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STARLIKE FUNCTIONS WITH A FIXED COEFFICIENT

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This paper establishes several conditions on the parameters A, B, b for the exact radius of convexity of the class

$$S^*_{k,b}(A,B) = \left\{ f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \dots;\ \frac{zf'(z)}{f(z)} \in P_{k,b}(A,B) \right\},$$

where

$$P_{k,b}(A,B) = \left\{ p(z) = 1 + b(A-B)z^k + p_{2k}z^{2k} + \dots;\ p(z) \prec \frac{1+Az^k}{1+Bz^k} \right\},$$

$k = 1, 2, 3, \dots, -1 \leqslant B < A \leqslant 1, 0 \leqslant b \leqslant 1$.

1. INTRODUCTION

Let $P_k(A,B)$, $-1 \leqslant B < A \leqslant 1$, $k = 1,2,3,\dots$, denote the class of functions $p(z) = 1 + p_kz^k + p_{2k}z^{2k} + \dots$ defined by

$$p(z) \prec \frac{1+Az^k}{1+Bz^k}$$

in the unit disc $\Delta = \{z; |z| < 1\}$, where $\prec$ means subordination. Then each $p(z)$ in $P_k(A, B)$ has a positive real part in Δ. As is well-known, a necessary and sufficient condition for a function $f(z) = z + a_2z^2 + \dots$ to be univalent starlike in Δ is

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad z \in \Delta.$$

This condition suggests that starlike functions can be defined in terms of functions of positive real part in the unit disc. In fact, a general class of starlike functions may be defined as

$$S^*_k(A, B) = \left\{ f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \dots;\ \frac{zf'(z)}{f(z)} \in P_k(A, B) \right\}.$$

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With $k = 1$, the following special cases of $S_k^*(A, B)$ are familiar:

$$S^*(1-2\alpha, -1) = \{f(z) = z + a_2 z^2 + \ldots;\ \mathrm{Re}[zf'(z)/f(z)] > \alpha,\ 0 \leqslant \alpha < 1\},$$
$$S^*(1, 1/\alpha - 1) = \{f(z) = z + a_2 z^2 + \ldots;\ |zf'(z)/f(z) - \alpha| < \alpha,\ \alpha > 1/2\},$$
$$S^*(\alpha, 0) = \{f(z) = z + a_2 z^2 + \ldots;\ |zf'(z)/f(z) - 1| < \alpha,\ 0 < \alpha \leqslant 1\},$$
$$S^*(\alpha, -\alpha) = \{f(z) = z + a_2 z^2 + \ldots;\ |zf'(z)/f(z) - 1| \,/\, |zf'(z)/f(z) + 1| < \alpha,\ 0 \leqslant \alpha < 1\}.$$

These classes of functions have been studied by many authors, starting with Robertson [**7**] for starlike functions of order α. The classes $S^*(1, 1/\alpha - 1)$, $S^*(\alpha, 0)$, $S^*(\alpha, -\alpha)$ were introduced by Janowski [**2**], MacGregor [**3**] and Padmanabhan [**5**] respectively.

In this paper, we study starlike functions with a fixed coefficient. For functions $p(z) = 1 + p_k z^k + p_{2k} z^{2k} + \ldots$ in $P_k(A, B)$, it is known that $|p_k| \leqslant A - B$, $k = 1, 2, 3, \ldots$ (see Anh [**1**], Theorem 4.1). We may therefore define the following subclass of $P_k(A, B)$:

$$P_{k,b}(A, B) = \{p(z) = 1 + b(A-B)z^k + \ldots \in P_k(A, B),\ 0 \leqslant b \leqslant 1\}.$$

We shall then consider the corresponding class of k-fold symmetric starlike functions with a real nonnegative second coefficient:

$$S_{k,b}^*(A, B) = \{f(z) = z + (b(A-B)/k)z^{k+1} + \ldots;\ zf'(z)/f(z) \in P_{k,b}(A, B)\}.$$

We shall investigate mainly how the second coefficient in the series expansion of functions in $S_{k,b}^*(A, B)$ affects their radius of convexity. This problem was studied in Tepper [**8**], McCarty [**4**], Tuan and Anh [**9**], among others. Tepper [**8**] obtained the radius of convexity of $S_{1,b}^*(1, -1)$. The results contained in McCarty [**4**] for $S_{1,b}^*(1-2\alpha, -1)$ and Tuan and Anh [**9**] for $S_{1,b}^*(A, B)$ are in fact achieved by functions in larger classes where the second coefficient is allowed to vary. It is more difficult to obtain sharp results within $S_{k,b}^*(A, B)$ where the second coefficient is assumed fixed, real and nonnegative. As far as we are aware, apart from Tepper's result for the simplest class $S_{1,b}^*(1, -1)$, no complete and accurate radius of convexity for any other class of starlike functions with a real fixed coefficient is available. It seems that the radius of convexity of $S_{k,b}^*(A, B)$ can be derived only with some restriction on the parameters A, B, b. In Section 2, we obtain the required conditions for an extremal problem on $P_{k,b}(A, B)$. The results play an essential rôle in the derivation of the radius of convexity for $S_{k,b}^*(A, B)$. This is investigated in Section 3. The conditions of Section 2 are established in the hope that they would be satisfied for some simpler cases of $S_{k,b}^*(A, B)$. This is indeed the case for $S_{k,b}^*(1, 1/\alpha - 1)$ whose radius of convexity is now completely determined. The radii of convexity of $S_{k,b}^*(\alpha, -\alpha)$ and $S_{k,b}^*(\alpha, 0)$ are also obtained for a certain range of α.

2. An Extremal Problem on $P_{k,b}(A, B)$

By definition, the radius of convexity of $S^*_{k,b}(A, B)$ is the smallest root in $(0, 1]$ of the equation

$$\min_{f(z)\in S^*_{k,b}(A,B)} \min_{|z|=r<1} \operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} = 0.$$

From the definition of $S^*_{k,b}(A, B)$, we derive that

$$1+\frac{zf''(z)}{f'(z)} = p(z)+\frac{zp'(z)}{p(z)}, \qquad P(z)\in P_{k,b}(A, B).$$

Thus the radius of convexity of $S^*_{k,b}(A, B)$ is obtained if we can solve the extremal problem

$$\min_{p(z)\in P_{k,b}(A,B)} \min_{|z|=r<1} \operatorname{Re}\left\{p(z)+\frac{zp'(z)}{p(z)}\right\}. \tag{2.1}$$

Problem (2.1) is studied in this section. We first obtain the growth theorem for functions in $P_{k,b}(A, B)$, which is required in the solution of (2.1). We need the following result:

Lemma 1. *For a given point z in Δ, let F be regular in a neighbourhood of each point $p(z)$, $p \in P_k(1, -1)$. Then the functional $\operatorname{Re} F(p(z))$, $z \in \Delta$, attains its maximum and minimum over the class $P_k(1, -1)$ only for functions of the form $(1+e^{-i\theta}z^k)/(1-e^{-i\theta}z^k)$.*

Proof: See Pfaltzgraff and Pinchuk [**6**, Theorem 7.3]. ∎

Theorem 1. *If $p(z)\in P_{k,b}(A, B)$, then on $|z| = r < 1$,*

$$|p(z)| \leqslant \frac{1+b(1+A)r^k+Ar^{2k}}{1+b(1+B)r^k+Br^{2k}},$$

$$\operatorname{Re}\{p(z)\} \geqslant \begin{cases} \frac{1+b(1-A)r^k-Ar^{2k}}{1+b(1-B)r^k-Br^{2k}}, & \text{for } k = 1, 3, 5, \ldots \\ \frac{1-b(1-A)r^k-Ar^{2k}}{1-b(1-B)r^k-Br^{2k}}, & \text{for } k = 2, 4, 6, \ldots. \end{cases}$$

The results are sharp.

Proof: We require a representation formula for $P_{k,b}(A, B)$. Denote by U the class of functions $\psi(z)$ regular in Δ and such that $|\psi(z)| \leqslant 1$ there. For $p(z) \in P_{k,b}(A, B)$, we define

$$\psi_1(z) = \frac{1}{z^k}\frac{1-p(z)}{Bp(z)-A}. \tag{2.2}$$

Then $\psi_1(z) \in U$ and $\psi_1(0) = b$. The function

$$\psi_2(z) = \frac{\psi_1(z) - b}{1 - b\psi_1(z)} \tag{2.3}$$

is therefore in U and $\psi_2(0) = 0$. Hence the function

$$\psi(z) = \frac{\psi_2(z)}{z^k} \tag{2.4}$$

belongs to U. From (2.2)–(2.4), it follows that a function $p(z) \in P_{k,b}(A, B)$ can be represented in the form

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \tag{2.5}$$

where

$$w(z) = z^k \cdot \frac{z^k\psi(z) + b}{1 + bz^k\psi(z)}, \qquad \psi(z) \in U. \tag{2.6}$$

As $|\psi(z)| \leqslant 1$ in Δ, we have $|w(z)| \leqslant C$ on $|z| = r < 1$, where $C = r^k(r^k + b)/(1 + br^k)$. An application of the subordination principle then implies that the image of $|z| \leqslant r$ under the transformation (2.5) is contained in the disc

$$|p(z) - a_{k,b}| \leqslant d_{k,b},$$

where

$$a_{k,b} = \frac{1 - ABC^2}{1 - B^2C^2}, \quad d_{k,b} = \frac{(A - B)C}{1 - B^2C^2}.$$

It follows immediately that for $p(z) \in P_{k,b}(A, B)$ and on $|z| = r < 1$,

$$\frac{1 - AC}{1 - BC} \leqslant \operatorname{Re}\{p(z)\} \leqslant |p(z)| \leqslant \frac{1 + AC}{1 + BC}. \tag{2.7}$$

The upper bound is sharp for the function $p(z)$ with $\psi(z) = 1$ in (2.6) and at $z = r$. The lower bound is attained for the function $p(z)$ with $\psi(z) = -1$ in (2.6), at $z = -r$ and for $k = 1, 3, 5, \ldots$. For k even, this lower bound is not achieved by a function within $P_{k,b}(A, B)$. For the sharp lower bound in the case of k even, we represent $p(z)$ in terms of function in $P_k(1, -1)$. As seen from (2.3), the function $\psi_2(z)$ satisfies the conditions of Schwarz's lemma. Therefore, the function

$$q(z) = \frac{1 + \psi_2(z)}{1 - \psi_2(z)} \tag{2.3a}$$

belongs to the unrestricted class $P_k(1, -1)$. From (2.2), (2.3) and (2.3a), a function $p(z) \in P_{k,b}(A, B)$ can be represented in the form

$$p(z) = \frac{(1+b)(1+Az^k)q(z) + (1-b)(1-Az^k)}{(1+b)(1+Bz^k)q(z) + (1-b)(1-Bz^k)}, \tag{2.3b}$$

where $q(z) \in P_k(1, -1)$. An application of Lemma 1 now yields that $\operatorname{Re}\{p(z)\}$ attains its minimum over $P_k(A, B)$ for a function of the form (2.3b), where $q(z)$ is given by $(1+\varepsilon z^k)/(1-\varepsilon z^k)$, $|\varepsilon| = 1$. For the time being, we consider a larger class, namely

$$\tilde{P}_{k,b}(A, B) = \{p(z) = 1 + b(A-B)e^{i\theta}z^k + \ldots \in P_k(A, B),\ \theta \text{ real}\}.$$

Then, since $P(e^{i\theta}z)$ belongs to $\tilde{P}_{k,b}(A, B)$ if $p(z)$ is in $\tilde{P}_{k,b}(A, B)$, we may assume, without loss of generality, that the minimum of $\operatorname{Re}\{p(z)\}$ over $\tilde{P}_{k,b}(A, B)$ is attained on the real axis at $z = -r$. Now, using (2.3b) with $q(z) = (1+\varepsilon z^k)/(1-\varepsilon z^k)$, we get

$$\begin{aligned} p(-r) &= \frac{1 + bAr^k + (Ar^k + b)\varepsilon r^k}{1 + bBr^k + (Br^k + b)\varepsilon r^k} \\ &= \frac{1 + bAr^k}{1 + bBr^k} \cdot \frac{1 + M\varepsilon r^k}{1 + N\varepsilon r^k}, \end{aligned}$$

where $M = (Ar^k + b)/(1 + bAr^k)$, $N = (Br^k + b)/(1 + bBr^k)$. It can be checked that $-1 \leqslant N < M \leqslant 1$. Thus, the minimum of $\operatorname{Re}\{p(-r)\}$ corresponds to $\varepsilon = -1$, that is

$$\operatorname{Re}\{p(z)\} \geqslant \frac{1 - b(1-A)r^k - Ar^{2k}}{1 - b(1-B)r^k - Br^{2k}} \tag{2.3c}$$

over $\tilde{P}_{k,b}(A, B)$. However with $\varepsilon = -1$, the extremal function becomes

$$\begin{aligned} p(z) &= \frac{(1+b)(1+Az^k)(1-z^k)/(1+z^k) + (1-b)(1-Az^k)}{(1+b)(1+Bz^k)(1-z^k)/(1+z^k) + (1-b)(1-Bz^k)} \\ &= \frac{1 - b(1-A)z^k - Az^{2k}}{1 - b(1-B)z^k - Bz^{2k}} \\ &= 1 + b(A-B)z^k + \ldots \end{aligned}$$

which belongs to $P_{k,b}(A, B)$. Consequently, the bound given by (2.3c) is the best possible bound over $P_{k,b}(A, B)$, $k = 2, 4, 6, \ldots$. ∎

For the solution of (2.1), we require the following lemma. For $k = 1, 2, 3, \ldots$, let B_k be the class of functions $w(k) = b_k z^k + b_{2k} z^{2k} + \ldots$ regular in Δ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ in Δ.

Lemma 2. *If $w(z) \in B_k$, then for $z \in \Delta$,*

$$|zw'(z) - kw(z)| \leqslant \frac{k\left(|z|^{2k} - |w(z)|^2\right)}{1 - |z|^{2k}}. \tag{2.8}$$

Proof: We have $|w(z)| \leqslant |z|^k$ for $w(z) = b_k z^k + b_{2k} z^{2k} + \ldots \in B_k$ in view of Schwarz's lemma. Therefore, we may write

$$w(z) = z^k \psi(z^k),\ z \in \Delta$$

for $\psi(z) \in U$. Then

$$zw'(z) - kw(z) = kz^{2k}\psi'(z^k).$$

From Carathéodory's inequality

$$|\psi'(z)| \leqslant \frac{1 - |\psi(z)|^2}{1 - |z|^2},\ z \in \Delta,\ \psi(z) \in U,$$

we obtain (2.8) directly. Equality in (2.8) occurs for functions of the form $z^k(z^k - c)/(1 - cz^k)$, $|c| \leqslant 1$. ∎

We now prove

Theorem 2. *Let* $\alpha \geqslant 0, \beta \geqslant 0, k = 1, 2, 3, \ldots, |z| = r < 1$, $L = \beta k(1 - A)(1 + Ar^{2k})$, $K = \alpha(A - B)(1 - r^{2k}) + \beta k(1 - B)(1 + Br^{2k})$. *If* $p(z) \in P_{k,b}(A, B)$, *then on* $|z| = r$,

$$\operatorname{Re}\left\{\alpha p(z) + \beta\frac{zp'(z)}{p(z)}\right\} \geqslant \beta\frac{A + B}{A - B} + \frac{1}{(A - B)(1 - r^{2k})}\left[L \cdot \frac{1 - BC}{1 - AC} + K \cdot \frac{1 - AC}{1 - BC} - 2\beta k(1 - ABr^{2k})\right]$$

under the condition that

$$\frac{L}{K} \leqslant \left(\frac{1 - AC}{1 - BC}\right)^2. \tag{2.9}$$

The result is sharp.

Proof: From the representation formula (2.5), we may write

$$\alpha p(z) + \beta\frac{zp'(z)}{p(z)} = \alpha\frac{1 + Aw(z)}{1 + Bw(z)} + \beta\frac{(A - B)zw'(z)}{[1 + Aw(z)][1 + Bw(z)]},$$

$w(z)$ being defined by (2.6). Applying (2.8) to the second term of the right-hand side, we find

$$\begin{aligned}\operatorname{Re}\left\{\alpha p(z)+\beta\frac{zp'(z)}{p(z)}\right\} &\geqslant \operatorname{Re}\left\{\alpha\frac{1+Aw(z)}{1+Bw(z)}+\beta\frac{(A-B)kw(z)}{(1+Aw(z))(1+Bw(z))}\right\}\\ &\quad -\frac{k\beta(A-B)\left(|z|^{2k}-|w(z)|^2\right)}{\left(1-|z|^{2k}\right)|1+Aw(z)|\,|1+Bw(z)|}.\end{aligned}\tag{2.10}$$

From (2.5), we also have

$$w(z)=\frac{p(z)-1}{A-Bp(z)}.$$

Hence, in terms of $p(z)$, the above inequality becomes

$$\begin{aligned}\operatorname{Re}\left\{\alpha p(z)+\beta\frac{zp'(z)}{p(z)}\right\} &\geqslant \beta k\frac{A+B}{A-B}+\frac{1}{A-B}\operatorname{Re}\left\{[\alpha(A-B)-\beta kB]p(z)-\frac{\beta kA}{p(z)}\right\}\\ &\quad -\frac{k\beta\left(r^{2k}|A-Bp(z)|^2-|p(z)-1|^2\right)}{(A-B)(1-r^{2k})\,|p(z)|}.\end{aligned}\tag{2.11}$$

Put $p(z)=a_{k,b}+u+iv$, $|p(z)|=R$, and denote the RHS of (2.11) by $S(u,v)$. Then, as

$$\begin{aligned}r^{2k}|A-Bp(z)|^2-|p(z)-1|^2 &= r^{2k}\left(A^2-2AB(a_{k,b}+u)+B^2R^2\right)\\ &\quad -R^2+2(a_{k,b}+u)-1\\ &= -\left(1-B^2r^{2k}\right)R^2+2\left(1-ABr^{2k}\right)(a_{k,b}+u)\\ &\quad -\left(1-A^2r^{2k}\right)\\ &= -\left(1-B^2r^{2k}\right)R^2+2a_{k,1}\left(1-B^2r^{2k}\right)(a_{k,b}+u)\\ &\quad -\left(1-B^2r^{2k}\right)\left(a_{k,1}^2-d_{k,1}^2\right),\end{aligned}$$

we get

$$\begin{aligned}S(u,v) &= \beta\frac{A+B}{A-B}+\frac{1}{A-B}\left\{[\alpha(A-B)-\beta kB](a_{k,b}+u)-\frac{\beta kA(A_{k,b}+u)}{R^2}\right.\\ &\quad \left.+\beta k\frac{\left(1-B^2r^{2k}\right)}{1-r^{2k}}\left[R-2a_{k,1}\frac{a_{k,b}+u}{R}+\frac{a_{k,1}^2-d_{k,1}^2}{R}\right]\right\}\\ &= \beta\frac{A+B}{A-B}+\frac{1}{A-B}\left\{\left[\alpha(A-B)-\beta kB-\frac{\beta kA}{R^2}\right](a_{k,b}+u)\right.\\ &\quad \left.+\beta k\frac{1-B^2r^{2k}}{1-r^{2k}}\cdot\frac{1}{R}\left[(a_{k,b}+u-a_{k,1})^2+v^2-d_{k,1}^2\right]\right\}.\end{aligned}$$

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Now,

$$\frac{\partial S}{\partial v} = \frac{\beta k}{A-B} \cdot \frac{v}{R^4} T(u, v), \tag{2.12}$$

where

$$\begin{aligned} T(u,v) &= 2A(a_{k,b}+u) + \frac{1-B^2r^{2k}}{1-r^{2k}}\left[R^3 - R(a_{k,1}^2 - 2a_{k,1}(a_{k,b}+u) - d_{k,1}^2)\right] \\ &= 2(a_{k,b}+u)\left(A + \frac{1-B^2r^{2k}}{1-r^{2k}}a_{k,1}R\right) + \frac{1-B^2r^{2k}}{1-r^{2k}}\left(R^3 - R\left(a_{k,1}^2 - d_{k,1}^2\right)\right). \end{aligned}$$

Since

$$\frac{dC}{db} = \frac{r^k(1-r^{2k})}{(1+br^k)^2} > 0, \quad \frac{d(a_{k,b}-d_{k,b})}{db} = -\frac{A-B}{(1-BC)^2}\cdot\frac{dC}{db} < 0,$$
$$\frac{d(a_{k,b}+d_{k,b})}{db} = \frac{A-B}{(1+BC)^2}\cdot\frac{dC}{db} > 0,$$

we have $a_{k,b} - d_{k,b} \geqslant a_{k,1} - d_{k,1}$, $a_{k,b} + d_{k,b} \geqslant a_{k,0} - d_{k,0}$. Also $R \geqslant a_{k,1} - d_{k,1}$. It follows that

$$\begin{aligned} A + \frac{1-B^2r^{2k}}{1-r^{2k}}\cdot a_{k,1}R &\leqslant A + \frac{1-B^2r^{2k}}{1-r^{2k}}(a_{k,1}-d_{k,1})^2 \\ &= \frac{(1+B)(1-Ar^k)^2 + (A-B)(1-ABr^{2k})}{(1-Br^k)^2} > 0. \end{aligned}$$

In view of this inequality and the result that

$$a_{k,b} + u = \operatorname{Re}\{p(z)\} \geqslant a_{k,b} - d_{k,b} \geqslant a_{k,1} - d_{k,1},$$

we then have

$$T(u, v) \geqslant G(R),$$

where

$$G(R) = 2(a_{k,1} - d_{k,1})\left[A + \frac{1-B^2r^{2k}}{1-r^{2k}}a_{k,1}R\right] + \frac{1-B^2r^{2k}}{1-r^{2k}}\left[R^3 - R\left[a_{k,1}^2 - d_{k,1}^2\right]\right].$$

Now,

$$\frac{dG}{dR} = \frac{1-B^2r^{2k}}{1-r^{2k}}\left((a_{k,1}-d_{k,1})^2 + 3R^2\right) > 0.$$

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Therefore,

$$G(R) \geqslant G(a_{k,1}-d_{k,1}) = 2(a_{k,1}-d_{k,1})\left[A+\frac{1-B^2r^{2k}}{1-r^{2k}}(a_{k,1}-d_{k,1})^2\right]$$
$$\geqslant 2(a_{k,1}-d_{k,1})^2[A+(a_{k,1}-d_{k,1})^2]$$
$$> 0 \qquad \text{as seen above.}$$

Summing up, we have $T(u,v) > 0$, and it follows from (2.12) that the minimum of $S(u, v)$ on the disc $|p(z)-a_{k,b}| \leqslant d_{k,b}$ occurs at $v = 0$ and for some u in $[-d_{k,b}, d_{k,b}]$. Setting $v = 0$ in the expression for $S(u, v)$, we get

$$S(u, 0) = \beta\frac{A+B}{A-B} + \frac{1}{(A-B)(1-r^{2k})}\left[\frac{L}{a_{k,b}+u} + K(a_{k,b}+u) - 2\beta k\frac{1-ABr^{2k}}{1-r^{2k}}\right],$$

where

$$L = \beta k(1-A)(1+Ar^{2k}),\; K = \alpha(A-B)(1-r^{2k}) + \beta k(1-B)(1+Br^{2k}).$$

It is seen that

$$\frac{dS(u, 0)}{du} = \frac{1}{(A-B)(1-r^{2k})}\left[-\frac{L}{(a_{k,b}+u)^2} + K\right]$$

vanishes at the point $u_0 = (L/K)^{1/2} - a_{k,b}$. Now,

$$(a_{k,b}+u_0)^2 \leqslant \frac{(1-A)(1+Ar^{2k})}{(1-B)(1+Br^{2k})}$$
$$< \frac{1+Ar^{2k}}{1+Br^{2k}}$$
$$= a_{k,0}+d_{k,0} \leqslant a_{k,b}+d_{k,b} \leqslant (a_{k,b}+d_{k,b})^2.$$

Thus, $u_0 < d_{k,b}$. However, it is not necessary that $u_0 > -d_{k,b}$. It is seen that the condition $u_0 \leqslant -d_{k,b}$ is equivalent to $(u_0+a_{k,b})^2 \leqslant (a_{k,b}-d_{k,b})^2$; that is,

$$\frac{L}{K} \leqslant \left[\frac{1-AC}{1-BC}\right]^2.$$

Thus, under the above condition, the minimum of $S(u, 0)$ occurs at the end point $u = -d_{k,b}$, its value being

$$S(-d_{k,b}, 0) = \beta\frac{A+B}{A-B} + \frac{1}{(A-B)(1-r^{2k})}$$
$$\left[L\cdot\frac{1-BC}{1-AC} + K\cdot\frac{1-AC}{1-BC} - 2\beta k(1-ABr^{2k})\right].$$

We have seen earlier that the lower bound $a_{k,b} - d_{k,b}$ of $\operatorname{Re}\{p(z)\}$ is attained for the function

$$p_0(z) = \frac{1 - b(1 - A)z^k - Az^{2k}}{1 - b(1 - B)z^k - Bz^{2k}} \tag{2.13}$$

at $z = -r$ obtained by taking $\psi(z) = -1$ in (2.6). To show that the result of this theorem is sharp, we need only to show that inequality (2.10) is an equality for the same function $p_0(z)$ at $z = -r$. In fact, with $\psi(z) = -1$ in (2.6), direct calculation gives

$$\begin{aligned} zw'(z) &= kw(z) - \frac{k\left(1 - b^2\right)z^{2k}}{\left(1 - bz^k\right)^2} \\ &= kw(z) - \frac{k\left(z^{2k} - w(z)^2\right)}{1 - z^{2k}}, \end{aligned}$$

which yields equality in (2.10). ∎

Remark 1. To obtain some condition simpler than (2.9), we note that in a more symmetrical form, condition (2.9) holds if

$$\frac{(1 - A)\left(1 + Ar^{2k}\right)}{(1 - B)\left(1 + Br^{2k}\right)} \leqslant \left[\frac{1 - AC}{1 - BC}\right]^2. \tag{2.14}$$

Also, as

$$\frac{1 + Ar^{2k}}{1 + Br^{2k}} \leqslant \frac{1 + A}{1 + B},$$

condition (2.14) holds if

$$\frac{1 - A^2}{1 - B^2} \leqslant \left[\frac{1 - AC}{1 - BC}\right]^2.$$

This is equivalent to

$$A + B - 2(1 + AB)C + (A + B)C^2 \geqslant 0. \tag{2.15}$$

We note that, apart from the simple case $A = 1$, $B = -1$, inequality (2.15) is not satisfied if $A + B \leqslant 0$, $AB \neq -1$. This has eliminated several interesting cases such as the class $S^*(\alpha, -\alpha)$, where $A + B = 0$. Our next theorem will give conditions which cover this case.

Remark 2. As mentioned in the proof of Theorem 2, the minimum of $S(u, 0)$ can occur within the interval $[-d_{k,b}, d_{k,b}]$. In that case, the minimum value is

$$S(u_0, 0) = \beta\frac{A + B}{A - B} + \frac{2}{(A - B)(1 - r^{2k})}\left[(LK)^{1/2} - \beta k\left(1 - ABr^{2k}\right)\right].$$

It seems that this bound is not achieved by any function in $P_{k,b}(A, B)$.

THEOREM 3. *Let* $\alpha \geqslant 0$ $,\beta \geqslant 0$, $k = 1, 3, 5, \ldots$, $|z| = r < 1$, $D = (r^k + b)/(1 + br^k)$, $C = r^k D$. *Under the following conditions:*

(i) $A + B \geqslant 0$, $AB < A + B$,

(ii) $1 - r^{2k} + (A+B)r^{2k} - 2r^k(1 + ABr^{2k})D + r^{2k}(A + B - AB(1 - r^{2k}))D^2 > 0$, $0 < r < 1$, *we have for* $p(z) \in P_{k,b}(A, B)$ *that*

$$\mathrm{Re}\left\{\alpha p(z) + \beta\frac{zp'(z)}{p(z)}\right\} \geqslant \alpha\frac{1 - AC}{1 - BC} - \frac{(A-B)\beta k r^k}{1 - r^{2k}} \cdot \frac{r^k + (1 - r^{2k})D - r^k D^2}{1 - (A+B)r^k D + ABr^{2k}D^2}.$$

The result is sharp.

PROOF: Write

$$\varphi(z) = \frac{z^k\psi(z) + b}{1 + bz^k\psi(z)}, \qquad \psi(z) \in U.$$

Then, for $p(z) \in P_{k,b}(A, B)$, we have

$$zp'(z) = \frac{(A-B)z^k[k\varphi(z) + z\varphi'(z)]}{[1 + Bz^k\varphi(z)]^2}$$

which yields

$$\begin{aligned}
(2.16)\qquad \mathrm{Re}\left\{\frac{zp'(z)}{p(z)}\right\} &\geqslant -\frac{(A-B)\,|z|^k\,|k\varphi(z) + z\varphi'(z)|}{|p(z)|\,|1 + Bz^k\varphi(z)|^2} \\
&\geqslant -\frac{(A-B)\,|z|^k\,|k\varphi(z) + z\varphi'(z)|}{\mathrm{Re}\{p(z)\}\,|1 + Bz^k\varphi(z)|^2} \\
&= -\frac{(A-B)\,|zw'(z)|}{1 + (A+B)\,\mathrm{Re}\{z^k\varphi(z)\} + AB\,|z^k\varphi(z)|^2},
\end{aligned}$$

where $w(z) = z^k\varphi(z)$. Under the condition $A + B \geqslant 0$, $(A+B)\,\mathrm{Re}\{z^k\varphi(z)\} \geqslant -(A+B)\,|z^k|\,|\varphi(z)|$. In this case, (2.16) becomes

$$\mathrm{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant -\frac{(A-B)\,|zw'(z)|}{1 - (A+B)\,|z^k|\,|\varphi(z)| + AB\,|z^{2k}|\,|\varphi(z)|^2}.$$

Also, in view of Lemma 2,

$$|zw'(z)| \leqslant k\,|w(z)| + \frac{k\left(|z|^{2k} - |w(z)|^2\right)}{1 - |z|^{2k}};$$

thus,

$$(2.17)\qquad \mathrm{Re}\{\frac{zp'(z)}{p(z)}\} \geqslant -\frac{(A-B)k\,|z|^k}{1 - |z|^{2k}} \cdot \frac{\left(1 - |z|^{2k}\right)|\varphi(z)| + |z|^k - |z|^k\,|\varphi(z)|^2}{1 - (A+B)\,|z|^k\,|\varphi(z)| + AB\,|z|^{2k}\,|\varphi(z)|^2}.$$

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Put $|\varphi(z)| = x$, and denote the second factor on the RHS by $F(x)$; then

$$\frac{dF}{dx} = \frac{N(x)}{\left(1 - (A+B)r^k x + ABr^{2k}x^2\right)^2},$$

where

$$N(x) = 1 - r^{2k} + (A+B)r^{2k} - 2r^k\left(1 + ABr^{2k}\right)x + r^{2k}[A + B - AB\left(1 - r^{2k}\right)]x^2.$$

Now,

$$\frac{dN}{dx} = -2r^k\left(1 + ABr^{2k}\right) + 2r^{2k}[A + B - AB\left(1 - r^{2k}\right)]x.$$

Under the conditions $AB < A + B$, $A + B \geqslant 0$, we have $A + B - AB\left(1 - r^{2k}\right) > 0$. Thus, in this case,

$$\begin{aligned}\frac{dN}{dx} &< -2r^k\left(1 + ABr^{2k} - \left(A + B - AB + ABr^{2k}\right)\right)\\ &= -2r^k(1 - A)(1 - B) < 0.\end{aligned}$$

As a result, we have $N(x) \geqslant N(D)$ since $|\varphi(z)| \leqslant D$ on $|z| = r$. Consequently, $dF/dx > 0$ if $N(D) > 0$. The condition $N(D) > 0$ will then yield that $F(x) \leqslant F(D)$, that is,

$$\text{(2.18)} \qquad \operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant -\frac{(A-B)kr^k}{1 - r^{2k}} \cdot \frac{\left(1 - r^{2k}\right)D + r^k - r^k D^2}{1 - (A+B)r^k D + ABr^{2k}D^2}.$$

Putting (2.7) and (2.18) together, we obtain the lower bound for $\operatorname{Re}\{\alpha p(z) + \beta zp'(z)/p(z)\}$ of the theorem. As noted before, the lower bound of $\operatorname{Re}\{p(z)\}$ is achieved by taking $\psi(z) = -1$ in (2.6) (that is, by choosing $\varphi(z) = \left(b - z^k\right)/\left(1 - bz^k\right)$) and at the point $z = -r$. It is a simple exercise to check that every inequality used in this proof becomes an equality at $z = -r$, $k = 1, 3, 5, \ldots$ and for $\psi(z) = -1$. Hence the result is sharp. ∎

3. Radii of Convexity

As noted at the beginning of Section 2, the radius of convexity of $S^*_{k,b}(A, B)$ is given by the smallest root in $(0, 1]$ of the equation $M(r) = 0$, where

$$M(r) = \min_{p(z)\in P_{k,b}(A,B)} \min_{|z|=r<1} \operatorname{Re}\left\{p(z) + \frac{zp'(z)}{p(z)}\right\}.$$

An application of Theorem 2 with $\alpha = 1$, $\beta = 1$ gives $M(r)$, and solving $M(r) = 0$ we obtain

COROLLARY 1. *Let A, B, b be such that*

(3.1)
$$k(1-A)(1+Ar^{2k})(1-BC)^2 \leqslant [(A-B)(1-r^{2k})+k(1-B)(1+Br^{2k})](1-AC)^2$$

for $0<r<1$. Then the radius of convexity of $S^*_{k,b}(A,\,B)$ is given by the smallest root in $(0,\,1]$ of the equation

(3.2)
$$\begin{aligned}&[(A+B)\ (1-r^{2k})-2k(1-ABr^{2k})]\ (1+b(1-A)r^k-Ar^{2k})(1+b(1-B)r^k-Br^{2k})\\ &\quad +k(1-A)(1+Ar^{2k})(1+b(1-B)r^k-Br^{2k})^2\\ &\quad +[(A-B)(1-r^{2k})+k(1-B)(1+Br^{2k})](1+b(1-A)r^k-Ar^{2k})^2=0.\end{aligned}$$

It can be checked that the LHS is equal to $(2-k)A-kB$ at $r=0$ and it is equal to 0 at $r=1$. Thus, the above equation has at least one root within $(0,\,1]$. For the class $S^*_{k,b}(1,\,1/\alpha-1)$, it is seen immediately that condition (3.1) is satisfied for any b and for any $\alpha>1/2$. Thus, the radius of convexity for this class is determined completely as

COROLLARY 2. *The radius of convexity of $S^*_{k,b}(1,\,1/\alpha-1)$ is given by the smallest root in $(0,\,1]$ of equation (3.2) with $A=1$, $B=1/\alpha-1$.*

For the class $S^*_{k,b}(\alpha,\,0)$, condition (2.15) becomes $\alpha\geqslant 2C/(1+C^2)$. Thus, for this class, we have

COROLLARY 3. *The radius of convexity of $S^*_{k,b}(\alpha,\,0)$ is given by the smallest root in $(0,\,1]$ of equation (3.2) with $A=\alpha$, $B=0$ and $\alpha\geqslant 2C/(1+C^2)$.*

Using Theorem 3, we have

COROLLARY 4. *Let A, B, b satisfy conditions (i) and (ii) of Theorem 3. Then the radius of convexity of $S^*_{k,b}(A,\,B)$ is given by the smallest root in $(0,\,1]$ of the equation*

(3.3)
$$\begin{aligned}&(1-r^{2k})(1+b(1-A)r^k-Ar^{2k})[1+(2b-A-B)r^k\\ &\quad +(b^2(1-A)(1-B)-A-B)r^{2k}-b(A+B-2AB)r^{3k}+ABr^{4k}]\\ &\quad -(A-B)kr^k(1+b(1-B)r^k-Br^{2k})(1+2r^k-(1-b)r^{2k}-2r^{3k}-br^{4k})=0.\end{aligned}$$

Again, it can be checked that the LHS is equal to 1 at $r=0$ and 0 at $r=1$. Thus, the equation has at least one root within $(0,\,1]$. For the class $S^*_{k,b}(\alpha,\,0)$, condition (i) of Theorem 3 is obvious, while condition (ii) becomes

$$1-2r^k+(2\alpha-1)r^{2k}>0,\qquad 0<r<1.$$

Consequently, for this class, we have

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Corollary 5. *The radius of convexity of* $S^*_{k,b}(\alpha,\, 0)$ *is given by the smallest root* r_0 *in* $(0,\, 1]$ *of equation (3.3) with* $A = \alpha$, $B = 0$ *for such* α *that*

$$1 - 2r_0^k + (2\alpha - 1)r_0^{2k} > 0.$$

For the class $S^*_{k,b}(\alpha,\, -\alpha)$, we note that condition (i) of Theorem 3 is always satisfied, while conditions (ii) becomes

$$1 - 2r^k - \left(1 - \alpha^2\right)r^{2k} + 2\alpha^2 r^{3k} - \alpha^2 r^{4k} > 0, \quad 0 < r < 1.$$

Thus for this class, we get

Corollary 6. *The radius of convexity of* $S^*_{k,b}(\alpha,\, -\alpha)$ *is given by the smallest root* r_1 *in* $(0,\, 1]$ *of equation (3.3) with* $A = \alpha$, $B = -\alpha$ *for such* α *that*

$$1 - 2r_1^k - \left(1 - \alpha^2\right)r_1^{2k} + 2\alpha^2 r_1^{3k} - \alpha^2 r_1^{4k} > 0.$$

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