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On Hausdorff dimension of invariant sets for expanding maps of a circle

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Abstract. Given an orientation preserving C^2 expanding mapping $g: S^1 \to S^1$ of a circle we consider the family of closed invariant sets $K_g(\varepsilon)$ defined as those points whose forward trajectory avoids the interval $(0, \varepsilon)$. We prove that topological entropy of $g|K_g(\varepsilon)$ is a Cantor function of ε. If we consider the map $g(z) = z^q$ then the Hausdorff dimension of the corresponding Cantor set around a parameter ε in the space of parameters is equal to the Hausdorff dimension of $K_g(\varepsilon)$. In § 3 we establish some relationships between the mappings $g\,|\,K_g(\varepsilon)$ and the theory of β-transformations, and in the last section we consider DE-bifurcations related to the sets $K_g(\varepsilon)$.

0. *Introduction*

First we give the following:

Definition 1. Let $g: S^1 \to S^1$ be a C^2 expanding map (i.e. such that there exists $n \geq 1$ for which $|(f^n)'(x)| > 1$ for every $x \in S^1$) which preserves orientation. Let $0 \leq \varepsilon \leq 1$ and let $(0, \varepsilon)$ denote the open interval on S^1 of length ε whose left endpoint is one of the fixed points for g. We choose an orientation and suppose that the whole length of S^1 is equal to 1. Now we define the set

$$K_g(\varepsilon) = \bigcap_{n=0}^{\infty} g^{-n}(S^1 \backslash (0, \varepsilon)).$$

It is easy to see that $K_g(\varepsilon)$ is a closed, invariant set for g, that is, $g(K_g(\varepsilon)) \subset K_g(\varepsilon)$ and furthermore $g(K_g(\varepsilon)) = K_g(\varepsilon)$. However, we remark that the inclusion $g^{-1}(K_g(\varepsilon)) \subset K_g(\varepsilon)$ does not hold except for $K_g(\varepsilon) = \varnothing$ or S^1.

Let $\{K_\lambda\}_{\lambda \in \Lambda}$ be a continuous family of mixing repellers for a real analytic family $\{f_\lambda : S^1 \to S^1\}_{\lambda \in \Lambda}$ of real analytic mappings and let $\{\varphi_\lambda : S^1 \to \mathbb{R}\}_{\lambda \in \Lambda}$ be a real analytic family of real analytic functions. Then as Ruelle [**R**] proved, the pressure function $\Lambda \ni \lambda \to P_{f_\lambda|K_\lambda}(\varphi_\lambda\,|\,K_\lambda)$ is real analytic. For $\varphi_\lambda \equiv 0$ this means that the topological entropy is a real analytic function. Ruelle proved also that Hausdorff dimension of these sets K_λ is real analytic. Our mappings $\{g\,|\,K_g(\varepsilon)\}_{\varepsilon \in [0,1]}$ need not be repellers, in fact for certain $\varepsilon \in [0, 1]$ they are not locally maximal and in this case the topological entropy is no longer analytic. We have:

THEOREM 1. *The function* $[0, 1] \ni \varepsilon \to h_{\text{top}}(g|K_g(\varepsilon)) = h(\varepsilon)$ *is continuous. The set* $C(g)$ *of those parameters which have no neighbourhood on which our function is constant, is homeomorphic to the Cantor set and has Lebesgue measure equal to zero.*

(We shall use also $C_+(g)$, the set of those points which have no right-side neighbourhood on which h is constant.)

Remark. In fact we will show that around any parameter from $[0, 1]\backslash C(g)$ even the sets $K_g(\varepsilon)$ are constant, not only the topological entropy. So the set $C(g)$ can also be defined as the set of parameters without any neighbourhood on which the function $\varepsilon \to K_g(\varepsilon)$ is constant. Moreover it is just the set of points for which $K_g(\varepsilon)$ is not locally maximal.

COROLLARY 1. *If g is of the form $z \to z^q$, $q \geq 2$, then the same holds for the function $\varepsilon \to \mathrm{HD}\,(K_g(\varepsilon))$, where $\mathrm{HD}(X)$ denotes Hausdorff dimension of the set X. Moreover*

$$\mathrm{HD}\,(K_g(\varepsilon)) = h_{\mathrm{top}}(g \mid K_g(\varepsilon))/\log(q).$$

In the special case $g(z) = z^q$ we prove additionally one result about the local metric structure of the Cantor set $C(g)$. First we give:

Definition 2. Let $\varepsilon \in C(g)$. We define the *local Hausdorff dimension* at the point ε as $H(\varepsilon) = \lim_{r \to 0} \mathrm{HD}\,(B(\varepsilon, r) \cap C(g))$. This limit exists because the function $r \to \mathrm{HD}\,(B(\varepsilon, r) \cap C(g))$ is decreasing.

Now we can formulate:

THEOREM 2. *If $g(z) = z^q$, $q > 1$ and $\varepsilon \in C(g)$, then $H(\varepsilon) = \mathrm{HD}\,(K_g(\varepsilon))$.*

Remark. It will follow from the proof that theorem 1 is in fact a theorem about left shift dynamics on the space of one-sided sequences of q symbols. z^q is considered in this theorem just to realise the lexicographical order geometrically.

1. *Pressure, entropy and Hausdorff dimension*

In the proofs of our theorems we shall use the following versions of theorems of McCluskey & Manning and Lai-Sang Young.

THEOREM 3 (**[McC–M]**, see also [**B**$_2$], [**R**]). *Let K be a mixing repeller* ([**R**]) *for a C^2 map $f: S^1 \to S^1$ (i.e. in particular f is expanding on K) and $L \subset K$ be a closed invariant subset for f. Then there is a unique number $0 \leq t \leq 1$ such that $P_{f|L}(-t \log Df \mid L) = 0$. This number is equal to the Hausdorff dimension of L.*

THEOREM 4 ([**Y**]). *If a C^2 expanding map $g: S^1 \to S^1$ preserves an ergodic Borel probability measure μ with Lyapunov exponent χ_μ, then $h_\mu(g) = \mathrm{HD}\,(\mu)\chi_\mu$, where $h_\mu(g)$ is the measure-theoretic entropy of g and $\mathrm{HD}(\mu)$ is the dimension of the measure μ, i.e. $\mathrm{HD}\,(\mu) = \inf\{\mathrm{HD}(X)\colon X \subset S^1, \mu(X) = 1\}$.*

The proofs of these theorems are almost the same as in the original papers. To prove the first theorem it is necessary to know that the map $f \mid K$ has a Markov partition which consists of the intersections of K with some intervals.

From these theorems due to expansiveness we easily obtain the following:

COROLLARY 2. *Assuming the same as in theorem 3 we get additionally that there is an $(f \mid L)$-invariant Borel ergodic measure μ so that $\mathrm{HD}\,(L) = h_\mu(f \mid L)/\chi_\mu$.*

Now we shall consider the sets $K_g(\varepsilon)$ as in theorem 1. We prove the following:

PROPOSITION 1. (i) *The function* $[0,1]\ni\varepsilon\to h(\varepsilon)$ *is left-side continuous.*

(ii) *The function* $[0,1]\ni\varepsilon\to \mathrm{HD}\,(K_g(\varepsilon))$ *is also left-side continuous.*

Proof. For every $\varepsilon\in[0,1]$ consider the function of t given by $\varphi_\varepsilon(t)=P(-t\log Dg\,|\,K_g(\varepsilon))$ on R. This is a decreasing family of functions when ε increases. It is also left-side upper semi-continuous.

Indeed, since g is expansive $h_{(\cdot)}g$ is upper semi-continuous as a function of measure. Thus for every fixed t, if $\varepsilon\nearrow\varepsilon_0$ we have for equilibrium states $\mu_{\varepsilon,t}$ and for μ^*-any weak limit of $\{\mu_{\varepsilon,t}\}$

$$\lim_{\varepsilon\nearrow\varepsilon_0} P(-t\log Dg\,|\,K_g(\varepsilon))=\lim_{\varepsilon\nearrow\varepsilon_0}\Big(h_{\mu_{\varepsilon,t}}(g\,|\,K_g(\varepsilon))+\int -t\log Dg\,|\,K_g(\varepsilon)\,d\mu_{\varepsilon,t}\Big)$$
$$\le h_{\mu^*}(g\,|\,K_g(\varepsilon_0)+\int -t\log Dg\,|\,K_g(\varepsilon)\,d\mu^*$$
$$\le P(-t\log Dg\,|\,K_g(\varepsilon_0)).$$

So the family of functions $\{\varphi_\varepsilon\}$ is left-side continuous in the topology of pointwise convergence. For $t=0$ we get exactly proposition 1 (i). Using theorem 3 we get proposition 1 (ii).

2. *Proofs of theorems*

Before we give the proof of theorem 1 we recall the well-known:

LEMMA 1. *If a sequence* $\{a_n\}_{n=1}^{\infty}$ *with positive elements is given by a recurrence formula* $a_n=c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}$, *where* $k\ge1$ *is a constant integer,* $c_1,\ldots,c_k\ge0$ *are constant coefficients, then the following limit exists:*

$$\lim_{n\to\infty}\frac{a_{n-1}}{a_n},$$

and it is equal to the unique positive root of the equation

$$1=c_1x+c_2x^2+\cdots+c_kx^k. \tag{1}$$

Hint for the proof: Consider the action of the matrix

$$\begin{pmatrix} 0 & 1 & \cdot\cdot & \cdot & \cdot\cdot & 0\\ 0 & 0 & 1 & \cdot\cdot & \cdot & 0\\ \vdots & & & & & \\ 0 & \cdot & \cdot & \cdot\cdot & 0 & 1\\ c_k & c_{k-1} & \cdot & \cdot\cdot & c_2 & c_1\end{pmatrix}.$$

Proof of theorem 1. If $\deg(g)=q\ge2$ then g is topologically conjugate to the map $S^1\to S^1$, $z\to z^q$ and to prove that our function is a Cantor one we can assume $g(z)=z^q$. We define

$$Z=\{\varepsilon\in[0,1]\colon g^n(\varepsilon)\in(0,\varepsilon)\quad\text{for some } n\ge1\}.$$

(We make the convention $0=1$ as points on the circle, $[0,1]$ means the whole circle and $[0,0]=\{0\}$.)

Let $\varepsilon\in Z$. Since g is continuous, there exist numbers $\varepsilon_1<\varepsilon<\varepsilon_2$ such that for every $\theta\in[\varepsilon_1,\varepsilon_2]$, $g^n(\theta)\in(0,\varepsilon_1)$. Now, if $z\notin K_g(\varepsilon_2)$ then there exists a number

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$m\geq 0$ so that $g^m(z)\in(0,\varepsilon_2)$. If moreover $g^m(z)\in(0,\varepsilon_1)$ then $z\notin K_g(\varepsilon_1)$ and if $g^m(z)\in[\varepsilon_1,\varepsilon_2]$ then $g^{m+n}(z)\in(0,\varepsilon_1)$. It means $z\notin K_g(\varepsilon_1)$ too. Hence we obtained $K_g(\varepsilon_2)\subset K_g(\varepsilon_1)\subset K_g(\varepsilon_2)$. In particular $h|[\varepsilon_1,\varepsilon_2]$ is constant. Thus we proved:

LEMMA 2. *At every point of Z the function h is continuous and locally constant.*

It will turn out that $C(g)=S^1\backslash Z$. Now, let

$$Z_1=\{\varepsilon\in(0,1]\colon g^n(\varepsilon)=0\quad\text{for some } n\geq 0\}.$$

By arguments analogous to those in the proof of lemma 2 for every $\varepsilon\in Z_1$ there exists a number $\varepsilon<\varepsilon_1<1$ such that

$$K_g(\varepsilon_1)\subset K_g(\varepsilon)\subset K_g(\varepsilon_1)\cup\bigcup_{n=0}^{\infty}g^{-n}(\varepsilon).$$

Since the set $\bigcup_{n=0}^{\infty}g^{-n}(\varepsilon)$ is countable, it follows that $\mathrm{HD}|[\varepsilon,\varepsilon_1]$ is a constant function. In our case for every Borel ergodic g-invariant probability measure μ on S^1, $\chi_\mu(g)=\log(q)$, i.e. the Lyapunov exponent is independent of the measure. So corollary 2 and theorem 4 imply that, for every $0\leq\theta\leq 1$, $\mathrm{HD}(K_g(\theta))=\sup\{\mathrm{HD}(\mu)$: μ is a Borel ergodic g-invariant probability measure on $K_g(\theta)\}=\sup\{h_\mu/\chi_\mu\colon \mu$ is a Borel $\cdots\}=h_\nu/\log(q)$ where ν is one of the measures with maximal entropy for $g|K_g(\theta)$. This means that

$$\mathrm{HD}(K_g(\theta))=h_{\mathrm{top}}(g|K_g(\theta))/\log q.$$

Therefore the function $h|[\varepsilon,\varepsilon_1]$ is constant and using proposition 1 (i) we get the following:

LEMMA 3. *At every point of Z_1 the function h is continuous and locally right-side constant.*

Let $\Sigma_q^+=\{0,1,\ldots,q-1\}^\infty$ be a metric space with the standard metric $\rho(\{a_n\}_{n=1}^\infty,\{b_n\}_{n=1}^\infty)=q^{-m}$ where $m=\min\{n\colon a_n\neq b_n\}-1$, and let σ denote the shift transformation of Σ_q^+. We define $\varphi\colon\Sigma_q^+\to[0,1]/(0=1)=S^1$ as follows

$$\varphi(\{x_n\}_{n=1}^\infty)=\sum_{n=1}^{\infty}\frac{x_n}{q^n}. \tag{2}$$

This function is continuous and it is at most 2-to-1 at each point of S^1. Moreover the following diagram is commutative:

$$\begin{array}{ccc}\Sigma_q^+ & \xrightarrow{\ \sigma\ } & \Sigma_q^+\\ \varphi\downarrow & & \downarrow\varphi\\ S^1 & \xrightarrow{\ g\ } & S^1\end{array}$$

Thus $h_{\mathrm{top}}(g|K_g(\varepsilon))=h_{\mathrm{top}}(\sigma|\varphi^{-1}(K_g(\varepsilon)))$. The maximal number of elements of a $(1/q,n)$-separated set for $\sigma|\varphi^{-1}(K_g(\varepsilon)))$ is equal to the number $R_n(\varepsilon)$ of all sequences of length n that one can extend to a sequence belonging to $\varphi^{-1}(K_g(\varepsilon)))$. Call the set of such sequences $A_n(\varepsilon)$. Then Bowen's theorem [**B**$_1$] says that

$$h(\varepsilon)=\lim_{n\to\infty}\frac{1}{n}(\log R_n(\varepsilon)) \tag{3}$$

because $1/q$ is an expansive constant for σ.

In view of lemmas 2 and 3 to prove continuity of the function h it is sufficient to consider $\varepsilon \in [0, 1]\backslash(Z \cup Z_1)$. Suppose first $\varepsilon \neq 0$. These conditions imply that $\varphi^{-1}(\varepsilon)$ is exactly one sequence. Hence one can write $\varphi^{-1}(\varepsilon) = \varepsilon_1\varepsilon_2\ldots$. Moreover for every $n \geq 1$

$$\varepsilon_n q^{-1} + \varepsilon_{n+1}q^{-2} + \cdots \geq \varepsilon_1 q^{-1} + \varepsilon_2 q^{-2} + \cdots = \varepsilon. \tag{4}$$

Now we shall calculate the asymptotic behaviour of $R_n(\varepsilon)$. For $k \geq 1$, $n \geq k$ let

$$Q_{k,n}(\varepsilon) = \{\{x_j\}_{j=1}^n \in A_n(\varepsilon) \colon x_1 = \varepsilon_1, \ldots x_k = \varepsilon_k\},$$

and for $i = \varepsilon_k + 1, \ldots, q-1$,

$$Q_{k,n}^i(\varepsilon) = \{\{x_j\}_{j=1}^n \in A_n(\varepsilon) \colon x_1 = \varepsilon_1, \ldots, x_{k-1} = \varepsilon_{k-1}, x_k = i\}.$$

It is easy to see that

$$A_n(\varepsilon) = \left(\bigcup_{j=1}^{k} \bigcup_{i=\varepsilon_j+1}^{q-1} Q_{j,n}^i(\varepsilon)\right) \cup Q_{k,n}(\varepsilon), \tag{5}$$

and all the sets in this union are pairwise disjoint. Moreover observe that the number of elements of the set $Q_{j,n}^i(\varepsilon)$ is equal to $R_{n-j}(\varepsilon)$ because the sequence $\varepsilon_1 \ldots \varepsilon_{j-1} i a_{j+1} \cdots a_{j_n}$ belongs to $A_n(\varepsilon)$ iff $a_{j+1} \cdots a_{j_n} \in A_{n-j}(\varepsilon)$. Therefore

$$\sum_{j=1}^{k} \sum_{i=\varepsilon_j+1}^{q-1} R_{n-j}(\varepsilon) \leq R_n(\varepsilon) \leq \sum_{j=1}^{k} \sum_{i=\varepsilon_j+1}^{q-1} R_{n-j}(\varepsilon) + R_{n-k}(\varepsilon). \tag{6}$$

Let $\alpha_k(\varepsilon)$ and $\beta_k(\varepsilon)$ denote the unique positive roots of the equations

$$1 = \sum_{j=1}^{k} c_j(\varepsilon)x^j, \qquad 1 = \sum_{j=1}^{k} c_j(\varepsilon)x^j + x^k,$$

where $c_j(\varepsilon) = q - j - \varepsilon_j$. Observe that the numbers $c_j(\varepsilon)$ are independent of k. Therefore by lemma 1 and formula (3) we have

$$\log(1/\alpha_k(\varepsilon)) \leq h(\varepsilon) \leq \log(1/\beta_k(\varepsilon)). \tag{7}$$

The sequence $\{\alpha_k(\varepsilon)\}_{k=1}^{\infty}$ is obviously decreasing and $\alpha_k(\varepsilon) \geq \beta_k(\varepsilon)$ for every $k \geq 1$.

Let $\alpha(\varepsilon) = \lim_{k\to\infty} \alpha_k(\varepsilon)$. Since, for $k \geq 1$, $0 \leq c_k(\varepsilon) \leq q-1$ and, for k large enough ($k \geq k_0$), $\alpha_k(\varepsilon) \leq \alpha_{k_0}(\varepsilon) < 1$ the polynomials

$$F_{k,\varepsilon}(x) = \sum_{j=1}^{k} c_j(\varepsilon)x^j \quad \text{and} \quad G_{k,\varepsilon}(x) = F_{k,\varepsilon}(x) + x^k,$$

restricted to the interval $[0, \alpha_{k_0}(\varepsilon)]$ converge uniformly to the common limit $F_\varepsilon(x) = \sum_{j=1}^{\infty} c_j(\varepsilon)x^j$. Thus we obtain

$$F_\varepsilon(\alpha(\varepsilon)) = \lim_{k\to\infty} F_{k,\varepsilon}(\alpha_k(\varepsilon)) = 1$$

and if β is the limit of an arbitrary converging subsequence $\{\beta_{k_m}\}_{m=1}^{\infty}$ of the sequence $\{\beta_k\}_{k=1}^{\infty}$ then

$$F_\varepsilon(\beta) = \lim_{m\to\infty} G_{k_m,\varepsilon}(\beta_{k_m}(\varepsilon)) = 1.$$

Therefore $\beta = \alpha(\varepsilon)$ since obviously $F_\varepsilon - 1$ has exactly one positive root. Thus the limit $\lim_{k\to\infty} \beta_k(\varepsilon)$ exists, is equal to $\alpha(\varepsilon)$ and moreover it is the unique positive root of the following equation

$$1 = F_\varepsilon(x) = \sum_{j=1}^{\infty} c_j(\varepsilon)x^j. \tag{8}$$

Thus by (7)

$$h(\varepsilon) = -\log \alpha(\varepsilon). \tag{9}$$

Now we can prove the following facts about the structure of the set $C(g)$.

LEMMA 4. (i) $C_+(g) = S^1 \backslash (Z \cup Z_1)$.

(ii) $C(g) = \mathrm{cl}\,(C_+(g))$.

(iii) *For every $\varepsilon \in C_+(g)$ there exists a decreasing sequence $\{\varepsilon^{(n)}\}_{n=1}^{\infty}$ of non-periodic points belonging to $C_+(g)$, greater than ε and tending to ε.*

Proof. Lemmas 2 and 3 imply that $C_+(g) \subset S^1 \backslash (Z \cup Z_1)$. For $\varepsilon \in S^1 \backslash (Z \cup Z_1)$ we set

$$b_{n,k} = \varepsilon_1 \cdots \varepsilon_n (q-1)^k,$$

where for every finite sequence a, a^m denotes concatenation of m copies of sequence a, $m = 1, 2, \ldots, \infty$.

Let k_n denote the number of symbols $(q-1)$ following immediately after the sequence $\varepsilon_1 \cdots \varepsilon_n$ and let

$$\varepsilon^{(n)} = \varphi(b_{n,k_n+1} b_{n,k_n+2} \cdots).$$

It is easy to see that for n large enough $\varepsilon^{(n)}$ is a non-periodic point greater than ε. Also it belongs to $S^1 \backslash (Z \cup Z_1)$ and the sequence $\{\varepsilon^{(n)}\}_{n=1}^{\infty}$ is decreasing and tends to ε. Thus, using (8) and (9) we obtain that ε and indeed all $\varepsilon^{(n)}$ belong to $C_+(g)$. This proves (i) and (iii).

Now, let $\varepsilon \in C(g)$ and given $r > 0$ pick $x \in (\varepsilon - r, \varepsilon)$. If $x \in C_+(g)$ then the proof of (ii) is finished. In the other case let y denote the maximal number such that $h \,|\, [x, y] = h(x)$. Thus by definition $y \in C_+(g)$ and $y \le \varepsilon$ since $\varepsilon \in C(g)$. Consequently the proof of (ii) is finished.

We return to the proof of theorem 1. Fix $0 \ne \varepsilon \in S^1 \backslash (Z \cup Z_1) = C_+(g)$ and choose the sequence $\{\varepsilon^{(n)}\}_{n=1}^{\infty}$ as in lemma 4(iii). Since h is decreasing function and by formula (9) the sequence $\{\varepsilon^{(n)}\}_{n=1}^{\infty}$ is also decreasing. Therefore there is $n_0 \ge 1$ so that for $n \ge n_0$, $\alpha(\varepsilon^{(n)}) < \alpha(\varepsilon^{(n_0)}) < 1$. In view of the definitions of $c_j(\cdot)$ and $\varepsilon^{(n)}$, for $n \ge n_0$ and $x \in [0, \alpha_{n_0}(\varepsilon)]$,

$$|F_{\varepsilon^{(n)}}(x) - F_\varepsilon(x)| \le 2 \sum_{j=n}^{\infty} (q-1) \alpha_{n_0}^j(\varepsilon).$$

Thus $F_{\varepsilon^{(n)}}$ converges uniformly to F_ε on the interval $[0, \alpha_{n_0}(\varepsilon)]$. As in the proof of formula (9) it implies that $\lim_{n\to\infty} \alpha(\varepsilon^{(n)}) = \alpha(\varepsilon)$ and consequently

$$\lim_{n\to\infty} h(\varepsilon^{(n)}) = h(\varepsilon).$$

For $\varepsilon = 0$, $K_g(\varepsilon) = S^1$ and hence $h(0) = \log q$. If we consider the sequence $\varepsilon^{(n)} = \varphi(0^n x)$, $n = 1, 2, \ldots$ where x is an arbitrary element from $\varphi^{-1}((0, 1] \backslash (Z \cup Z_1))$, then also $\varepsilon^{(n)} \in [0, 1] \cup (Z \cup Z_1)$. Moreover $\varepsilon^{(n)} \searrow 0$ and now it easily follows from (8) and (9) that $h(\varepsilon^{(n)}) \nearrow h(\varepsilon)$. This completely proves continuity of the function h.

From the properties of h we see that the set $C(g)$ is perfect. It is non-empty because $h(0) = \log q \ne 0 = h(1)$. By lemma 2 and density of Z it has empty interior. Thus this set is homeomorphic to the Cantor set.

To calculate its Lebesgue measure we must go back to the arbitrary C^2 expanding map $g: S^1 \to S^1$. Then there exists (see [K]) a g-invariant Borel ergodic probability measure μ equivalent to the Lebesgue measure λ. Since $g^{-1}(S^1\backslash K_g(\varepsilon))\subset S^1\backslash K_g(\varepsilon)$ and $\lambda(S^1\backslash K_g(\varepsilon))\geq\varepsilon>0$ for $\varepsilon\in(0,1]$, $\mu(S^1\backslash K_g(\varepsilon))=1$. Thus for $\varepsilon\neq 0$, $\lambda(K_g(\varepsilon))=0$. Let $0\neq\varepsilon_n\to 0$. From the definition of Z it is easy to see that

$$C(g)\subset\{0\}\cup\bigcup_{n=1}^{\infty}K_g(\varepsilon_n),$$

and consequently $\lambda(C(g))=0$. This completes the proof of theorem 1.

In the proof of lemma 3 we obtained the formula

$$\mathrm{HD}\,(K_g(\varepsilon))=h(\varepsilon)/\log q \text{ if } g(z)=z^q.$$

This and theorem 1 prove corollary 1.

Observe that the function $\varepsilon\to\mathrm{HD}\,(K_g(\varepsilon))$ is left-side continuous for every C^2 expanding g by proposition 1(ii).

Remark. Continuity of the function h implies precisely the continuity of the family $\{\varphi_\varepsilon(0)\}_{\varepsilon\in[0,1]}$ from the proof of proposition 1. Continuity of $\{\varphi_\varepsilon(t)\}$ and more generally of the pressure of any fixed Hölder continuous function as a function of ε without the assumption that $g(z)=z^q$ is an open question. A positive answer obviously implies the continuity of $\varepsilon\to\mathrm{HD}\,(K_g(\varepsilon))$ for any expanding g.

Proof of theorem 2. Let $\varepsilon\in C(g)$. Since $C(g)\cap B(\varepsilon,r)\subset K_g(\varepsilon-r)$, corollary 1 implies

$$H(\varepsilon)\leq\mathrm{HD}\,(K_g(\varepsilon)). \tag{10}$$

First we suppose that $\varepsilon\in C_+(g)$ and it is a non-periodic point for g. Since for every $r>0$

$$K_g(\varepsilon)=\left(\bigcup_{n=0}^{\infty}g^{-n}([\varepsilon,\varepsilon+r))\right)\cap K_g(\varepsilon)\cup K_g(\varepsilon+r),$$

it follows that

$$\mathrm{HD}\,([\varepsilon,\varepsilon+r)\cap K_g(\varepsilon))=\mathrm{HD}\bigcup_{n=0}^{\infty}g^{-n}([\varepsilon,\varepsilon+r))\cap K_g(\varepsilon)=\mathrm{HD}\,(K_g(\varepsilon)). \tag{11}$$

Denote by $n(r)\geq 1$ the minimal number such that $g^{n(r)}([\varepsilon-r,\varepsilon])\cap(\varepsilon-r,\varepsilon)\neq\emptyset$. Observe that for $i=1,\dots,n(r)-1$, $g^i((\varepsilon-r,\varepsilon])\cap(0,\varepsilon)=\emptyset$. Set $\varepsilon_1=(\varepsilon-r,\varepsilon]\cap g^{-n(r)}(\varepsilon)$. This intersection is non-empty and moreover $\varepsilon_1<\varepsilon$ because $\varepsilon<g^{n(r)}(\varepsilon)$. Therefore

$$(\varepsilon_1,\varepsilon]\cap g^{-n(r)}(K_g(\varepsilon))\subset C(g)\cap B(\varepsilon,r).$$

Thus by (11), $\mathrm{HD}\,(C(g)\cap B(\varepsilon,r))\geq\mathrm{HD}\,(K_g(\varepsilon))$ and consequently

$$H(\varepsilon)\geq\mathrm{HD}\,(K_g(\varepsilon)). \tag{12}$$

Now the proof of theorem 2 follows from inequalities (10), (12), corollary 1 and lemma 4(ii), (iii).

3. *Connections with β-transformations*

In this section inspired by M. Misiurewicz we will give another way to obtain (8) and (9) of § 2 using β-transformations.

Recall that we still work with the map $z \to z^q$ and hence the following diagram is commutative:

$$\begin{array}{ccc} S^1 & \xrightarrow{g} & S^1 \\ {\scriptstyle 1-}\downarrow & & \downarrow{\scriptstyle 1-} \\ S^1 & \xrightarrow{g} & S^1 \end{array}$$

where $1-$ denotes the subtraction from 1 in the additional notation. It immediately implies that the following diagram is also commutative:

$$(*) \qquad \begin{array}{ccc} K_g(\varepsilon) & \xrightarrow{g} & K_g(\varepsilon) \\ {\scriptstyle 1-}\downarrow & & \downarrow{\scriptstyle 1-} \\ 1-K_g(\varepsilon) & \xrightarrow{g} & 1-K_g(\varepsilon) \end{array}$$

Moreover

$$\begin{aligned} 1-K_g(\varepsilon) &= \{z \in S^1 : 0 \le g^n(z) \le 1-\varepsilon \quad \text{for every } n \ge 0\} \\ &= S^1 \setminus \bigcup_{n=0}^{\infty} g^{-n}((1-\varepsilon, 1)). \end{aligned}$$

Let $\varepsilon \in C_+(g)$ (in fact this assumption will be necessary in the proof of proposition 2). There exist two sequences $\{a_i\}_{i=1}^{\infty}$, $\{b_i\}_{i=1}^{\infty}$ such that $S^1 \setminus (1-K_g(\varepsilon)) = \bigcup_{i=1}^{\infty} (a_i, b_i)$ because the set $1-K_g(\varepsilon)$ is closed. If for every $i \ge 1$ we identify the endpoints a_i and b_i in the space $1-K_g(\varepsilon)$, we obtain a space S_ε homeomorphic to the circle. Denote by π the corresponding projection of $1-K_g(\varepsilon)$ onto S_ε.

Observe now that if $(a_i, b_i) \ne (1-\varepsilon, 1)$ (hence $(a_i, b_i) \cap (1-\varepsilon, 1) = \emptyset$) then there is $j \ge 1$ such that $(g(a_i), g(b_i)) = (a_j, b_j)$.

Indeed $g(a_i)$, $g(b_i) \in K_g(\varepsilon)$ and for $x \in (a_i, b_i)$ there exists $n \ge 1$ such that $g^n(x) \in (1-\varepsilon, 1)$, so $g(x) \in 1-K_g(\varepsilon)$. Therefore for every $x \in S_\varepsilon \setminus \{\pi(0) = \pi(\varepsilon)\}$ we can define

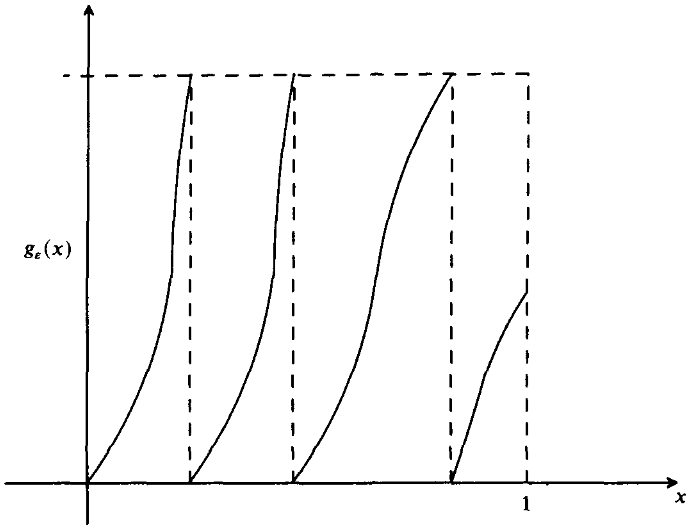
$$\bar{g}(x) = \pi(g(\bar{x})) \qquad \text{where } \bar{x} \in \pi^{-1}(x).$$

Moreover putting $\bar{g}(\pi(0)) = \pi(g(0)) = \pi(0)$ we get the map $\bar{g} : S_\varepsilon \to S_\varepsilon$ continuous at every point except $\pi(0) = \pi(\varepsilon)$ and the following diagram is commutative

$$(**) \qquad \begin{array}{ccc} 1-K_g(\varepsilon) & \xrightarrow{g|1-K_g(\varepsilon)} & 1-K_g(\varepsilon) \\ \downarrow & & \downarrow \\ S_\varepsilon & \xrightarrow{\bar{g}} & S_\varepsilon \end{array}$$

Let I denote the unit interval $[0, 1]$. There exists an orientation preserving continuous function $u : I \to S$ such that $u(0) = u(1) = \pi(0)$ and $u|(0, 1)$ is a homeomorphism onto $S_\varepsilon \setminus \pi(0)$. We define the mapping $g_\varepsilon : I \to I$ putting

$$g_\varepsilon(x) = \begin{cases} u^{-1} \circ \bar{g} \circ u(x), & \text{if } \bar{g} \circ u(x) \ne \pi(0), \\ 0, & \text{if } \bar{g} \circ u(x) = \pi(0),\ x \ne 1, \\ u^{-1} \circ \bar{g}(\varepsilon), & \text{if } x = 1. \end{cases}$$



Hence the following diagram is commutative except at the point 1

$$(***)\qquad \begin{array}{ccc} I & \xrightarrow{g_\varepsilon} & I \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ S_\varepsilon & \xrightarrow{\bar g} & S_\varepsilon \end{array}$$

Now we establish some properties of g_ε. The proof is easy, so we omit it.

PROPOSITION 2. *Let $\varepsilon \in C_+(g)$, $c_0 = 0$ and $c_i = u^{-1} \circ \pi(i/q)$ for $i = 1, \ldots, i(\varepsilon)$ where $i(\varepsilon)/q$ is the greatest pre-image of $0 = 1$ less than $1 - \varepsilon$. Then for every $0 \le i \le i(\varepsilon) - 1$, $g_\varepsilon(c_i) = 0$, $\lim_{x \nearrow c_{i+1}} g_\varepsilon(x) = 1$, $g_\varepsilon|[c_i, c_{i+1})$ is strictly increasing and continuous and $g_\varepsilon|[c_{i(\varepsilon)}, 1)$ is also strictly increasing and continuous.*

Looking at the diagrams (*), (**), (***) we see that up to a countable set the map $u^{-1} \circ \pi \circ 1-$ is well-defined, 1-1, onto, continuous and the following diagram is commutative:

$$\begin{array}{ccc} K_g(\varepsilon) & \xrightarrow{g|K_g(\varepsilon)} & K_g(\varepsilon) \\ {\scriptstyle u^{-1}\circ\pi\circ 1-}\downarrow & & \downarrow{\scriptstyle u^{-1}\circ\pi\circ 1-} \\ I & \xrightarrow[g_\varepsilon]{} & I \end{array}$$

Therefore we have

THEOREM 5. *Denote by $H(K_g(\varepsilon))(H(g_\varepsilon))$ the set of all $g|K_g(\varepsilon)(g_\varepsilon)$-invariant Borel ergodic probability measures with positive entropy. Since such measures have no atoms, if $\mu \in H(g|K_g(\varepsilon))$ then the map*

$$(g|K_g(\varepsilon): K_g(\varepsilon) \to K_g(\varepsilon), \mu) \xrightarrow{u^{-1}\circ\pi\circ 1-} (g_\varepsilon : I \to I, (u^{-1} \circ \pi \circ 1-)_*\mu)$$

is a metric isomorphism and consequently the map

$$H(g\,|\,K_g(\varepsilon)) \ni \mu \to (u^{-1} \circ \pi)_* \mu \in H(g_\varepsilon)$$

is bijective.

COROLLARY 3. $h_{\text{top}}(g\,|\,K_g(\varepsilon)) = \sup_{\mu \in H(g_\varepsilon)} h_\mu(g_\varepsilon)$.

Now we recall a theorem of Parry [**P**].

THEOREM 6. *If a mapping $f: J \to J$ of an interval with or without each endpoint satisfies the following conditions:*

(i) $\bigcup_{i=1}^{s} I_i = J$, *where* I_i, $i = 1, 2, \ldots, s$ *are non-trivial disjoint intervals and* $f\,|\,I_i$ *is continuous and strictly monotone*;

(ii) *for every* $i, j = 1, \ldots, s$, $f(I_i) \cap f(I_j) \neq \varnothing$;

(iii) *f is strongly transitive, i.e. for every non-empty open set U there exists an integer m such that* $\bigcup_{i=0}^{m} f^i(U) = J$;

then f is topologically conjugate to a transformation $T(f): J \to J$ such that for some $\beta > 1$ and $\{\alpha_i\}_{i=1}^{s}$, $T(f)(x) = \alpha_i \pm \beta x$, $x \in I_i$.

In view of proposition 2 the interval $[0, 1)$ is invariant for the map g and we will check that the map $\bar{g}_\varepsilon = g_\varepsilon\,|\,[0, 1)$ satisfies the assumptions of this theorem. (Observe here that we cannot use the function g_ε because in general $g_\varepsilon([0, 1]) \subset [0, 1)$ and thus (iii) of Parry's theorem cannot hold.)

To prove this condition for $\bar{g}_\varepsilon$ we first observe that the mapping $g\,|\,1 - K_g(\varepsilon)$ is strongly transitive because for every point $x \in 1 - K_g(\varepsilon)$ the 'tree' of pre-images $\bigcup_{j=0}^{\infty} (g\,|\,1 - K_g(\varepsilon))^{-j}(x)$ of this point is dense in $1 - K_g(\varepsilon)$ and the space $1 - K_g(\varepsilon)$ is compact.

Since π is a surjection, the commutativity of the diagram (**) implies that the same holds for the mapping $\bar{g}: S_\varepsilon \to S_\varepsilon$. And since the diagram

$$\begin{array}{ccc} [0,1) & \xrightarrow{\bar{g}_\varepsilon} & [0,1) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ S_\varepsilon & \xrightarrow{\bar{g}} & S_\varepsilon \end{array}$$

obtained from (**) is commutative, the map $u\,|\,[0, 1)$ is univalent, $u\,|\,(0, 1)$ is open, the mapping $\bar{g}_\varepsilon: [0, 1) \to [0, 1)$ is also strongly transitive.

Moreover proposition 2 immediately implies that for $\bar{g}_\varepsilon$ the condition (i) holds. It implies also, that for $0 \le i \le i(\varepsilon) - 1$, $\bar{g}_\varepsilon([c_i, c_{i+1})) = [0, 1)$ which gives the condition (ii).

By this proposition we now get that the map $T(\bar{g}_\varepsilon)$ obtained from Parry's theorem must be of the form $T(\bar{g}_\varepsilon)(x) = \beta x (\text{mod } 1)$.

It is well known that $\sup_{H(T(\bar{g}_\varepsilon))} h_\mu T(\bar{g}_\varepsilon) = \log \beta$. From this and theorem 5,

$$h_{\text{top}}(g\,|\,K_g(\varepsilon)) = -\log \beta^{-1}. \tag{13}$$

Now we must find a formula defining β in terms of the code of ε with respect to the partition given by the points $0/q, 1/q, \ldots, q/q$. But first we will do it for the point $1 - \varepsilon$. (Observe that since $\varepsilon \in C_+(g)$, for every $m \ge 1$, $g^m(\varepsilon) \neq i/q$, $i = 0, \ldots, q - 1$ and so the same holds for $1 - \varepsilon$.)

And indeed in view of proposition 2 the code of $1-\varepsilon$ is the same as the code of 1 under $T(\bar{g}_\varepsilon)$ with respect to the partition $[0, 1/\beta), [1/\beta, 2/\beta), \ldots, [([\beta]-1)/\beta, [\beta]/\beta), [[\beta]/\beta, 1)$

Putting, for $x \in [0, 1)$,

$$S(x) = \begin{cases} j & \text{if } x \in [j/\beta, (j+1)/\beta), j \le [\beta]-1, \\ [\beta] & \text{if } x \in [[\beta]/\beta, 1), \end{cases}$$

we have $1 = \sum_{j=0}^{\infty} S(T^j(\bar{g}_\varepsilon)(1))/\beta^{j+1}$. But we observe that $S(T^j(\bar{g}_\varepsilon)(1)) = (1-\varepsilon)_{j+1}$. This implies that $1/\beta$ satisfies the following equation:

$$1 = \sum_{j=1}^{\infty} (1-\varepsilon)_j x^j.$$

We remark now that $(1-\varepsilon)_j = q-1-\varepsilon_j$ and so we get the equation

$$1 = \sum_{j=1}^{\infty} (q-1-\varepsilon_j) x^j.$$

This and (13) give (8) and (9).

P. Walters [$\mathbf{W}_1$] proved that for every β-transformation there exists a unique measure with maximal entropy. Therefore by theorem 5 the same holds for the maps $g \,|\, K_g(\varepsilon)$. Let μ_ε denote this unique measure on $K_g(\varepsilon)$. Using upper-semicontinuity of the function $\mu \to h_\mu(g)$, theorem 1 implies that if $\varepsilon \to \varepsilon_0$ then $\mu_\varepsilon \to \mu_{\varepsilon_0}$ and hence $\chi_{\mu_\varepsilon} \to \chi_{\mu_{\varepsilon_0}}$. Now from the formula HD $(\mu_\varepsilon) = h(\varepsilon)/\chi_{\mu_\varepsilon}$ we get the following:

THEOREM 7. *For every orientation preserving C^2 mapping $g: S^1 \to S^1$ the function $\varepsilon \to \mathrm{HD}\,(\mu_\varepsilon)$ is continuous.*

Remark. Developing the ideas presented in this section we are able to investigate more general invariant subsets than $K_g(\varepsilon)$. Namely let $\mathcal{K}_n$ $(n > 1)$ be the class of all invariant subsets arising by omitting at most n open disjoint intervals. In a later paper we will prove in particular that topological entropy on the subsets in $\mathcal{K}_n$ is continuous. $\mathcal{K}_n$ is considered here as a space with the Hausdorff metric.

4. DE-*perturbations*

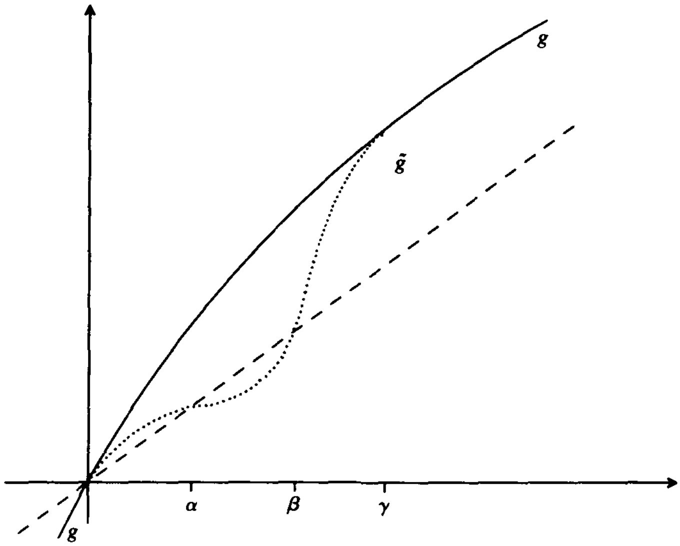
In this section we prove a few results about DE-perturbations and use some facts obtained in the previous sections.

Definition 3. We say that a C^2 mapping $\tilde{g}: S^1 \to S^1$ is a one-sided DE-perturbation obtained from an orientation preserving C^2 expanding map $g: S^1 \to S^1$ if the following conditions are satisfied:

(a) there exist numbers $0 < \alpha < \beta < 1$ such that $\tilde{g}(\alpha) = \alpha$, $\tilde{g}(\beta) = \beta$ and for every $x \in (0, \beta)$ $\lim_{n\to\infty} (\tilde{g}^n(x)) = \alpha$;

(b) there exists a number γ: $\beta < \gamma < 1$ such that $\tilde{g} \,|\, [\gamma, 0] = g \,|\, [\gamma, 0]$ and $D\tilde{g} \,|\, [\beta, 0] > 1$;

(c) the length of $g((\beta, \gamma))$ is less than $1/2$.



For definitions of related perturbations and their properties see for instance [S_1], [S_2], [W_2].

THEOREM 8. *If* DE (g) *denotes the set of all* DE*-perturbations obtained from an orientation preserving* C^2 *expanding mapping* $g: S^1 \to S^1$ *then the function* $\mathrm{DE}(g) \ni \tilde{g} \to \mathrm{HD}(\Omega(\tilde{g}))$, *where* $\Omega(g)$ *is the set of non-wandering points for* $\tilde{g}$, *is continuous if we consider the* C^1 *topology on* DE (g).

Since all the maps from DE (g) are topologically conjugate, the proof of this theorem is similar to the proof of the analogous for horseshoes on surfaces (see [**McC–M**]), use the function $P(-t \log D\tilde{g})$.

THEOREM 9. *With the same assumptions as before, the function* $\mathrm{DE}(g) \ni \tilde{g} \to \mathrm{HD}(\mu(\tilde{g}))$, *where* $\mu(\tilde{g})$ *is the unique g-invariant Borel probability measure with maximal entropy, is continuous.*

Proof. Let $\tilde{g}_n \to \tilde{g}$, $\tilde{g}_n$, $\tilde{g} \in \mathrm{DE}(g)$ and $p_n: S^1 \to S^1$ establishes topological conjugacy between $\tilde{g}_n$ and $\tilde{g}$. Then $p_{n*}(\mu(\tilde{g}))$ is a measure with maximal entropy for $\tilde{g}_n$ and $\lim_{n\to\infty} p_{n*}(\mu(\tilde{g})) = \mu(\tilde{g})$ in the weak topology on measures because $\lim_{n\to\infty} p_n = \mathrm{Id}$ in the C^0 topology. Moreover $\lim_{n\to\infty} \chi_{p_n^*}(\mu(\tilde{g})) = \chi_{\mu(\tilde{g})}$ and the theorem follows from theorem 4.

This theorem is also true if we replace measures with maximal entropy by equilibrium states for an arbitrarily fixed Hölder continuous function on S^1.

PROPOSITION 3. *If* $\tilde{g}_n \to \tilde{g}$ *in the* C^0 *topology in such a way that* $\gamma_n \to 0$ *then* $\liminf_{n\to\infty} \mathrm{HD}(\Omega(\tilde{g}_n)) \geq \mathrm{HD}(\mu)$, *where* μ *is the measure with maximal entropy for g.*

Proof. If by μ_n we denote the measure with maximal entropy for the map $g \mid K_g(\varepsilon_n)$ (see theorem 7) then

$$\mathrm{HD}(\Omega(\tilde{g}_n)) \geq \mathrm{HD}(K_g(\gamma_n)) \geq \mathrm{HD}(\mu_n)$$

because $\Omega(\tilde{g}_n) \supset K_g(\gamma_n)$. Thus using theorem 7 we get

$$\liminf_{n\to\infty} \mathrm{HD}\,(\Omega(\tilde{g}_n)) \ge \mathrm{HD}\,(\mu).$$

THEOREM 10. *If a one parameter, continuous (in the C^0 topology), family $\{g_\lambda\}_{\lambda\in[0,1)}$ of maps belonging to* DE(g) *satisfies the following conditions*:

(a) $\lim_{\lambda\to 1} \beta(\lambda) = \lim_{\lambda\to 1} \gamma(\lambda) \overset{\text{def}}{=} \gamma$ *and for every* $\lambda \in [0, 1)$, $\gamma(\lambda) \le \gamma$;

(b) $\gamma \notin K_g(\gamma)$ *or* $\gamma \in K_g(\gamma)$ *and it is not periodic*;

(c) *there exists a point* $\psi_\lambda \in (\beta(\lambda), \gamma(\lambda))$ *such that* $\lim_{\lambda\to 1} (g_\lambda(\psi_\lambda)) = g(\gamma)$;

(d) $\lim_{\lambda\to 1} \inf \{g'_\lambda(z)\colon z \in [\beta(\lambda), \psi_\lambda] = \infty\}$;

(e) $\alpha(x)$ *is a constant function* α.

Then $\lim_{\lambda\to 1} (\mathrm{HD}\,(\Omega(g_\lambda)) = \mathrm{HD}\,(K_g(\gamma))$.

Proof. Let $\Omega_\lambda = \Omega(g_\lambda)\backslash\{\alpha\}$. Since $\Omega_\lambda \subset [\beta(\lambda), 0]$ is a g_λ-invariant compact set, condition (b) from definition 3 implies that Ω_λ is a mixing repeller for g_λ. Thus by corollary 2 there exists a g_λ-invariant Borel ergodic probability measure μ_λ on Ω_λ such that $\mathrm{HD}(\Omega_\lambda) = h_{\mu_\lambda}/\chi_{\mu_\lambda}$. We choose a sequence $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \mu_{\lambda_n} = \mu$ in the weak topology. Moreover we can require that $\limsup_{\lambda\to 1} \mathrm{HD}\,(\Omega_\lambda) = \lim_{n\to\infty} \mathrm{HD}\,(\Omega_{\lambda_n})$.

Now we shall prove that if $\mathrm{HD}\,(K_g(\gamma)) > 0$ then

$$\lim_{n\to\infty} \mu_{\lambda_n}([\beta(\lambda_n), \psi_{\lambda_n}]) = 0. \tag{14}$$

Indeed, suppose to the contrary that for some increasing subsequence $\{n_k\}_{k=1}^{\infty}$ and $\delta > 0$, $\mu_k([\beta(\tau_k), \psi_{\tau_k}]) \ge \delta$, where $\tau_k \overset{\text{def}}{=} \lambda_{n_k}$ and $\mu_k \overset{\text{def}}{=} \mu_{\tau_k}$. Now, condition (d) implies that for $k = 1, \ldots$

$$\chi_{\mu_k} \ge \delta \inf \{g'_{\tau_k}(z)\colon z \in [\beta(\tau_k), \psi_{\tau_k}] \to \infty \qquad \text{as } k \to \infty,$$

and from this

$$\lim_{k\to\infty} \mathrm{HD}\,(\Omega_{\tau_k}) = \lim_{k\to\infty} h_{\mu_k}/\chi_{\mu_k} = 0,$$

since, as is easy to see,

$$h_{\mu_k} < h_{\mathrm{top}}(g_{\tau_k}|\Omega_{\tau_k}) = \log\,(\deg g).$$

But this is impossible because $\Omega_{\tau_k} \supset K_g(\gamma)$ by (a).

These considerations also show that if $\mathrm{HD}\,(K_g(\gamma)) = 0$ and equality (14) does not hold then $\limsup_{\lambda\to 1} \mathrm{HD}\,(\Omega_\lambda) = 0$ and in this case the theorem is proved. Thus we can assume that equality (14) is satisfied. We have also

$$\lim_{n\to\infty} \mu_{\lambda_n}([\psi_{\lambda_n}, \gamma]) = 0, \tag{15}$$

because conditions (a), (b) and (c) imply for every $k \ge 1$ that for n large enough the sets

$$[\psi_{\lambda_n}, \gamma], g_{\lambda_n}^{-1}([\psi_{\lambda_n}, \gamma]), \ldots, g_{\lambda_n}^{-k}([\psi_{\lambda_n}, \gamma])$$

are pairwise disjoint.

Now we can prove that μ is a g-invariant measure. Indeed, let $\varphi: S^1 \to R$ be an arbitrary continuous function. We have:

$$\begin{aligned}
\int_{S^1} \varphi\, d\mu &= \lim_{n\to\infty} \int_{S^1} \varphi\, d\mu_{\lambda_n} = \lim_{n\to\infty} \int_{S^1} \varphi \circ g_{\lambda_n}\, d\mu_{\lambda_n} \\
&= \lim_{n\to\infty} \left(\int_{[\beta(\lambda_n),\gamma)} \varphi \circ g_{\lambda_n}\, d\mu_{\lambda_n} + \int_{[\gamma,0]} \varphi \circ g_{\lambda_n}\, d\mu_{\lambda_n} \right) \\
&= \lim_{n\to\infty} \left(\int_{[\beta(\lambda_n),\gamma)} \varphi \circ g_{\lambda_n}\, d\mu_{\lambda_n} + \int_{[\gamma,0]} \varphi \circ g\, d\mu_{\lambda_n} \right) \\
&= \lim_{n\to\infty} \int_{[\beta(\lambda_n),\gamma)} (\varphi \circ g_{\lambda_n} - \varphi \circ g)\, d\mu_{\lambda_n} + \lim_{n\to\infty} \int_{[\beta(\lambda_n),0]} \varphi \circ g\, d\mu_{\lambda_n} \\
&= \lim_{n\to\infty} \int_{[\beta(\lambda_n),\gamma)} (\varphi \circ g_{\lambda_n} - \varphi \circ g)\, d\mu_{\lambda_n} + \lim_{n\to\infty} \int_{S^1} \varphi \circ g\, d\mu_{\lambda_n}.
\end{aligned}$$

Since φ is a bounded function, the first term of the last expression converges to zero by (14) and (15) and the second one converges to $\int_{S^1} \varphi \circ g\, d\mu$. This means that μ is a g-invariant measure. Equalities (14) and (15) prove also that supp $(\mu) \subset [\gamma, 0]$ and thus supp $(\mu) \subset K_g(\gamma)$.

By changing a bit the classical proof that the function $\mu \to h_\mu(f)$ is upper semi-continuous for an expansive mapping f we can show that

$$h_\mu(g) \geq \limsup_{n\to\infty} h_{\mu_n}(g_{\lambda_n}) \qquad \text{where we put } \mu_n = \mu_{\lambda_n}.$$

Therefore, by theorem 4,

$$\text{HD}\,(K_g(\gamma)) \geq h_\mu(g)/\chi_\mu(g) \geq \limsup_{n\to\infty} h_{\mu_n}/\chi_{\mu_n} = \lim_{n\to\infty} \text{HD}\,(\Omega_{\lambda_n}) = \limsup_{\lambda\to 1} \text{HD}\,(\Omega_\lambda).$$

This and the inequality $\liminf_{\lambda\to 1} \text{HD}\,(\Omega_\lambda) \geq \text{HD}\,(K_g(\gamma))$ which holds because for every $\lambda \in [0, 1)$, $\Omega_\lambda \subset K_g(\gamma)$, prove our theorem.

This theorem permits us to estimate the Hausdorff dimension of the sets $\Omega(g)$ by HD $(K_g(\gamma))$, and as was shown in the proof of theorem 1 this number for the map $g(z) = z^q$ is given by an actual formula.

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