9

Inversion Formulas and Range

Let (M, g) be a simple two-dimensional manifold, and let $f \in C^\infty(M)$. We already know that f is determined by its geodesic X-ray transform uniquely and stably. In this chapter we will discuss the issues of reconstruction and range characterization, i.e. how to determine f from $I_0 f$ in a constructive way and how to decide which functions in $\partial_+ SM$ are of the form $I_0 f$ for some f.

In fact we will prove reconstruction formulas that allow one to exactly recover f from $I_0 f$ when (M, g) has constant curvature, and lead to approximate recovery with error terms given by Fredholm operators when (M, g) is a general simple surface. For the unit disk in the plane, the reconstruction formula is equivalent to the filtered backprojection formula (Theorem 1.3.3) after a suitable transformation is applied.

9.1 Motivation

This section motivates the derivation of the reconstruction formulas and introduces the operator W that will appear. Let (M, g) be a simple surface and let $f \in C^\infty(M)$ be real valued. We would like to reconstruct the function f in M from the knowledge of its geodesic X-ray transform $I_0 f$ on $\partial_+ SM$. Recall from Lemma 4.2.2 that the X-ray transform is characterized as $I_0 f = u^f|_{\partial_+ SM}$, where u^f solves the transport equation

$$Xu^f = -f \text{ in } SM, \qquad u^f|_{\partial_- SM} = 0.$$

The function u^f has the minor problem of not being smooth near $\partial_0 SM$, but this can be rectified by considering its odd part u^f_-. Since f is even, u^f_- is in $C^\infty(SM)$ by Theorem 5.1.2, and it satisfies

$$Xu^f_- = -f \text{ in } SM, \qquad u^f_-|_{\partial SM} = (I_0 f)_-,$$

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where $(I_0 f)_-$ is the odd part of the zero extension of $I_0 f$ to ∂SM, i.e.

$$(I_0 f)_-(x,v) := \begin{cases} \frac{1}{2} I_0 f(x,v), & (x,v) \in \partial_+ SM, \\ -\frac{1}{2} I_0 f(x,-v), & (x,v) \in \partial_- SM. \end{cases} \tag{9.1}$$

Now, if we could determine the solution u_-^f in SM from the knowledge of its boundary value $(I_0 f)_-$ on ∂SM, then we could reconstruct f just by using the equation $f = -Xu_-^f$. Of course an arbitrary solution of $Xu = -f$ is not determined uniquely by its boundary values (the solution u is only unique up to adding solutions of $Xr = 0$, i.e. invariant functions). However, uniqueness may follow if we impose additional conditions on u. One useful condition is that u is *holomorphic* in the angular variable.

We consider the following scheme:

$$\begin{cases} \textit{Produce a holomorphic odd function } u^* \in C^\infty(SM) \textit{ so that} \\ Xu^* = -f \textit{ in } SM \textit{ and } u^*|_{\partial SM} \textit{ is determined by } I_0 f. \end{cases} \tag{9.2}$$

If such a function u^* could be found, we could reconstruct a real f from $u^*|_{\partial SM}$ as follows: since $X(\mathrm{Im}(u^*)) = 0$, the function $\mathrm{Im}(u^*)$ is determined in SM by the boundary values $u^*|_{\partial SM}$. By holomorphicity u^* is determined by $\mathrm{Im}(u^*)$ (in principle, up to a real additive constant, but the fact that u^* is odd implies that $u_0^* = 0$ so this constant does not appear). We could then recover f from the equation $f = -Xu^*$.

Recall that u_-^f is a smooth odd solution of $Xu = -f$ and that $u_-^f|_{\partial SM}$ is determined by $I_0 f$. The first naive attempt to implement (9.2) would be to choose u^* to be (twice) the holomorphic projection of u_-^f, i.e.

$$u^* := (\mathrm{Id} + iH)u_-^f = 2(u_1^f + u_3^f + u_5^f + \cdots). \tag{9.3}$$

It turns out that this attempt already works if (M, g) has constant curvature. We formulate a related lemma.

Lemma 9.1.1 (Holomorphic projection of u_-^f) *Let (M, g) be a compact non-trapping surface with strictly convex boundary. If $f \in C^\infty(M)$, then $u^* := (\mathrm{Id} + iH)u_-^f \in C^\infty(SM)$ satisfies*

$$Xu^* = -f - iWf,$$

where W is the operator

$$W\colon C^\infty(M) \to C^\infty(M), \quad Wf = (X_\perp u^f)_0.$$

Proof To see this, recall from Definition 6.1.4 the Guillemin–Kazhdan operators $\eta_\pm = \frac{1}{2}(X \pm iX_\perp)$ that satisfy $\eta_\pm\colon \Omega_k \to \Omega_{k\pm 1}$. Using the decompositions

$$X = \eta_+ + \eta_-, \qquad iX_\perp = \eta_+ - \eta_-$$

together with the equation $Xu_-^f = -f$, we see that

$$\begin{aligned} Xu^* &= 2\eta_- u_1^f = \big(\eta_- u_1^f + \eta_+ u_{-1}^f\big) + \big(\eta_- u_1^f - \eta_+ u_{-1}^f\big) \\ &= -f - iWf. \end{aligned}$$

□

The operator W will be important for the reconstruction formulas. We will prove that it has the following three properties:

(1) If (M,g) has constant curvature, then $W \equiv 0$.
(2) If g is C^3-close to a metric of constant curvature, then W has small norm (Krishnan, 2010).
(3) If (M,g) is a general simple surface, then W is a smoothing operator (Pestov and Uhlmann, 2004).

By (1) we see that if (M,g) has constant curvature, then $Wf = 0$ and $Xu^* = -f$. Therefore the scheme (9.2) with the choice of u^* given in (9.3) allows us to reconstruct f from $I_0 f$. In the general case Wf is an error term. We may iterate the construction once more using the anti-holomorphic function

$$u^{**} := (\mathrm{Id} - iH)u_-^{f+iWf} = 2\big(u_{-1}^{f+iWf} + u_{-3}^{f+iWf} + \cdots\big).$$

Note that $X(u^* - u^{f+iWf}) = 0$, and $u^* - u^{f+iWf}|_{\partial_- SM} = u^*|_{\partial_- SM}$ is determined by $I_0 f$. Thus $I_0 f$ determines $u^{f+iWf}|_{\partial SM}$ and hence also $u^{**}|_{\partial SM}$. Now a computation as above yields that

$$Xu^{**} = -f - W^2 f.$$

It follows that the function $f + W^2 f$ can be reconstructed from $I_0 f$.

In the following sections we will prove the properties (1)–(3) of the operator W in detail. We will give a slightly different argument for reconstructing $f + W^2 f$ from $I_0 f$, based on using the fibrewise Hilbert transform H and the commutator formula $[H,X]u = X_\perp u_0 + (X_\perp u)_0$. To conclude this section, it is instructive to see why $W \equiv 0$ in the Euclidean case.

Example 9.1.2 (W in the Euclidean case) Let (M,g) be the Euclidean unit disk and let $f \in C_c^\infty(M^{\mathrm{int}})$. Then we may write

$$u^f(x,\theta) = \int_0^\infty f(x + tv_\theta)\,dt,$$

where $v_\theta = (\cos\theta, \sin\theta)$. Since $X_\perp = (v_\theta)_\perp \cdot \nabla_x$, we have

$$X_\perp u^f(x,\theta) = \int_0^\infty (v_\theta)_\perp \cdot \nabla_x f(x + tv_\theta)\,dt.$$

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We may then compute

$$\begin{aligned} Wf(x) = (X_\perp u^f)_0(x) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty (v_\theta)_\perp \cdot \nabla_x f(x+tv_\theta)\,dt\,d\theta \\ &= -\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_\varepsilon^\infty\int_0^{2\pi}\frac{\partial_\theta(f(x+tv_\theta))}{t}\,d\theta\,dt. \end{aligned}$$

One has $Wf(x) \equiv 0$ since $\int_0^{2\pi}\partial_\theta(f(x+tv_\theta))\,d\theta = 0$.

9.2 Properties of Solutions of the Jacobi Equation

Let (N,g) be a closed oriented two-dimensional manifold. We have seen in Section 3.7.2 that Jacobi fields on N are completely described by the smooth functions $a,b\colon SN\times\mathbb{R}\to\mathbb{R}$ that satisfy the Jacobi equation in the t-variable,

$$\ddot{a} + K(\gamma_{x,v}(t))a = 0, \qquad \ddot{b} + K(\gamma_{x,v}(t))b = 0,$$

with initial conditions $a(x,v,0) = 1$, $\dot{a}(x,v,0) = 0$, and $b(x,v,0) = 0$, $\dot{b}(x,v,0) = 1$.

The functions a and b have the following properties.

Proposition 9.2.1 *There exist smooth functions $R, P \in C^\infty(TN)$ such that*

$$a(x,v,t) = 1 + t^2R(x,tv), \tag{9.4}$$

$$b(x,v,t) = t + t^3P(x,tv). \tag{9.5}$$

Moreover, we have

$$b(x,v,t) = t\ \det(d\exp_x|_{tv}).$$

Proof We first consider $a(x,v,t)$. The initial conditions $a(x,v,0) = 1$ and $\dot{a}(x,v,0) = 0$ together with Taylor's formula imply that

$$a(x,v,t) = 1 + t^2c(x,v,t), \tag{9.6}$$

where $c \in C^\infty(SN\times\mathbb{R})$. By differentiating the equation $\ddot{a} + Ka = 0$ repeatedly we obtain

$$\partial_t^{k+2}a(x,v,0) = -\sum_{j=0}^{k}\binom{k}{j}(X^jK)(x,v)\partial_t^{k-j}a(x,v,0),$$

where X^j is the geodesic vector field applied j times. Using induction and the fact that $1 = g_{jk}v^jv^k$, we see that $\partial_t^{k+2}a(x,v,0)$ is a homogeneous polynomial of degree k in v. Thus by (9.6), $\partial_t^k c(x,v,0)$ is a homogeneous polynomial of degree k in v.

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We use Borel summation and define

$$c_1(x,v,t) := \sum_{k=0}^{\infty} \frac{\partial_t^k c(x,v,0)}{k!} t^k \chi(t/\varepsilon_k), \tag{9.7}$$

where $\chi \in C_c^\infty(\mathbb{R})$ satisfies $0 \le \chi \le 1$, $\chi = 1$ for $|t| \le 1/2$, and $\chi = 0$ for $|t| \ge 1$, and ε_k are chosen so that $c_1 \in C^\infty(SN \times \mathbb{R})$. Then $c = c_1 + c_2$ where $c_2 \in C^\infty(SN \times \mathbb{R})$ satisfies

$$\partial_t^k c_2(x,v,0) = 0, \qquad k \ge 0. \tag{9.8}$$

The formula (9.7) together with the fact that $\partial_t^k c(x,v,0)$ is a homogeneous polynomial of order k in v shows that $c_1(x,v,t) = R_1(x,tv)$ where $R_1 \in C^\infty(TN)$. Moreover, using (9.8) one can directly check that $R_2(x,w) := c_2(x, w/|w|, |w|)$ is smooth in TN with vanishing Taylor series when $w = 0$. Thus we have

$$a(x,v,t) = 1 + t^2 R(x,tv),$$

where $R := R_1 + R_2 \in C^\infty(TN)$.

The proof for $b(x,v,t)$ is analogous. First we observe that $b(x,v,t) = t + t^3 d(x,v,t)$ where d is smooth. By induction $\partial_t^{k+3} b(x,v,0)$, and hence also $\partial_t^k d(x,v,0)$, is a homogeneous polynomial of degree k in v. Thus $d(x,v,t) = P(x,tv)$ where P is smooth in TN. The formula $b(x,v,t) = t\ \det(d\exp_x|_{tv})$ follows from Remark 8.1.11 and Lemma 3.7.7. □

Remark 9.2.2 By differentiating the equations $\ddot{a} + Ka = 0$ and $\ddot{b} + Kb = 0$, it is easy to obtain the expansions

$$a = 1 - \frac{1}{2}Kt^2 - \frac{1}{6}dK|_x(v)t^3 + O(t^4),$$
$$b = t - \frac{1}{6}Kt^3 + O(t^4).$$

We also recall from Section 3.7.2 that the Jacobi equation $\ddot{y} + K(t)y = 0$ determines the differential of the geodesic flow φ_t: if we fix $(x,v) \in SM$ and $T_{(x,v)}(SM) \ni \xi = -\xi_1 X_\perp + \xi_2 V$ then

$$d\varphi_t(\xi) = -y(t)X_\perp(\varphi_t(x,v)) + \dot{y}(t)V(\varphi_t(x,v)), \tag{9.9}$$

where $y(t)$ is the unique solution to the Jacobi equation with initial conditions $y(0) = \xi_1$ and $\dot{y}(0) = \xi_2$ and $K(t) = K(\pi \circ \varphi_t(x,v))$ (cf. Section 3.7.2). The differential of the geodesic flow thus determines an $SL(2,\mathbb{R})$-cocyle Ψ over φ_t with infinitesimal generator

$$\mathcal{A} := \begin{pmatrix} 0 & -1 \\ K & 0 \end{pmatrix}.$$

This means that Ψ is the solution of the matrix ODE

$$\frac{d}{dt}\Psi(x,v,t) + \mathcal{A}(\varphi_t(x,v))\Psi(x,v,t) = 0, \qquad \Psi(x,v,0) = \mathrm{Id},$$

and satisfies the cocycle property

$$\Psi(x,v,t+s) = \Psi(\varphi_t(x,v),s)\,\Psi(x,v,t)$$

for all $(x,v) \in SN$ and $s,t \in \mathbb{R}$. We may write Ψ using the functions a,b above as

$$\Psi(x,v,t) = \begin{pmatrix} a & b \\ \dot{a} & \dot{b} \end{pmatrix}.$$

Clearly the cocycle Ψ can be identified with $d\varphi_t$ acting on the kernel of the contact 1-form of the geodesic flow (i.e. the 2-plane spanned by $X_\perp$ and V).

9.3 The Smoothing Operator *W*

Let (M,g) be a non-trapping surface with strictly convex boundary. We consider as usual (M,g) sitting inside a closed oriented surface (N,g). We shall define an operator $W\colon C^\infty(M) \to C^\infty(M)$ following our discussion at the begining of the chapter. This operator will have the property that it extends as a smoothing operator $W\colon L^2(M) \to C^\infty(M)$ when M is free of conjugate points, and it will play an important role in the Fredholm inversion formulas in the next section.

Given $f \in C^\infty(M)$, define for any $x \in M$,

$$(Wf)(x) := \big(X_\perp u^f\big)_0(x) = \frac{1}{2\pi}\ell_0^*\big(X_\perp u^f\big)(x).$$

In the definition above we may replace u^f by u_-^f and we have seen that the latter is smooth (cf. Theorem 5.1.2). Hence we have

$$Wf = \big(X_\perp u_-^f\big)_0 \in C^\infty(M).$$

Exercise 9.3.1 Show that $Wf = i(\eta_- u_1^f - \eta_+ u_{-1}^f)$.

We now give an integral representation for W when (M,g) is a simple surface. We will use the functions a and b introduced in the previous section. Note that (M,g) has no conjugate points if and only if $b(x,v,t) \neq 0$ for $t \in [-\tau(x,-v),\tau(x,v)]$, $t \neq 0$ and $(x,v) \in SM$.

Proposition 9.3.2 *Let (M,g) be a simple surface. The function*

$$w(x,v,t) := V\left(\frac{a(x,v,t)}{b(x,v,t)}\right)$$

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is smooth for $(x,v) \in SM$ *and* $t \in [-\tau(x,-v), \tau(x,v)]$*, and has the form*

$$w(x,v,t) = tQ(x,tv),$$

where Q *is smooth. The operator* W *has the expression*

$$(Wf)(x) = \frac{1}{2\pi} \int_{S_x M} \int_0^{\tau(x,v)} w(x,v,t) f(\gamma_{x,v}(t))\, dt\, dS_x(v).$$

The function $w = w(x,v,t)$ *also has the formula*

$$w = -\frac{1}{b(t)^2} \int_0^t g(t,s) b(s) \left[a(s)b(t) - b(s)a(t)\right] dK|_{\gamma_{x,v}(s)}(\dot{\gamma}_{x,v}(s)^{\perp})\, ds,$$

with $a(t) = a(x,v,t)$*,* $b(t) = b(x,v,t)$*,* $g(t,s) = b(\gamma_{x,v}(s), \dot{\gamma}_{x,v}(s), t-s)$*. In particular,* $W \equiv 0$ *if* (M,g) *has constant curvature.*

Proof By simplicity $b(x,v,t) \neq 0$ for $t \in [-\tau(x,-v), \tau(x,v)]$ and $t \neq 0$. Thus w is smooth for $t \neq 0$. Using the definition of w we have that

$$w(x,v,t) = \frac{V(a)}{b} - \frac{aV(b)}{b^2}.$$

Proposition 9.2.1 gives that

$$a(x,v,t) = 1 + t^2 R(x,tv),$$
$$b(x,v,t) = t + t^3 P(x,tv),$$

where R and P are smooth. Since $V = (0, v^{\perp})$ in the splitting (3.12), we have in the notation of Section 3.6 and in terms of the Sasaki metric on TM that

$$V(R(x,tv)) = \left\langle \nabla R|_{(x,tv)}, (0, tv^{\perp}) \right\rangle = \left\langle \mathrm{K}(\nabla R|_{(x,tv)}), tv^{\perp} \right\rangle$$
$$= \langle \mathrm{K}(\nabla R|_{(x,tv)})_{\perp}, tv \rangle.$$

Performing a similar computation for $V(P(x,tv))$, it follows that

$$V(a) = t^2 \hat{R}(x,tv),$$
$$V(b) = t^3 \hat{P}(x,tv),$$

where $\hat{R}$ and $\hat{P}$ are smooth (and $\hat{R}(x,0) = \hat{P}(x,0) = 0$). Since $b = t \det(d\exp_x|_{tv})$, we have that $V(a)/b$ and $aV(b)/b^2$ are of the form $tS(x,tv)$ for some smooth S (for the latter we also use $t^2 = g_{jk} tv^j tv^k$). It follows that

$$w(x,v,t) = tQ(x,tv)$$

for some smooth Q.

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To derive the integral formula for W we use its definition and write

$$(Wf)(x) = \frac{1}{2\pi}\int_{S_xM} X_\perp\left[\int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\,dt\right] dS_x(v). \tag{9.10}$$

Let us assume first that f has compact support contained in the interior of M. Then,

$$X_\perp \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\,dt = \int_0^{\tau(x,v)} X_\perp(f(\gamma_{x,v}(t)))\,dt.$$

Now observe that

$$X_\perp(f(\gamma_{x,v}(t))) = df \circ d\pi \circ d\varphi_t(X_\perp(x,v)),$$

and similarly

$$V(f(\gamma_{x,v}(t))) = df \circ d\pi \circ d\varphi_t(V(x,v)).$$

But by (9.9),

$$d\pi \circ d\varphi_t(X_\perp(x,v)) = -a\dot{\gamma}_{x,v}(t)^\perp$$

and

$$d\pi \circ d\varphi_t(V(x,v)) = b\dot{\gamma}_{x,v}(t)^\perp,$$

therefore for $t \neq 0$,

$$X_\perp(f(\gamma_{x,v}(t)) = df(-a\dot{\gamma}_{x,v}(t)^\perp) = -\frac{a}{b}V(f(\gamma_{x,v}(t)).$$

Inserting the last expression into (9.10) we derive

$$(Wf)(x) = \frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{S_xM}\int_\varepsilon^{\tau(x,v)} -\frac{a}{b}V(f(\gamma_{x,v}(t)))\,dt\,dS_x(v).$$

Since

$$\int_{S_xM} V\left(\int_\varepsilon^{\tau(x,v)} \frac{a}{b} f(\gamma_{x,v}(t))\,dt\right) dS_x(v) = 0,$$

and since $V(a/b)$ is smooth, we finally obtain

$$(Wf)(x) = \frac{1}{2\pi}\int_{S_xM}\int_0^{\tau(x,v)} V\left(\frac{a}{b}\right) f(\gamma_{x,v}(t))\,dt\,dS_x(v)$$

as desired.

Next, differentiating the ODEs for $a(t) = a(x,v,t)$ and $b(t) = b(x,v,t)$ yields

$$(Va)''(t) + K(\gamma_{x,v}(t))Va(t) = -dK|_{\gamma_{x,v}(t)}(d\pi \circ d\varphi_t(V(x,v)))a(t),$$
$$(Vb)''(t) + K(\gamma_{x,v}(t))Vb(t) = -dK|_{\gamma_{x,v}(t)}(d\pi \circ d\varphi_t(V(x,v)))b(t).$$

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But we saw that $d\pi \circ d\varphi_t(V(x,v)) = b(t)\dot{\gamma}_{x,v}(t)^{\perp}$. Duhamel's principle gives

$$Va(t) = -\int_0^t b(\gamma_{x,v}(s), \dot{\gamma}_{x,v}(s), t-s)a(s)b(s)dK|_{\gamma_{x,v}(s)}\big(\dot{\gamma}_{x,v}(s)^{\perp}\big)\,ds,$$

$$Vb(t) = -\int_0^t b(\gamma_{x,v}(s), \dot{\gamma}_{x,v}(s), t-s)b(s)b(s)dK|_{\gamma_{x,v}(s)}\big(\dot{\gamma}_{x,v}(s)^{\perp}\big)\,ds.$$

Now

$$w(x,v,t) = V\left(\frac{a(x,v,t)}{b(x,v,t)}\right) = \frac{(Va)b - a(Vb)}{b^2},$$

and the required formula for $w(x,v,t)$ follows.

The proof above was done assuming that $f \in C_c^\infty(M^{\mathrm{int}})$ but we could have carried out the same proof with $f \in C^\infty(M)$, i.e. smooth and supported all the way to the boundary. This would have produced two additional boundary terms:

$$X_\perp(\tau)f(\gamma_{x,v}(\tau(x,v))) \quad \text{and} \quad V(\tau)\frac{a(x,v,\tau(x,v))}{b(x,v,\tau(x,v))}f(\gamma_{x,v}(\tau(x,v))).$$

However these two terms cancel out due to the following fact, which is easily checked:

$$a(x,v,\tau(x,v))V(\tau) + b(x,v,\tau(x,v))X_\perp(\tau) = 0. \tag{9.11}$$

Hence we get the same integral formula for $f \in C^\infty(M)$. □

Exercise 9.3.3 Prove identity (9.11).

Exercise 9.3.4 Use Proposition 9.3.2 to show that if g is sufficiently C^3-close to a metric of constant curvature, then $\|W\|_{L^2} < 1$ (cf. Krishnan (2010)).

Exercise 9.3.5 Let $F := b^2 w$. Show that F satisfies the ODE (in time)

$$\dddot{F} + 4K(\gamma_{x,v}(t))\dot{F} + 2dK(\dot{\gamma}_{x,v}(t))F = -2V(K(\gamma_{x,v}(t))).$$

Show that $W = 0$ if and only if K is constant.

We now prove that W is smoothing on simple surfaces.

Proposition 9.3.6 *Let (M,g) be a simple surface. The operator W extends to a smoothing operator $W\colon L^2(M) \to C^\infty(M)$.*

Proof We will make a change of variables that transforms the integral expression for W into something of the form

$$(Wf)(x) = \int_M k(x,y)f(y)\,dV^2(y),$$

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with k smooth. The change of variables is exactly the same we used in the proof of Theorem 8.1.1. We set $\psi_x(v,t) := y = \exp_x(tv)$ and we see that

$$(Wf)(x) = \int_M k(x,y) f(y)\, dV^2(y),$$

where

$$k(x,y) := \frac{w(x,\psi_x^{-1}(y))}{b(x,\psi_x^{-1}(y))}.$$

Using Proposition 9.3.2 we can rewrite this as

$$k(x,y) = \frac{Q(x,\, \exp_x^{-1}(y))}{\det(d\exp_x|_{\exp_x^{-1}(y)})},$$

which clearly exhibits k as a smooth function. □

9.3.1 The Adjoint W^*

The adjoint of W with respect to the L^2-inner product of M can be easily computed:

Lemma 9.3.7 *Given $h \in C_c^\infty(M^{\text{int}})$ we have*

$$W^* h = \left(u^{X_\perp h}\right)_0.$$

Before proving the lemma we establish an auxiliary result that holds in any dimensions.

Lemma 9.3.8 *If $f \in L^2(SM)$ is even and $g \in L^2(SM)$ is odd, then*

$$(If, Ig)_{L^2_\mu} = 0.$$

Proof It suffices to check the claim when f and g are smooth and with compact support in M^{int}. We have

$$(If, Ig)_{L^2_\mu} = \int_{\partial_+ SM} \mu\, If\, Ig\, d\Sigma^{2n-2} = 2\int_{\partial SM} \mu u_+^f u_-^g\, d\Sigma^{2n-2}.$$

Since $Xu_+^f = Xu_-^g = 0$ we have $X(u_+^f u_-^g) = 0$, and using Proposition 3.5.12 we obtain

$$\int_{\partial SM} \mu u_+^f u_-^g\, d\Sigma^{2n-2} = 0.$$ □

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Proof of Lemma 9.3.7 Given $f, h \in C_c^\infty(M^{\text{int}})$ we compute

$$\begin{aligned}2\pi(Wf,h)_{L^2(M)} &= 2\pi\left(\left(X_\perp u^f\right)_0, h\right)_{L^2(M)}\\ &= (X_\perp u^f, h)_{L^2(SM)}\\ &= -(u^f, X_\perp h)_{L^2(SM)}\\ &= (u^f, X(u^{X_\perp h}))_{L^2(SM)}\\ &= -(Xu^f, u^{X_\perp h})_{L^2(SM)} - (If, I(X_\perp h))_{L^2_\mu}\\ &= (f, u^{X_\perp h})_{L^2(SM)}\\ &= 2\pi\left(f, \left(u^{X_\perp h}\right)_0\right)_{L^2(M)},\end{aligned}$$

where in the penultimate line we used Lemma 9.3.8. □

9.4 Fredholm Inversion Formulas

In this section we establish an inversion formula for I_0 up to a Fredholm error using the smoothing operator W. This formula was proved in Pestov and Uhlmann (2004), and we partly follow the presentation in Monard (2016b). We begin by proving the following result.

Theorem 9.4.1 *Let (M, g) be a compact non-trapping surface with strictly convex boundary. Then given $f \in C^\infty(M)$ we have*

$$f + W^2 f = -(X_\perp w^\sharp)_0,$$

where

$$w := [H\,(I_0 f)_-]|_{\partial_- SM} \circ \alpha,$$

and $(I_0 f)_-$ denotes the odd part of the zero extension of $I_0 f$ to ∂SM as in (9.1).

Proof The proof essentially consists in applying the Hilbert transform H twice to the equation $Xu_-^f = -f$ and using Proposition 6.2.2.

Applying H once we derive (since $Hf = 0$):

$$XHu_-^f = -Wf, \tag{9.12}$$

since $(u_-^f)_0 = 0$. Applying H again we obtain

$$XH^2 u_-^f + \left(X_\perp H u_-^f\right)_0 = 0,$$

and using that $H^2 u_-^f = -u_-^f$ we derive

$$-f = Xu_-^f = \left(X_\perp H u_-^f\right)_0. \tag{9.13}$$

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Using (9.12) we see that

$$Hu_-^f = u^{Wf} + w^\sharp,$$

where $w := [Hu_-^f]|_{\partial_- SM} \circ \alpha \in C^\infty(\partial_+ SM)$. Inserting this expression into (9.13) yields

$$-f - W^2 f = (X_\perp w^\sharp)_0,$$

and the proof is completed by observing that

$$u_-^f|_{\partial SM} = (I_0 f)_-. \qquad \square$$

Exercise 9.4.2 Using (9.12) show that $I_0(Wf) = 0$ if $I_0 f = 0$.

The term $(X_\perp w^\sharp)_0$ appearing in the formula in Theorem 9.4.1 can be interpreted as the adjoint of a suitable X-ray transform.

Definition 9.4.3 Let (M, g) be a non-trapping surface with strictly convex boundary. We set $I_\perp : C^\infty(M) \to C^\infty(\partial_+ SM)$ as

$$I_\perp(f) := I(X_\perp \ell_0 f).$$

Exercise 9.4.4 Let (M, g) be a simple surface. Show that $I_\perp(f) = 0$ if and only if f is constant.

By Proposition 3.5.12 we know that $X_\perp^* = -X_\perp$ if we let $X_\perp$ act on C^1-functions that are zero on ∂SM. Hence the formal adjoint $I_\perp^*$ is given by

$$I_\perp^*(w) = -\ell_0^* X_\perp I^*(w) = -2\pi (X_\perp w^\sharp)_0. \tag{9.14}$$

Next we shall reinterpret the term

$$w = [H\,(I_0 f)_-]|_{\partial_- SM} \circ \alpha$$

using suitable boundary operators. For this we need to have a preliminary discussion on objects at the boundary.

9.4.1 Boundary Operators

Let (M, g) be a non-trapping manifold with strictly convex boundary. We introduce the operators of even and odd continuation with respect to α:

$$A_\pm w(x, v) := \begin{cases} w(x, v) & \text{if } (x, v) \in \partial_+ SM, \\ \pm w(\alpha(x, v)) & \text{if } (x, v) \in \partial_- SM. \end{cases}$$

Recall that the operator A_+ already appeared in Section 5.1. Clearly $A_\pm : C(\partial_+ SM) \to C(\partial SM)$. We will examine next the boundedness properties of $A_\pm$.

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Lemma 9.4.5 *$A_\pm\colon L^2_\mu(\partial_+ SM) \to L^2_{|\mu|}(\partial SM)$ are bounded.*

Proof We compute

$$\begin{aligned}\|A_\pm w\|^2_{L^2_{|\mu|}(\partial SM)} &= \int_{\partial_+ SM} |w|^2 \mu\, d\Sigma^{2n-2} + \int_{\partial_- SM} |\alpha^* w|^2 \big(-\mu\, d\Sigma^{2n-2}\big)\\ &= \int_{\partial_+ SM} |w|^2 \mu\, d\Sigma^{2n-2} + \int_{\partial_+ SM} |w|^2 \alpha^*\big(\mu\, d\Sigma^{2n-2}\big).\end{aligned}$$

In the second term we used that α reverses orientation. By Proposition 3.6.8 we know that

$$\alpha^*\big(\mu\, d\Sigma^{2n-2}\big) = \mu\, d\Sigma^{2n-2},$$

and the lemma follows. □

The adjoint $A^*_\pm\colon L^2_{|\mu|}(\partial SM) \to L^2_\mu(\partial_+ SM)$ satisfies

$$\begin{aligned}(A_\pm w, u)_{L^2_{|\mu|}(\partial SM)} &= \int_{\partial_+ SM} w\bar{u}\mu\, d\Sigma^{2n-2} \pm \int_{\partial_- SM} (w\circ\alpha)\bar{u}\big(-\mu\, d\Sigma^{2n-2}\big)\\ &= \int_{\partial_+ SM} w(\bar{u} \pm \bar{u}\circ\alpha)\mu\, d\Sigma^{2n-2},\end{aligned}$$

so

$$A^*_\pm u = (u \pm u\circ\alpha)|_{\partial_+ SM}. \tag{9.15}$$

The boundary operator A^*_- can be used to give a very simple description of the range of I.

Proposition 9.4.6 *Let (M, g) be a non-trapping surface with strictly convex boundary. A function $q \in C^\infty(\partial_+ SM)$ belongs to the range of*

$$I\colon C^\infty(SM) \to C^\infty(\partial_+ SM)$$

if and only if there is $w \in C^\infty(\partial SM)$ such that $q = A^*_- w$.

Proof If $q \in C^\infty(\partial_+ SM)$ is in the the range of I, there is a smooth $f \in C^\infty(SM)$ such that $If = q$. Using Proposition 3.3.1 we know there is $u \in C^\infty(SM)$ such that $Xu = f$ and integrating this equation between boundary points we obtain $u\circ\alpha - u = If$. Thus if we set $w = -u|_{\partial SM}$, then $q = If = A^*_- w$.

Conversely, if $q = A^*_- w$ for $w \in C^\infty(\partial SM)$, we extend w to a smooth function on SM, still denoted by w. Now set $f := -Xw$ and once again, integrating between boundary points we see that $If = A^*_- w = q$ as desired. □

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Remark 9.4.7 Note that the previous proposition holds in any dimension with the operator A_-^* defined by (9.15).

9.4.2 Symmetries in Data Space

Let $\mathrm{a}\colon SM \to SM$ denote the antipodal map on each fibre, $\mathrm{a}(x,v) := (x,-v)$. Clearly $\mathrm{a}\colon \partial SM \to \partial SM$. Define a new involution combining the scattering relation with a as

$$\alpha_{\mathrm{a}} := \alpha \circ \mathrm{a} = \mathrm{a} \circ \alpha.$$

From the definitions we see that

$$\alpha_{\mathrm{a}}\colon \partial_\pm SM \to \partial_\pm SM.$$

Lemma 9.4.8 *Let (M,g) be a non-trapping manifold with strictly convex boundary and let $f \in C^\infty(SM)$. Then*

$$I(f) \circ \alpha_{\mathrm{a}} = I(f \circ \mathrm{a}).$$

Proof Using that $\mathrm{a} \circ \varphi_t = \varphi_{-t} \circ \mathrm{a}$ and $\tau \circ \alpha_{\mathrm{a}} = \tau$, we write for $(x,v) \in \partial_+ SM$,

$$\begin{aligned} I(f) \circ \alpha_{\mathrm{a}}(x,v) &= \int_0^{\tau(\alpha_{\mathrm{a}}(x,v))} f(\varphi_t(\alpha_{\mathrm{a}}(x,v))\,dt \\ &= \int_0^{\tau(x,v)} f \circ \mathrm{a}(\varphi_{-t} \circ \alpha(x,v))\,dt \\ &= \int_0^{\tau(x,v)} f \circ \mathrm{a}(\varphi_{\tau(x,v)-t}(x,v))\,dt \quad = I(f \circ \mathrm{a})(x,v) \end{aligned}$$

as desired. □

This lemma motivates the following decomposition in data space:

$$C^\infty(\partial_+ SM) = \mathcal{V}_+ \oplus \mathcal{V}_-, \tag{9.16}$$

where

$$\mathcal{V}_\pm = \{f \in C^\infty(\partial_+ SM) : \ f \circ \alpha_{\mathrm{a}} = \pm f\}.$$

Lemma 9.4.9 *Given $h \in C^\infty(\partial_+ SM)$ we have*

$$h^\sharp \circ \mathrm{a} = (h \circ \alpha_{\mathrm{a}})^\sharp.$$

In particular, if $h \in \mathcal{V}_+$ ($\mathcal{V}_-$), then the function $h^\sharp$ is even (odd) in SM.

Proof Using the definition of $h^\sharp$ and α we write

$$\begin{aligned}(h\circ\alpha_{\mathtt{a}})^\sharp &= h(\alpha(\mathtt{a}(\varphi_{-\tau(x,-v)}(x,v))))\\ &= h(\alpha(\varphi_{\tau(x,-v)}(x,-v)))\\ &= h(\varphi_{-\tau(x,v)}(x,-v))\\ &= h^\sharp\circ\mathtt{a}\end{aligned}$$

as claimed. □

Exercise 9.4.10 Show that the decomposition (9.16) is orthogonal with respect to the L^2_μ-inner product on ∂_+SM.

We are now ready to prove the following inversion formula up to the Fredholm error W^2.

Theorem 9.4.11 *Let (M,g) be a compact non-trapping surface with strictly convex boundary. Then given $f\in C^\infty(M)$ we have*

$$f+W^2f=\frac{1}{8\pi}I_\perp^*A_+^*HA_-I_0(f).$$

Proof As in Theorem 9.4.1 we let

$$w:=\alpha^*H(I_0f)_-|_{\partial_+SM}.$$

Using Lemma 9.4.8 we see that

$$A_-(I_0(f))=2(I_0f)_- \tag{9.17}$$

and hence by (9.15) we may write

$$w=\alpha^*H(I_0f)_-|_{\partial_+SM}=\frac{1}{4}(A_+^*-A_-^*)HA_-I_0(f).$$

A simple inspection using (9.17) reveals that

$$A_-^*HA_-I_0(f)\in\mathcal{V}_+$$

and hence by Lemma 9.4.9 and (9.14) this function is annihilated by $I_\perp^*$ (note that $X_\perp$ maps even functions to odd functions). This yields

$$I_\perp^*(w)=\frac{1}{4}I_\perp^*A_+^*HA_-I_0(f).$$

The claimed formula now follows from Theorem 9.4.1 and (9.14). □

Exercise 9.4.12 Let (M,g) be a non-trapping surface with strictly convex boundary. Show that given $f\in C^\infty(M)$ such that $f|_{\partial M}=0$, we have

$$f+(W^*)^2f=-\frac{1}{8\pi}I_0^*A_+^*HA_-I_\perp(f).$$

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Does the equation hold if we do not require $f|_{\partial M} = 0$? (Hint: consider the case of the Euclidean disk.)

Remark 9.4.13 The equations in Theorem 9.4.11 and Exercise 9.4.12 provide approximate inversion formulas for I_0 and $I_\perp$. The formulas become exact only in constant curvature. The boundary operator $A_+^* H A_-$ could be interpreted as a filter that is applied to the data $I_0(f)$, before the backprojection operation of applying $I_\perp^*$. In this sense the analogy with the filtered backprojection formula in Theorem 1.3.3 for the Euclidean case is evident. Note that the formulas are valid on any non-trapping surface with strictly convex boundary. The absence of conjugate points (i.e. simplicity) is only used when claiming that W is a smoothing operator.

The fact that the formulas become exact in constant curvature, and in particular in the case of the unit disk in the plane, raises the question (with stentorian voice) as to how the inversion formula given by Theorem 9.4.1 relates to the filtered backprojection formula (FBP) in Theorem 1.3.3. In the next section we shall see how to derive Theorem 1.3.3 from Theorem 9.4.1 when f is supported in the interior of the unit disk in $\mathbb{R}^2$. This will be achieved by introducing a suitable transformation between fan-beam geometry and parallel-beam geometry. But first we give some general remarks concerning the Hilbert transform.

9.4.3 Alternative Expressions for the Hilbert Transform

We let (M, g) be a non-trapping surface with strictly convex boundary. The fibrewise Hilbert transform was introduced in Definition 6.2.1. There is an alternative way of writing the transform in terms of the principal value of an integral over each $S_x M$. More precisely we may write:

$$Hu(x,w) = \frac{1}{2\pi}\,\text{p.v.}\int_{S_x M} \frac{1 + \langle v, w\rangle}{\langle v, w_\perp\rangle} u(x,v)\, dS_x(v). \tag{9.18}$$

Exercise 9.4.14 Prove that (9.18) is equivalent to Definition 6.2.1.

The next lemma provides an integral expression for the function Hu_-^f, where $f \in C^\infty(M)$. Recall that $u_-^f|_{\partial SM} = (I_0 f)_-$.

Lemma 9.4.15 *We have for* $(x, w) \in SM$*,*

$$Hu_-^f(x,w) = \frac{1}{2\pi}\,\text{p.v.}\int_{S_x M} \frac{1}{\langle v, w_\perp\rangle}\left(\int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt\right) dS_x(v).$$

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Remark 9.4.16 If we use the special coordinates in Lemma 3.5.6 and think of v as an angle $\theta \in [0, 2\pi]$ and w also as an angle $\eta \in [0, 2\pi]$, we may alternatively write

$$Hu_-^f(x,\eta) = \frac{1}{2\pi}\,\text{p.v.}\int_0^{2\pi} \frac{1}{\sin(\eta-\theta)} \left(\int_0^{\tau(x,\theta)} f(\gamma_{x,\theta}(t))\,dt \right) d\theta.$$

Proof of Lemma 9.4.15 The following is true for any u:

$$H_- u(x,w) = \frac{1}{2\pi}\,\text{p.v.}\int_{S_x M} \frac{u(x,v)}{\langle v, w_\perp\rangle}\,dS_x(v),$$

where $H_- u := H(u_-)$. This follows from (9.18) by observing that the kernel of the Hilbert transform splits into odd and even (in v) as

$$\frac{1+\langle v,w\rangle}{\langle v, w_\perp\rangle} = \frac{1}{\langle v, w_\perp\rangle} + \frac{\langle v,w\rangle}{\langle v, w_\perp\rangle}.$$

The proof of the lemma is completed by recalling that

$$u^f(x,v) = \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\,dt. \qquad \square$$

9.5 Revisiting the Euclidean Case

In this section we let $M = \overline{\mathbb{D}}$ be the closure of the unit disk in $\mathbb{R}^2$. Suppose f is a smooth function supported inside the disk. We use the notation $Rf(s,w)$ to indicate the Radon transform of f in parallel-beam coordinates as in Section 1.1. In other words,

$$Rf(s,w) := \int_{-\infty}^{\infty} f(sw + tw^\perp)\,dt,$$

where $(s,w) \in \mathbb{R} \times S^1$. Note that $Rf(s,w) = 0$ for s outside $[-1,1]$. We let H^s denote the standard Hilbert transform in the variable s:

$$(H^s g)(s,w) = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{g(t,w)}{s-t}\,dt.$$

Our first task is to introduce a suitable transformation mapping from SM (and ∂SM in particular) to the parallel-beam coordinates $(s,w) \in [-1,1] \times S^1$.

Define $\mathbf{h}\colon SM \to [-1,1] \times S^1$ by

$$\mathbf{h}(x,w) := (\langle x, w_\perp\rangle, w_\perp).$$

We also define

$$h := \mathbf{h}|_{\partial SM}.$$

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Since the geodesic flow is $\varphi_t(x,v) = (x+tv, v)$ we see that $\mathbf{h}\circ\varphi_t = \mathbf{h}$. In terms of $(x,w) \in \partial SM$, we may express the scattering relation quite nicely as

$$\alpha(x,w) = (x - 2\langle x,w\rangle w, w).$$

We may check directly that $h\circ\alpha = h$ (obviously it also follows from the fact that $\mathbf{h}$ remains constant along geodesics).

The next lemma is an important observation to relate the Pestov–Uhlmann formula with the FBP formula in Theorem 1.3.3 (compare with Boman and Strömberg (2004, equation (2.12))).

Lemma 9.5.1 *We have*

$$Hu_-^f = -\frac{1}{2}\mathbf{h}^* H^s Rf.$$

Proof Using Lemma 9.4.15 we may write

$$Hu_-^f(x,w) = \frac{1}{2\pi}\,\text{p.v.}\int_{S_xM}\frac{1}{\langle v,w_\perp\rangle}\left(\int_0^\infty f(x+tv)\,dt\right)dS_x(v). \quad (9.19)$$

The key change of variables is given as follows. Given $y\in\mathbb{R}^2$ we write it as

$$y = x+tv = r_1 w + r_2 w^\perp, \quad (9.20)$$

taking advantage of the fact that $\{w, w^\perp\}$ is an oriented orthonormal basis of $\mathbb{R}^2$. The change of variables $(t,v)\mapsto(r_1,r_2)$ relates the area elements as

$$t\,dt\,dS_x(v) = dr_1\,dr_2.$$

From (9.20) we see that

$$\langle x, w^\perp\rangle + t\langle v, w^\perp\rangle = r_2,$$

and thus we may transform the integral in (9.19) to

$$\begin{aligned} Hu_-^f(x,w) &= \frac{1}{2\pi}\,\text{p.v.}\int_{-\infty}^\infty \frac{dr_2}{\langle x,w^\perp\rangle - r_2}\left(\int_{-\infty}^\infty f\big(r_1 w + r_2 w^\perp\big)\,dr_1\right) \\ &= \frac{1}{2\pi}\,\text{p.v.}\int_{-\infty}^\infty \frac{Rf(-r_2, w_\perp)}{\langle x,w^\perp\rangle - r_2}\,dr_2 \\ &= -\frac{1}{2}H^s Rf(\langle x,w_\perp\rangle, w_\perp). \end{aligned}$$

□

Remark 9.5.2 Since $h\circ\alpha = h$, the formula above implies that $H(I_0 f)_-$ is invariant under α. This is a peculiarity of constant curvature since, in general, $XHu_-^f = -Wf$ and $Wf = 0$ in constant curvature. (Recall that $u_-^f|_{\partial SM} = (I_0 f)_-$.)

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9.5.1 $X_\perp$ and $\frac{d}{ds}$

Given $p \in C^\infty([-1,1] \times S^1)$ we can pull it back via h to obtain $h^*p \in C^\infty(\partial SM)$. Moreover, $(h^*p) \circ \alpha = h^*p$ and thus by Theorem 5.1.1 this function gives rise to a smooth first integral on SM that we denote by $(h^*p)^\sharp$. Clearly $(h^*p)^\sharp = \mathbf{h}^*p$, which is very convenient.

Lemma 9.5.3 *We have*

$$X_\perp(h^*p)^\sharp = \left(h^*\frac{\partial p}{\partial s}\right)^\sharp.$$

Equivalently

$$X_\perp(\mathbf{h}^*p) = \mathbf{h}^*\frac{\partial p}{\partial s}.$$

Proof The flow of $X_\perp$ is simply $\psi_t(x,v) = (x + tv_\perp, v)$. Thus

$$\mathbf{h}^*p(\psi_t(x,v)) = p(\mathbf{h}(x+tv_\perp, v)) = p(\langle x, v_\perp\rangle + t, v_\perp).$$

Differentiating at $t = 0$ we obtain:

$$X_\perp(\mathbf{h}^*p)(x,v) = \frac{\partial p}{\partial s}(\mathbf{h}(x,v)) = \left(\mathbf{h}^*\frac{\partial p}{\partial s}\right)(x,v)$$

as desired. □

9.5.2 Deriving the FBP from Theorem 9.4.1

To finish off, we define $w := H(I_0 f)_-|_{\partial SM}$ and note that by Lemma 9.5.1 one has $w = -\frac{1}{2}h^*H^sRf$. Defining $p := -\frac{1}{2}H^sRf$ we have $w = h^*p$, so $w^\sharp = \mathbf{h}^*p$. Now Lemma 9.5.3 gives that

$$X_\perp w^\sharp = -\frac{1}{2}\mathbf{h}^*\left(\frac{d}{ds}H^sRf\right).$$

Theorem 9.4.1 in the constant curvature case (so that $W = 0$) together with Remark 9.5.2 will tell us that

$$f = -(X_\perp w^\sharp)_0.$$

Let $g := \frac{d}{ds}H^sRf$. Then performing the fibrewise average and using the definition of $\mathbf{h}$, we derive

$$\begin{aligned} f(x) &= \frac{1}{4\pi}\int_{S_xM} g(\mathbf{h}(x,v))\,dS_x(v) \\ &= \frac{1}{4\pi}\int_{S_xM} g(\langle x, v_\perp\rangle, v_\perp)\,dS_x(v) \\ &= \frac{1}{4\pi}\int_{S_xM} g(\langle x, v\rangle, v)\,dS_x(v). \end{aligned}$$

Therefore using the definition of the backprojection operator R^* given in Section 1.3, we obtain

$$f = \frac{1}{4\pi} R^* \left(\frac{d}{ds} H^s Rf \right), \tag{9.21}$$

which is a well-known form of the FBP formula.

Exercise 9.5.4 Show that (9.21) is equivalent to the FBP formula from Theorem 1.3.3. (Hint: use that $|\sigma| = (i\sigma)(\mathrm{sgn}(\sigma)/i)$ and identify the operators associated with each factor as a Fourier multiplier.)

9.5.3 Holomorphic Integrating Factors

Continuing with the Euclidean unit disk M, we know from Remark 9.5.2 that in the flat case Hu_-^f is a first integral, thus $w := (I + iH)u_-^f$ has the property that $Xw = -f$ and moreover it is holomorphic and odd. Similarly, $\tilde{w} = (I - iH)u_-^f$ is odd, anti-holomorphic and solves $X\tilde{w} = -f$. Such functions are called *holomorphic integrating factors*. Proving their existence in the simple case will be very important and the subject of discussion in subsequent chapters. Here we simply wish to point out that their existence in the Euclidean case is quite straightforward.

For completeness we note:

Lemma 9.5.5 $u_+^f = \frac{1}{2}\mathbf{h}^* Rf$.

Exercise 9.5.6 Prove the lemma.

Remark 9.5.7 The function $g := \frac{1}{2}(I + iH^s)Rf$ appears prominently in the classical literature on the attenuated Radon transform. Lemmas 9.5.1 and 9.5.5 tell us that $u^f - w = \mathbf{h}^* g$ and the holomorphicity of w in the angular variable is extensively used, see, for instance, Finch (2003, Lemma 2.1).

9.6 Range

We will describe the range of I_0 and $I_\perp$ following Pestov and Uhlmann (2004). To do this we shall introduce a boundary operator that will naturally appear in the discussion below.

Let (M, g) be a non-trapping surface with strictly convex boundary. We define

$$P\colon C_\alpha^\infty(\partial_+ SM) \to C^\infty(\partial_+ SM)$$

as

$$P := A_-^* H A_+.$$

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We have:

Proposition 9.6.1 *Let (M,g) be a non-trapping surface with strictly convex boundary. Then*

$$P = \frac{1}{2\pi}\big(I_\perp I_0^* - I_0 I_\perp^*\big).$$

Proof Let $w \in C_\alpha^\infty(\partial_+SM)$ so that $w^\sharp \in C^\infty(SM)$. The proof is essentially a rewriting of the commutator formula between X and the Hilbert transform H given in Proposition 6.2.2. Indeed, apply H to $Xw^\sharp = 0$ to obtain

$$-XHw^\sharp = X_\perp\big((w^\sharp)_0\big) + \big(X_\perp w^\sharp\big)_0.$$

Since $I_\perp^* w = -2\pi(X_\perp w^\sharp)_0$ (cf. (9.14)) and $I_0^* w = 2\pi(w^\sharp)_0$, we deduce

$$-XHw^\sharp = \frac{1}{2\pi}\big(X_\perp I_0^* w - I_\perp^* w\big).$$

Integrating this equation along geodesic connecting boundary points (i.e. applying the X-ray transform I), we obtain

$$\big(-Hw^\sharp\circ\alpha + Hw^\sharp\big)|_{\partial_+SM} = \frac{1}{2\pi}\big(I_\perp I_0^* w - I_0 I_\perp^* w\big).$$

But the left-hand side is $A_-^* H(w^\sharp|_{\partial SM}) = Pw$ and the proposition is proved. □

It turns out that the symmetries that we have already discussed produce a further splitting of the formula above. Indeed observe that

$$I_0^*|_{\mathcal{V}_-} = 0; \quad I_\perp^*|_{\mathcal{V}_+} = 0.$$

These are naturally dual to

$$\text{range}\, I_0 \subset \mathcal{V}_+; \quad \text{range}\, I_\perp \subset \mathcal{V}_-$$

thanks to Exercise 9.4.10. Also note that $A_-^* u$ is in $\mathcal{V}_+$ (respectively $\mathcal{V}_-$) if u is odd (respectively even) on ∂SM. Hence if we split the Hilbert transform as $H = H_+ + H_-$ where $H_\pm u = Hu_\pm$ (as usual, $u_\pm$ denote the even and odd parts of u with respect to a), then the formula in Proposition 9.6.1 splits as $P = P_+ + P_-$ where

$$P_- := A_-^* H_- A_+ = -\frac{1}{2\pi} I_0 I_\perp^* \tag{9.22}$$

and

$$P_+ := A_-^* H_+ A_+ = \frac{1}{2\pi} I_\perp I_0^*. \tag{9.23}$$

These formulas imply right away the following range properties for I_0 and $I_\perp$. Recall that $I_0^*, I_\perp^* : C_\alpha^\infty(\partial_+SM) \to C^\infty(M)$.

Theorem 9.6.2 *Let (M,g) be a non-trapping surface with strictly convex boundary. Then*

(i) *A function $h \in C^\infty(\partial_+SM)$ is in the range of I_0:* range $I_\perp^* \to C^\infty(\partial_+SM)$ *if and only if there is $w \in C_\alpha^\infty(\partial_+SM)$ such that $h = P_-w$.*
(ii) *A function $h \in C^\infty(\partial_+SM)$ is in the range of $I_\perp$:* range $I_0^* \to C^\infty(\partial_+SM)$ *if and only if there is $w \in C_\alpha^\infty(\partial_+SM)$ such that $h = P_+w$.*

If, in addition, M is simple (i.e. there are no conjugate points), then I_0^* and $I_\perp^*$ are surjective and the items above give full characterization of the range of I_0 and $I_\perp$ exclusively in terms of the boundary operators $P_\pm$.

Proof Items (i) and (ii) are direct consequences of (9.22) and (9.23). In the simple case, surjectivity of I_0^* is proved in Theorem 8.2.1 and surjectivity of $I_\perp^*$ will be proved in Theorem 12.3.1. □

Remark 9.6.3 It is natural to ask whether the range conditions in Theorem 9.6.2 are related to the Helgason–Ludwig range conditions as described in Chapter 1, when one is considering compactly supported functions in the unit disk in $\mathbb{R}^2$. In Monard (2016a, Theorem 3) it is proved that these range conditions are equivalent once the transformation between fan-beam geometry and parallel-beam geometry is implemented.

9.7 Numerical Implementation

The Fredholm inversion formulas in Theorem 9.4.11 and Exercise 9.4.12 have been implemented in Monard (2014). In what follows we focus exclusively on the formula in Theorem 9.4.11 and for simplicity, we let F be the filter $F := \frac{1}{8\pi} A_+^* H A_-$, so the formula becomes

$$f + W^2 f = I_\perp^* F I_0(f).$$

From Proposition 9.3.2 we easily derive the observation that W becomes a contraction in L^2 whenever the metric g is C^3-close to a metric of constant curvature. Hence $\mathrm{Id} + W^2$ may be inverted by a Neumann series to obtain

$$f = \sum_{k=0}^{\infty} \big(- W^2\big)^k \big[I_\perp^* F I_0(f) \big].$$

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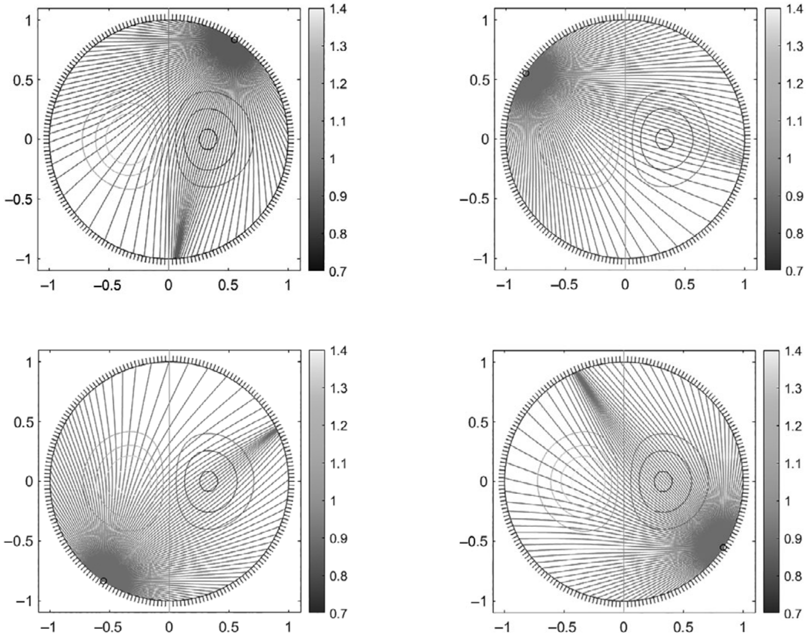

Figure 9.1 Geodesics of g.

It turns out that implementing this Neumann series does not require implementing the operator W^2 and this is a major advantage. Indeed writing $-W^2 = \mathrm{Id} - I_\perp^* F I_0$, we may rewrite the Neumann series as

$$f = \sum_{k=0}^{\infty} \big(\mathrm{Id} - I_\perp^* F I_0\big)^k \big[I_\perp^* F I_0(f)\big].$$

This suggests that a good approximation for the inversion of f in terms of $I_0 f$ is given in terms of the truncated series

$$f \approx \sum_{k=0}^{N} \big(\mathrm{Id} - I_\perp^* F I_0\big)^k \big[I_\perp^* F I_0(f)\big]. \tag{9.24}$$

Note that the computation of (9.24) only involves solving the forward problem iteratively and the approximate inversion given by $I_\perp^* F$. Several numerical experiments illustrating this inversion may be found in Monard (2014). Here we include one as follows, kindly provided to us by François Monard. The metric g on the unit disk has the form $e^{2\lambda}(dx_1^2 + dx_2^2)$ where

$$5\lambda = \exp\big(-\big((x_1 - 0.3)^2 + x_2^2\big)/2\sigma^2\big) - \exp\big(-\big((x_1 + 0.3)^2 + x_2^2\big)/2\sigma^2\big),$$

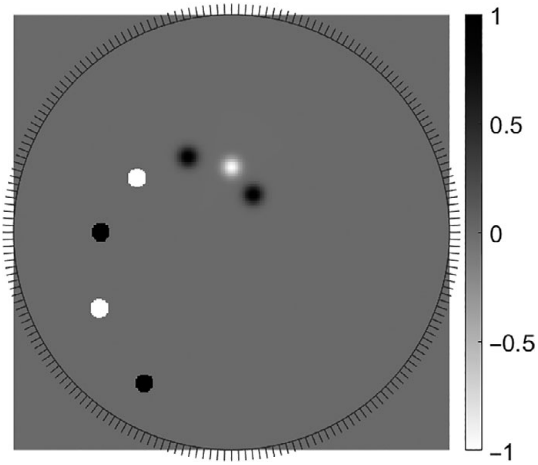

Figure 9.2 The function f.

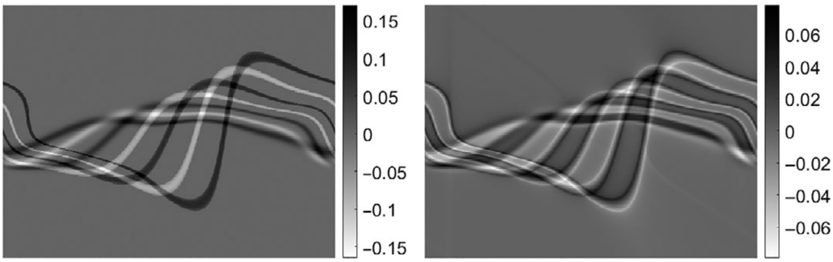

Figure 9.3 The left figure depicts $I_0 f$ and the right one depicts $F I_0 f$.

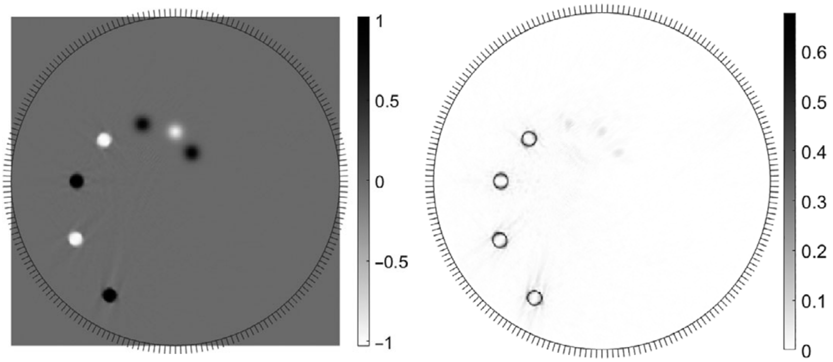

Figure 9.4 Reconstruction and error after no iterations.

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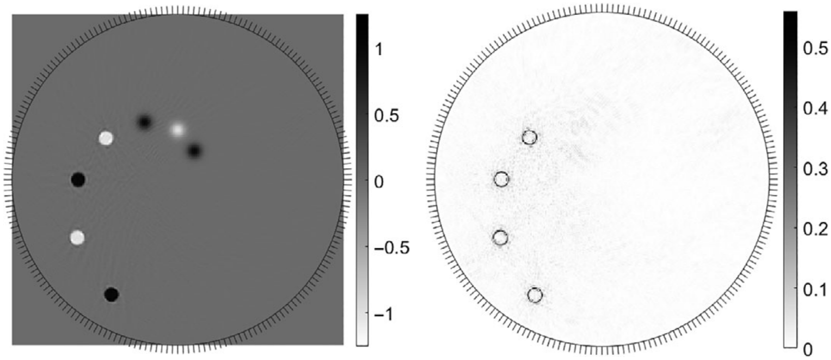

Figure 9.5 Reconstruction and error after five iterations.

with $\sigma = 0.25$. The metric is simple and has low sound speed and high sound speed regions; geodesics emanating from different boundary points are depicted in Figure 9.1.

The function f to be reconstructed is given in Figure 9.2 and it is a mix of Gaussians of various widths and weights.

Figure 9.3 shows $I_0 f$ and its filtered version $F I_0 f$. Figure 9.4 shows $I_\perp^* F I_0 f$ and the error and finally, Figure 9.5 shows (9.24) implemented after five iterations and the corresponding error. For more details on the algorithm and a thorough discussion we refer to Monard (2014).