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A GENERALIZATION OF LYNDON'S THEOREM ON THE COHOMOLOGY OF ONE-RELATOR GROUPS

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1. Introduction. In this paper we generalize a theorem of Lyndon's [**7**], which states that a one-relator group $G = F/(r)$ (F is free and $r \in F$) has cohomological dimension cd $(F/(r)) \leqq 2$ if and only if the relator r is not a proper power in F. His proof relies on the Identity Theorem and recently he has shown [**8**] how a generalized version of this theorem and a generalized version of the Freiheitsatz can be simultaneously obtained by the methods of combinatorial geometry. These generalizations refer to a situation where the free group F is replaced by a free product of subgroups of the additive group of real numbers.

In Theorem 1 below we deduce from the classical form of the Freiheitsatz, by straightforward and simple reasoning, that the Freiheitsatz remains valid if F is replaced by any free product of torsion-free abelian groups. Our main result, however, is Theorem 2, which describes the cohomology of $G = P/(r)$, where $P = \mathbf{Z}^{m_1} * \ldots * \mathbf{Z}^{m_n}$, $m_i \in \mathbf{N}$. If $m = \sup(m_1, \ldots, m_n)$, $n > 1$, and the word r, when cyclically reduced, contains at least one letter from each factor $\mathbf{Z}^{m_i}$, then $m \leqq \text{cd}\, G \leqq m + 1$ if and only if r is not a proper power in P. Our proof relies upon standard techniques of homological algebra and a generalization of Magnus' basic breakdown of one-relator groups. The Identity Theorem is not used at all, and, except for the case $m = 1$, the relationship between Theorem 2 and the generalized Identity Theorem is unclear. Our work fits into the general context of groups acting upon trees, for which we refer to Serre's notes [**11**]. For the homological algebra we shall refer to [**3**], where it is shown that if a group acts upon a tree, the cohomology of the group is determined by the cohomology of the stabilizers of the vertices and edges of the tree.

The reader who is acquainted with Magnus' proof of the Freiheitsatz [**9**] will see that its generalization to $P/(r)$ is also a simple consequence of what we called the basic breakdown of $P/(r)$. We omit the details.

2. The generalized Freiheitsatz.

THEOREM 1. *Suppose that $A_1, \ldots, A_n$ are torsion-free abelian groups and $r \in A_1 * \ldots * A_n$ is cyclically reduced and contains at least one letter from each*

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component A_i. *Then the composition*

$$\mu : A_1 * \ldots * A_{n-1} \rightarrow A_1 * \ldots * A_n \rightarrow A_1 * \ldots * A_n/(r)$$

of canonical maps is injective.

Proof. Suppose that $\mu(x) = 1$. Then write x as an ordered product:

$$x = \prod_{j=1}^{m} f_j r^{e_j} f_j^{-1}, \quad f_j \in P = A_1 * \ldots * A_n, \quad e_j = \pm 1,$$

where the words $f_1, \ldots, f_m, x, r$, when written in normal form, involve only finitely many (non-zero) letters from each A_i. Let $\{a_{i,1}, a_{i,2}, \ldots, a_{i,k_i}\}$ be the union of the set S_i of all these letters and the set $\{s - s' \neq 0 | s, s' \in S_i\}$. The letters $a_{i,1}, a_{i,2}, \ldots, a_{i,k_i}$ generate a free abelian group $\mathbf{Z}^{m_i}$ of rank m_i, say, in A_i, and x belongs to the normal subgroup generated by r in the subgroup $P' = \mathbf{Z}^{m_1} * \ldots * \mathbf{Z}^{m_n}$ of P. So, we are reduced to the case where the A_i are finitely generated. We now prove the somewhat trivial fact that for every $i = 1, \ldots, n$, there exists a homomorphism $\varphi_i : \mathbf{Z}^{m_i} \rightarrow \mathbf{Z}$ such that $\varphi_i(a_{i,j}) \neq 0$ for all $j = 1, \ldots, k_i$. We fix i, and write $a_{i,j} = (b_{j,1}, b_{j,2}, \ldots, b_{j,m_i}) \in \mathbf{Z}^{m_i}$. The union of the k_i hyperplanes

$$H_j = \left\{ (x_1, \ldots, x_{m_i}) \in \mathbf{Z}^{m_i} \,\middle|\, \sum_{h=1}^{m_i} x_h b_{j,h} = 0 \right\}, \quad j = 1, \ldots, k_i$$

is not all of $\mathbf{Z}^{m_i}$. Indeed, for every $z \in \mathbf{N}$, the cube

$$C = \{ (x_1, \ldots, x_{m_i}) \in \mathbf{Z}^{m_i} | 0 \leqq x_h < z, h = 1, \ldots, m_i \}$$

contains at most $z^{m_i - 1}$ points of H_j (at most $m_i - 1$ components of a point in H_j can be chosen freely), and for $z > k_i$ there exists a point $(z_1, \ldots, z_{m_i})$ in C, which is not in $\bigcup_{j=1}^{k_i} H_j$. The homomorphism $\varphi_i : \mathbf{Z}^{m_i} \rightarrow \mathbf{Z}$ defined by

$$\varphi_i(x_1, \ldots, x_{m_i}) = \sum_{h=1}^{m_i} x_h z_h,$$

now satisfies the condition: $\varphi_i(a_{i,j}) \neq 0$ for all $j = 1, \ldots, k_i$. These homomorphisms φ_i induce a homomorphism

$$\varphi = \varphi_1 * \ldots * \varphi_n : \mathbf{Z}^{m_1} * \ldots * \mathbf{Z}^{m_n} \rightarrow \mathbf{Z} * \ldots * \mathbf{Z} = F(x_1, \ldots, x_n).$$

Since none of the letters of $\mathbf{Z}^{m_i}$ involved in x are sent to 1 by φ_i, the length of x equals the length of $\varphi(x)$. Moreover, $x \in A_1 * \ldots * A_{n-1}$ implies that $\varphi(x)$ belongs to the subgroup of $F(x_1, \ldots, x_n)$ generated by $x_1, \ldots, x_{n-1}$. Since none of the letters appearing in r are sent to 1, with distinct letters in r having distinct images, and

$$\varphi(x) = \prod_{j=1}^{m} \varphi(f_j)\varphi(r)^{e_j}\varphi(f_j)^{-1}$$

belongs to the normal subgroup of $F(x_1, \ldots, x_n)$ generated by $\varphi(r)$, which is still cyclically reduced, it follows from the classical Freiheitsatz, applied to $\varphi(x)$ and $\varphi(r)$, that $\varphi(x) = 1$. Hence the length of x is 0, i.e. $x = 1$.

3. The basic breakdown of $\mathbf{Z}^{m_1} * \ldots * \mathbf{Z}^{m_n}/(r)$. Our purpose is to generalize the basic breakdown of one-relator groups (see [**10**, p. 252]), and to place this generalization in the context of "groups acting on trees".

PROPOSITION 1. *Let A, B and C be any groups, and let N be the normal subgroup of $A*(B \times C)$ generated by A and B. Then N is a free product $(*_C A) * B$, where $*_C A$ denotes the free product of copies of A indexed by C. Let the group C act upon this free product by acting trivially on B, and by means of a regular permutation of the copies of A, (described by left multiplication of their indices); then this action corresponds to conjugation of elements of N by elements of C in $A * (B \times C)$.*

Proof. Rather than refer to the Kurosh subgroup theorem, we present the following simple argument. Let $\mu_c : A = A_c \to (*_C A) * B$ be the canonical inclusion, $(c \in C)$, and let P be the semi-direct product of $(*_C A) * B$ and C with respect to the action

$$\theta : C \to \text{Aut}\ ((*_C A) * B),$$

defined by $\theta(c) \circ \mu_{c'} = \mu_{cc'}$, $\theta(c)(b) = b$ for all $b \in B$, $c, c' \in C$. We shall prove that P is isomorphic to the coproduct (free product) P' of A and $B \times C$. Define $\alpha : (*_C A) * B \to P'$ by:

$$\alpha(\mu_c(a)) = (1, c)a(1, c)^{-1}, \quad (a \in A_c)$$
$$\alpha(b) = (b, 1), \quad (b \in B)$$

and observe that $\alpha(cxc^{-1}) = (1, c)\alpha(x)(1, c)^{-1}$ for all $x \in (*_C A) * B$. This implies that the mapping $\beta : P \to P'$, which sends xc, $(x \in (*_C A) * B, c \in C)$ to $\alpha(x)(1, c)$, is a homomorphism of groups. The embedding μ_1 that sends $A = A_1$ into $(*_C A) * B \subset P$ and the map $\nu : B \times C \to P$ that sends (b, c) to bc together induce a map $\gamma : P' \to P$, which is easily seen to be a left and right inverse of β. The isomorphism γ obviously maps N onto $(*_C A) * B$.

THEOREM 2. *Let $r \in P = \mathbf{Z}^{m_1} * \ldots * \mathbf{Z}^{m_n}$, $m_i \in \mathbf{N}$, $G = P/(r)$, $m = \sup(m_1, \ldots, m_n)$, and suppose that r is cyclically reduced and contains at least one letter from each factor $\mathbf{Z}^{m_i}$ of P. Then*

(i) $\text{cd}\ G = \infty$ *if r is a proper power in P;*

(ii) $m \leqq \text{cd}\ G \leqq m + 1$ *if $n > 1$ and r is not a proper power in P;*

(iii) $\text{cd}\ G = m - 1$ *if $n = 1$ and r is not a proper power in P.*

LEMMA 1. *Consider the maps $\pi_i : P \to \mathbf{Z}^{m_i}$ that are defined by $\pi_i | \mathbf{Z}^{m_j} = \text{id}.$ if $j = i$, and 0 otherwise, (and that generalize the exponentsum maps of a free group). Let $m_i > 1$ for some i and let $h \in \mathbf{N}$. Suppose that Theorem 2 holds for all $r \in P$, in cyclically reduced form, and satisfying:*

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(i) *the length of* r *is* $\leqq h$

(ii) *one or more of the components of* $\pi_i(r)$ *is zero.*

Then Theorem 2 *is valid for all* $r \in P$ *satisfying* (*i*).

Proof of Lemma 1. Write $k = m_i$, $\pi_i(r) = (r_1, \ldots, r_k)$, where $r \in P$ is supposed to be of length $\leqq h$. Suppose that $d \neq 0$ is the g.c.d. of $r_1, \ldots, r_k$, and let $b_1, \ldots, b_k \in \mathbf{Z}$ be such that $d = b_1 r_1 + \ldots + b_k r_k$. Right multiplication by the product AB of the matrices

$$A = \begin{bmatrix} b_1 & 0 & \cdot & \mathbf{0} & \\ b_2 & 1 & 0 & & \\ & 0 & 1 & & \\ & \mathbf{0} & & \ddots & \\ b_k & & & & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -r_2/d & & & -r_k/d \\ 0 & 1 & 0 & \mathbf{0} & \\ 0 & 0 & 1 & & \\ & \mathbf{0} & & \ddots & \\ & & & & 1 \end{bmatrix}$$

defines a monomorphism $\mu : \mathbf{Z}^k \rightarrow \mathbf{Z}^k$, mapping $\pi_i(r)$ onto $(d, 0, \ldots, 0)$. We now extend μ in the obvious way to an injective endomorphism $\bar{\mu}$ on $P = \mathbf{Z}^{m_1} * \ldots * \mathbf{Z}^{m_n}$ and we write $\tilde{r} = \bar{\mu}(r)$. The map $\bar{\mu}$ induces $\tilde{\mu} : P/(r) \rightarrow P/(\tilde{r})$, which fits into the commutative square

$$\begin{array}{ccc} \mathbf{Z}^k & \xrightarrow{\alpha} & P/(r) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\tilde{\mu}} \\ \mathbf{Z}^k & \xrightarrow{\beta} & P/(\tilde{r}) \end{array}$$

where α and β are the obvious maps. We claim that this square is a push-out. First, every pair of maps $\gamma : \mathbf{Z}^k \rightarrow K$, $\delta : P/(r) \rightarrow K$, for which $\gamma\mu = \delta\alpha$, uniquely determine a pair of maps $\gamma : \mathbf{Z}^k \rightarrow K$, $\delta' : P \rightarrow K$ for which $\gamma \cdot \mu = \delta'|\mathbf{Z}^k$ and $\delta'(r) = 1$. The pair consisting of $\gamma : \mathbf{Z}^k \rightarrow K$ and the restriction $\delta'|\mathbf{Z}^{m_1} * \ldots * \mathbf{Z}^{m_{i-1}} * \mathbf{Z}^{m_{i+1}} * \ldots * \mathbf{Z}^{m_n}$ induce a map $\epsilon : P \rightarrow K$, such that $\delta' = \epsilon \cdot \bar{\mu}$, $\epsilon(\tilde{r}) = 1$, $\gamma = \epsilon|\mathbf{Z}^k$, and $\epsilon : P \rightarrow K$ now induces a unique map $\varphi : P/(\tilde{r}) \rightarrow K$ such that $\gamma = \varphi \cdot \beta$, $\delta = \varphi \cdot \tilde{\mu}$. If $n = 1$, there is nothing more to prove. So suppose $n > 1$ and the word r, of length $\leqq h$, satisfies the hypotheses of Theorem 2. We remind the reader that cd $\mathbf{Z}^{m_j} = m_j$ ([**4**, 8.8]). From the generalized Freiheitsatz (Theorem 1) we deduce the inequality $m \leqq$ cd $P/(r)$ as well as the injectivity of α and hence of β and $\tilde{\mu}$. The cohomology of $P/(r)$ and the cohomology of $P/(\tilde{r})$ are related by the Mayer-Vietoris sequence associated with our push-out, [**5**, p. 221], and we obtain cd $P/(r) \leqq$ cd $P/(\tilde{r})$. We observe finally that r and $\tilde{r}$ are of the same length, which completes our proof.

LEMMA 2. *Let* $h \in \mathbf{N}$, *and suppose that* $m = 1$; *i.e.* $P = F(x_1, \ldots, x_n)$. *Suppose that Theorem* 2 *holds for all* $r \in P$ *satisfying:*

(i) *the exponentsum* $\pi_n(r)$ *of* x_n *in* r *is zero;*

(ii) *when* r *is expressed in reduced form as a word in the free group*

$$*_{j\in\mathbf{Z}}\, x_n{}^j F(x_1, \ldots, x_{n-1}) x_n{}^{-j} = \ker \pi_n \quad (\textit{see Proposition } 1),$$

its length is $< h$.

Then Theorem 2 *holds for all* $r \in P$ *of length* $\leqq h$.

Proof. We suppose, without loss of generality, that $n > 1$, the exponentsums d and e of x_1 and x_n respectively in r are non-zero and r is cyclically reduced and of length $\leqq h$. As in Lemma 1, one has a push-out diagram

$$\begin{array}{ccccc} x_n & & F(x_n) & \longrightarrow & P/(r) \\ \downarrow & & \downarrow & & \downarrow \\ x_n{}^d & & F(x_n) & \longrightarrow & P/(\tilde{r}) \end{array}$$

where $\tilde{r}$ is the word

$$\tilde{r}(x_1, \ldots, x_n) = r(x_1, \ldots, x_{n-1}, x_n{}^d)$$

obtained from r by substituting $x_n{}^d$ for x_n. Next, write

$$\bar{r}(x_1, \ldots, x_n) = r(x_1 x_n{}^{-e}, x_2, \ldots, x_{n-1}, x_n{}^d).$$

The groups $P/(\tilde{r})$ and $P/(\bar{r})$ are isomorphic, the cohomology of $P/(r)$ and that of $P/(\tilde{r})$ are related by the Mayer-Vietoris sequence (*loc cit*), the exponentsum of $\bar{r}$ in x_n is 0 and when we express $\bar{r}$ in terms of the conjugates $x_n{}^j x_i x_n{}^{-j}$ $(i = 1, \ldots, n-1, j \in \mathbf{Z})$, we obtain a word of length $< h$ in the free group $*_{j\in\mathbf{Z}}\, x_n{}^j F(x_1, \ldots, x_{n-1}) x_n{}^{-j}$. The result follows. (The author learned about the push-out of this lemma from Baumslag [**2**, p. 172].)

Before proceeding any further, we recall the definitions of a *graph of groups* (G, X) and its *colimit* $\varinjlim(G, X)$. A *graph of groups* (G, X) is a graph X, together with a diagram of groups consisting of a group G_P for each vertex P of X, a group G_e for each edge e of X and a pair of monomorphism $\bar{e} : G_e \to G_P$, $e : G_e \to G_Q$ for every edge e with initial point P and terminal point Q, and we require that G_e and $\{e, \bar{e}\}$ remain the same if the edge e is inverted in X. The colimit of the diagram is denoted $\varinjlim(G, X)$.

We now suppose the following: $r \in P = \mathbf{Z}^{m_1} * \ldots * \mathbf{Z}^{m_n}$, $n > 1$, the word r is cyclically reduced and contains a letter from each factor $\mathbf{Z}^{m_i}$ of P, and the last component of the image $\pi_n(r)$ of r in $\mathbf{Z}^{m_n}$ is zero. Let N denote the kernel of

$$\epsilon : P \xrightarrow{\pi_n} \mathbf{Z}^{m_n} \xrightarrow{\rho} \mathbf{Z},$$

where ρ is the projection upon the last component of $\mathbf{Z}^{m_n}$. Let r_0 be the cyclically reduced word obtained from r by rewriting it as a word in

$$N = *_{\mathbf{Z}}(\mathbf{Z}^{m_1} * \ldots * \mathbf{Z}^{m_{n-1}}) * \mathbf{Z}^{m_n - 1}$$

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(Proposition 1). Let x_n be a (free) generator of the last factor of $\mathbf{Z}^{m_n}$, let P_0 be the free product of those factors of N that are involved in r_0 and let $V_0 = P_0 \cap x_n^{-1}P_0x_n$. We now consider the oriented graph Y, which has exactly one vertex Q and exactly one positive edge y. We define a graph of groups (G, Y), by first letting $G_Q = P_0/(r_0)$ and $G_y = V_0$. We then define $\bar{y} : G_y \to G_Q$ to be the inclusion $V_0 \subset P_0$ followed by the projection: $P_0 \to P_0/(r_0)$. Finally, we note that $x_nV_0x_n^{-1} \subset P_0$, so that it is meaningful to define $y : G_y \to G_Q$ as conjugation by x_n followed by the projection: $P_0 \to P_0/(r_0)$. By Theorem 1, y and $\bar{y}$ are injective. We let T be the maximal subtree of Y consisting of the vertex Q and no edges.

THEOREM 3. *With the above notation, the group $P/(r)$ is isomorphic to the fundamental group $\pi_1(G, Y, T)$* ([**11**, I-65]) *of the graph of groups (G, Y).*

Proof. For every $j \in \mathbf{Z}$, let $P_j = x_n^jP_0x_n^{-j}$, $V_j = x_n^jV_0x_n^{-j}$, $r_j = x_n^jr_0x_n^{-j}$ and $G_j = P_j/(r_j)$. The universal cover $\tilde{Y}$ of (Y, T) is the graph Γ $(F(x_n), \{x_n\})$ of the free group $F(x_n)$, and we now consider the graph of groups $(G, \tilde{Y})$:

$$\cdots \longrightarrow G_{-2} \xrightarrow{V_{-1}} G_{-1} \xrightarrow{V_0} G_0 \xrightarrow{V_1} G_1 \xrightarrow{V_2} G_2 \longrightarrow \cdots$$

where the maps: $V_{j+1} \to G_j$, $V_j \to G_j$ are the inclusions $V_{j+1} \to P_j$, $V_j \to P_j$ followed by the projection: $P_j \to P_j/(r_j)$. (By Theorem 1, the maps $V_{j+1} \to G_j$, $V_j \to G_j$ are injective.) The group $F(x_n)$ acts upon this graph of groups, and hence upon its colimit $K = \varinjlim(G, \tilde{Y})$ in the obvious way, and $\pi_1(G, Y, T)$ may be described as the semi-direct product of K and $F(x_n)$ ([**11**, I-69 exercise]).

According to Proposition 1, N is generated by $\bigcup_{j\in\mathbf{Z}}P_j$. The group $R = (r)$ is the normal subgroup of N generated by the elements $r_j (j \in \mathbf{Z})$, and the inclusions $P_j \subset N$ induce maps $G_j \to N/R$, which in turn induce a map $K \to N/R$. One easily sees that this map is an isomorphism compatible with the action of $F(x_n)$ (The action of $F(x_n)$ on N/R is induced by the action by conjugation of $F(x_n)$ on N). The isomorphism can therefore be extended to an isomorphism of the semi-direct product $\pi_1(G, Y, T)$ of K and $F(x_n)$ onto the semi-direct product P/R of N/R and $F(x_n)$.

THEOREM 4. *If* cd $(P_0/(r_0)) \leqq m + 1$, *then* cd $(P/(r)) \leqq m + 1$, *and if* cd $(P_0/(r_0)) = \infty$ *then* cd $(P/(r)) = \infty$.

Proof. By Theorem 3 and [**11**, § 5.3], $P/(r)$ can be made to act upon a tree $\tilde{X} = \tilde{X}(G, Y, T)$, called the *universal cover of* (G, Y, T), in such a way that the stabilizer of every vertex is isomorphic to $G_0 = P_0/(r_0)$, and the stabilizer of every edge is isomorphic to V_0. Therefore, if cd $(P_0/(r_0)) = \infty$ then cd $(P/(r)) = \infty$ and, since cd $V_0 \leqq m$, it follows from [**3**, Theorem 3], that if cd $(P_0/(r_0)) \leqq m + 1$ then cd $(P/(r)) \leqq m + 1$.

Proof of Theorem 2. The case (iii) is trivial, and we henceforth assume that $n > 1$. The inequality cd $(P/(r)) \geqq m$ follows from the generalized Freiheisatz. We now carry out an induction argument on m.

If $m = 1$, we are in the classical situation of a one-relator group. The argument we present here is therefore a new proof of the inequality cd $(P/(r)) \leqq 2$ when r is not a proper power in P, and cd $(P/(r)) = \infty$ when it is. This proof goes by induction on the word length h of r. We can have $h = 1$ only in the trivial case when $n = 1$. So we may suppose that $h > 1$, and by Lemma 2, we may also suppose that the exponentsum $\pi_n(r)$ of x_n in r is zero. By the induction hypothesis, cd $(P_0/(r_0)) \leqq 2$ if r_0 is not a proper power in P_0, and cd $(P_0/(r_0)) = \infty$ if it is a proper power in P_0. (Recall that r_0 is the element r, expressed as a (shorter) reduced word in the free generators $x_{i,j} = x_n^{\,j} x_i x_n^{-j}$ $(i = 1, \ldots, n-1; j \in \mathbf{Z})$ of ker π_n, and P_0 is the subgroup of ker π_n generated by those generators that appear in this reduced word). Clearly r is a proper power in P if and only if r_0 is a proper power in ker π_n if and only if r_0 is a proper power in P_0. The proof of the case $m = 1$ is now completed by a reference to Theorem 4.

Suppose now that $m > 1$. To treat this case, we argue by induction on the number $s(P, r, m)$ of letters in the reduced word r that belong to some factor $\mathbf{Z}^{m_i}$ of P, with $m_i = m$. Permuting the free factors of P if necessary, we may suppose without loss in generality that $m_n = m$. According to Lemma 1, we may also suppose that the last component of ker π_n is zero. We obtain a cyclically reduced word r_0, by rewriting r as a word in the free product

$$N = *_{\mathbf{Z}} (\mathbf{Z}^{m_1} * \ldots * {}^{m_{n-1}}) * \mathbf{Z}^{m_n - 1} \quad \text{(Proposition 1)},$$

and P_0 denotes the free product of those factors of N that appear in r_0. Clearly $s(P_0, r_0, m) < s(P, r, m)$. If $s(P_0, r_0, m) = 0$, the induction hypothesis on m applies to $P_0/(r_0)$, and if $s(P_0, r_0, m) > 0$, then m equals the supremum m_0 of the ranks of the factors of the free product P_0, and the induction hypothesis on $s(P_0, r_0, m_0)$ applies to $P_0/(r_0)$. We note that r is a proper power in P if and only if it is a proper power in N if and only if r_0 is a proper power in P_0. The result now follows from Theorem 4.

Remark. To substantiate our claim of having generalized Lyndon's theorem, we should still point out that every $r \in P = \mathbf{Z}^{m_1} * \ldots * \mathbf{Z}^{m_n}$ is conjugate to a cyclically reduced word, and the cases of P having infinitely many factors: $P = \mathbf{Z}^{m_1} * \mathbf{Z}^{m_2} * \ldots$, or of some factor not being involved in a cyclically reduced word r, are easily dispensed with by noting that $P/(r)$ is then isomorphic to a free product of the form $G' * P''$, where $G' = P'/(r')$ satisfies the hypotheses of Theorem 2, and P'' is a free product of finitely generated free abelian groups; hence

$$\text{cd } (P/(r)) = \sup\, (\text{cd } G', \text{cd } P'').$$

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