_{\alpha\beta}(q) = P_{\alpha\beta}(q)\widehat{d}(q) + \xi \frac{q_{\alpha}q_{\beta}}{q^4}$$
(5.19)

$$-i\widehat{\Delta}_{\alpha\beta}^{-1}(q) = P_{\alpha\beta}(q)\widehat{d}(q)^{-1} + \frac{1}{\xi}q_{\alpha}q_{\beta}.$$
(5.20)

Light-cone gauge

$$i\widehat{\Delta}_{\alpha\beta}(q) = Q_{\alpha\beta}\widehat{d}(q) + \eta \frac{q_{\alpha}q_{\beta}}{(n \cdot q)^2}$$
(5.21)

$$-\mathrm{i}\widehat{\Delta}_{\alpha\beta}^{-1}(q) = P_{\alpha\beta}(q)\widehat{d}(q)^{-1} + \frac{1}{\eta}n_{\alpha}n_{\beta}, \qquad (5.22)$$

where the gauge-fixing parameter η (which has dimensions of (mass)²) is set to zero at the end of calculations.

We repeat the definition of $P_{\mu\nu}$ and $Q_{\mu\nu}$ given in Chapter 1:

$$P_{\mu\nu}(q) = g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}$$

$$Q_{\alpha\beta} = g_{\alpha\beta} - \frac{n_{\alpha}q_{\beta} + n_{\beta}q_{\alpha}}{n \cdot q}.$$
(5.23)

In both cases, the PT self-energy is defined by

$$\hat{d}^{-1}(q) = q^2 + i\hat{\Pi}(q).$$
 (5.24)

The general Ward identity for the PT vertex relates the corrections to the free vertex to the proper self-energy and has already been found at the one-loop level in Chapter 1:

$$q_1^{\mu}\widehat{\Lambda}_{\mu\nu\alpha}(q_1, q_2, q_3) = \widehat{\Pi}_{\nu\alpha}(q_2) - \widehat{\Pi}_{\nu\alpha}(q_3), \qquad (5.25)$$

$$q_1^{\mu}\widehat{\Gamma}_{\mu\nu\alpha}(q_1, q_2, q_3) = \widehat{d}^{-1}(q_2)P_{\mu\nu}(q_2) - \widehat{d}^{-1}(q_3)P_{\mu\nu}(q_3).$$
(5.26)

In the light-cone gauge, the right-hand side is really the difference between two inverse propagators:

$$q_1^{\mu} \widehat{\Gamma}_{\mu\nu\alpha}(q_1, q_2, q_3) = \widehat{\Delta}_{\alpha\beta}^{-1}(q_2) - \widehat{\Delta}_{\alpha\beta}^{-1}(q_3).$$
 (5.27)

This is not so in covariant gauges, unless the vertex used is the Γ^{ξ} vertex.

5.3.1 The gauge technique in the light-cone gauge

Just as for scalar charged particles, there is one special case where Eq. (5.27) is an identity, and that is the case of free massive gauge bosons. We generate the massive propagator by keeping only the quadratic terms of the kinetic energy plus the gauged nonlinear sigma (GNLS) model (Eq. (2.7)), and the vertex by keeping only the free cubic vertex plus the cubic term of the GNLS model. For the propagator, we need only replace $\widehat{\Pi}$ by m^2 in Eq. (5.24), and the cubic vertex is

$$\widehat{\Gamma}_{\mu\nu\alpha}^{(m^2)}(q_1, q_2, q_3) = (q_1 - q_2)_{\alpha} g_{\mu\nu} + \frac{m^2}{2} \frac{q_{1\mu} q_{2\nu} (q_1 - q_2)_{\alpha}}{q_1^2 q_2^2} + \text{c.p.}, \quad (5.28)$$

where c.p. stands for *cyclic permutations* (of momenta and indices). This vertex, or the improper form factor \hat{F} defined in Eq. (5.30), identically satisfies its Ward identities for any mass *m*.

If the PT propagator satisfied a Källen–Lehmann representation, we could then proceed to write a spectral integral for the form factor, as in the Abelian gauge theories. It is not surprising that infrared slavery, with its so-called wrong signs, does not permit a positive spectral function, but there still is a dispersion relation of the type of Eq. (5.5). The dispersion relation is just an integral over the massive propagator (we omit writing the η term of Eq. (5.21) because, ultimately, η is set to zero – although this must not be done until the end of any calculation):

$$i\widehat{\Delta}_{\alpha\beta}(q) = \int d\sigma \ \rho(\sigma) Q_{\alpha\beta} \frac{1}{q^2 - \sigma}.$$
(5.29)

Spectral form of the gauge technique The one-dressed-loop version of this selfenergy has a term involving (schematically) an integral over $\Gamma_0 \widehat{\Delta} \widehat{\Delta} \widehat{\Gamma}$ plus a seagull term. Provided that the propagators are transverse and $\widehat{\Gamma}$ satisfies its Ward identity, the output self-energy is transverse (and gauge invariant because the inputs to it will all be gauge invariant). The integrand of the one-dressed-loop self-energy integral involves a partly improper form factor, which we define as

$$\widehat{F}_{\mu\nu\alpha}(q_1, q_2, q_3) = \widehat{\Delta}^{\rho}_{\mu}(q_2)\widehat{\Delta}^{\lambda}_{\nu}(q_3)\widehat{\Gamma}_{\rho\lambda\alpha}(q_2, q_3), \qquad (5.30)$$

with

$$q_1^{\mu} \widehat{F}_{\mu\nu\alpha}(q_1, q_2, q_3) = \widehat{\Delta}_{\nu\sigma}(q_3) - \widehat{\Delta}_{\mu\rho}(q_2).$$
(5.31)

The expression [9]

$$\widehat{F}_{\mu\nu\alpha}^{\rm GT}(q_1, q_2, q_3) = \int d\sigma \ \rho(\sigma) \frac{Q_{\mu}^{\rho}(q_2)}{q_2^2 - \sigma} \,\widehat{\Gamma}_{\rho\lambda\alpha}^{(\sigma)}(q_1, q_2, q_3) \frac{Q_{\nu}^{\lambda}(q_3)}{q_3^2 - \sigma} \tag{5.32}$$

satisfies the Ward identity of Eq. (5.27) for any spectral function $\rho(\sigma)$, that is, for any PT propagator.

Algebraic form of the gauge technique The algebraic form [10] of the gauge technique vertex for NAGTs is considerably more general than the spectral form and can be used not only for QCD-like theories but also for theories with symmetry breaking, at finite temperature, and, in fact, for any physically reasonable circumstances for NAGTs in three or four dimensions. (The three-dimensional version is very useful for the functional Schrödinger equation, and we deal with it in Chapter 6.) We give it here only for the simple circumstance that the gluon PT proper self-energy is diagonal in group space (with group indices assigned in accordance with the momentum argument) and only in d = 4; see [10] for generalizations and references to other circumstances. The gauge technique radiative correction to the free vertex (see Eq. (5.25)) is

$$\widehat{\Lambda}_{\mu\nu\alpha}(q_1, q_2, q_3) = -\frac{q_{1\mu}q_{2\nu}}{2q_1^2 q_2^2} (q_1 - q_2)^{\rho} \widehat{\Pi}_{\rho\alpha}(q_3) - \left[P_{\mu}^{\rho}(q_1)\widehat{\Pi}_{\rho\nu}(q_2) - P_{\nu}^{\rho}(q_2)\widehat{\Pi}_{\rho\mu}(q_1)\right] \frac{q_{3\alpha}}{q_3^2} + \text{c.p.}, \quad (5.33)$$

and this satisfies the Ward identity of Eq. (5.25). (The replacement of $\widehat{\Pi}_{\mu\nu}(q)$ by $m^2 P_{\mu\nu}$ gives – after some algebra – the previous result of Eq. (5.28).) Note the presence of massless longitudinal excitations in this vertex; these decouple if the self-energies vanish at zero momentum.

Both the spectral form and the algebraic form have been used in studies of dynamical gluon mass generation in the light-cone Schwinger–Dyson equations of the pinch technique [9, 10], and a simplified algebraic form has been used in the covariant pinch technique Schwinger–Dyson equations [12]. These equations are detailed in Chapter 6.

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