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A METASTABLE RESULT FOR THE FINITE MULTIDIMENSIONAL CONTACT PROCESS

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ABSTRACT. We prove that for a contact process restricted to the cube $[1,n]^d$ and initially fully occupied, the time to die out, when it is suitably normalized, converges to an exponential random variable as n tends to infinity.

We consider the supercritical contact process restricted to a finite cube. For any finite cube this process must eventually die out. We examine the time to die out starting with all sites (in the cube) occupied. We prove

THEOREM. *Let $\{\zeta_t^n : t \geq 0\}$ be the d-dimensional contact process restricted to $[1,n]^d$ with transmission rate $\lambda > \lambda_c^d$ and let τ^n be the hitting time of the trap state $\underline{0}$. If ζ_0^n is identically equal to one on $[1,n]^d$ for each n, then*

$$\frac{\tau^n}{E[\tau^n]} \xrightarrow{D} e,$$

where e is an exponential random variable of mean one.

Metastable behaviour for contact processes was first considered by Cassandro, Galves, Olivieri and Vares (1984), where it was shown that in one dimension the above result held when the infection parameter λ was sufficiently large. That paper presented an approach useful for other systems as well as the contact process. Schonmann (1985) extended this result to all supercritical contact processes in one dimension and proved that no such result was possible for subcritical processes. The arguments of the latter paper were simplified in Durrett and Schonmann (1988), where much more was proven (again in one dimension) than we shall prove here for higher dimensions.

The contact process restricted to $[1,n]^d$, is a process on $\{0,1\}^{[1,n]^d}$ with flip rates for $x \in [1,n]^d$

$$c(x,\zeta) = \begin{cases} 1 & \text{if } \zeta(x) = 1 \\ \lambda \sum_{\substack{y \in [1,n]^d \\ |x-y|=1}} \zeta(y) & \text{if } \zeta(x) = 0. \end{cases}$$

We will be interested in the process for $\lambda > \lambda_c^d$ the critical value for the unrestricted contact process in d-dimensions. For details of this process see Liggett (1985) or Durrett (1988). Our argument uses ideas for proving metastability found in Durrett and

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Schonmann (1988). We also use renormalization results proven by Bezuidenhout and Grimmett (1990) and oriented percolation methods which have become standard after Durrett (1984). In Section One we establish a criterion for metastability. In Section Two we use the oriented percolation methods to establish bounds on "regeneration" and terminal times for our process. Our Theorem is finally proven in the last section.

Throughout we will write out our proof for the case $d = 2$, but it will be clear that the proof extends to all higher dimensions.

Notation and construction. Our method is essentially that employed for Theorem Three of Durrett and Schonmann (1988).

We will consider our processes as being derived from a collection of Poisson processes as in the Harris representation. This representation enables us to couple contact processes with different initial configurations. We consider time axes to have been drawn up through each of the sites in $[1,n]^2$. Let $P(t,x)$ and $Q(t,x,y)$ be independent Poisson processes, the Ps of rate 1, the Qs of rate λ. We place δs on the time axis through site x at time points in $P(t,x)$ and we place a directed line from x to y at times $Q(t,x,y)$. Given these Poisson processes and therefore these markings, we construct a contact process ζ^A on $[1,n]^2$ with initial state A by taking $\zeta_t^A = \{x : \exists$ a path from a point in $A \times \{0\}$ to (x,t) which travels in a positive direction along time axes, across lines between time axes in the assigned direction and croses no δs$\}$. If A is a singleton $\{x\}$, we abuse notation and write ζ^x instead of $\zeta^{\{x\}}$. Given the Harris construction, it is clear that $\zeta_t^A = \bigcup_{x\in A}\zeta_t^x$. We write ζ^1 for the process obtained by taking $A = [1,n]^2$. The dual process $\hat{\zeta}_s^{A,t}$, $0 \le s \le t$ is obtained by defining $\hat{\zeta}_s^{A,t} = \{y :$ there exists a path from $(y, t-s)$ to (x,t) for some $x \in A\}$. If we reverse the direction of the time axes and the direction of the lines between axes then we have $\hat{\zeta}_s^{A,t} = \{y :$ there exists a path from (x,t) to $(y,t-s)$ for some $x \in A\}$. Given this, it is easy to see that $\hat{\zeta}^{A,t}$ is a contact process on $[0,t]$ with initial configuration A. It is also evident that

$$\zeta_t^A(x) = 1 \text{ iff for some (all) } s \in [0,t], \quad \hat{\zeta}_s^{x,t} \cap \zeta_{t-s}^A \neq \emptyset.$$

We will denote the probability law of ζ_t on $\{0,1\}^{[1,n]^2}$, with $\zeta_0 \equiv 1$, by P^1. We suppress the dependence on n. We say $\zeta_t^{A'}$ is the contact process *restricted to B* for B a subset of $[1,n]^2$ if

$$\zeta_t^{A'}(x) = 1 \text{ iff } \exists \text{ a path from } A \text{ to } x \text{ entirely contained in } B \times R_+.$$

We can think of the contact process restricted to B as being obtained by suppressing all particles and particle births outside B.

Clearly, if B_1 and B_2 are disjoint, then for any A the processes ζ^A restricted to B_1 and ζ^A restricted to B_2 are independent given A, and are both contained in the unrestricted contact process.

$\{F_t\}$ will denote the natural filtration of the totality of Poisson processes. Given a stopping time W with respect to this filtration, the corresponding σ-field will be denoted by F_W.

Section One. In this section we adapt the approach of Durrett and Schonmann (1988) to our purpose. We first state a simple lemma which follows simply from attractiveness.

LEMMA 1.1. *For each s and t in $(0,\infty)$, $P^1[\tau^n > s] \cdot P^1[\tau^n > t] \geq P^1[\tau^n > s+t]$.*

Before proving the next result, which is the purpose of this section, we require some more notation. Given t, A and the Harris system which yields our contact process, we define

$$\zeta_s^{t,A} = \{y : \exists x \in A \text{ so that there is path from } (x,t) \text{ to } (y, t+s)\}.$$

This process is a contact process with initial configuration A, which is independent of the contact process ζ run up to time t. As before, when $A = [1,n]^2$ we write $\zeta^{t,1}$.

The following result states that if the time for "regeneration" of our process is, in the limit, stochastically negligible compared to the time for our process to die, then the time of the killing must be exponential in the limit.

PROPOSITION 1.2. *Suppose there exist sequences $\{a(n)\}$ and $\{b(n)\}$, both tending to infinity, so that*

1. *$b(n)/a(n)$ tends to infinity,*
2. $\sup_{\zeta_0 \in \{0,1\}^{[1,n]^2}} P^{\zeta_0}[\zeta^1_{a(n)} = \zeta_{a(n)}$ *or* $\tau^n < a(n)] \longrightarrow 1$ *as n tends to infinity.*
3. $\lim_{n\to\infty} P^1[\tau^n < b(n)] = 0$,

then under P^1, $\frac{\tau^n}{E[\tau^n]} \longrightarrow e$.

PROOF. We fix $s \in (0,\infty)$. It will suffice to show that $P\Big[\tau^n/E[\tau^n] > s\Big]$ tends to e^{-s} as n tends to infinity. To this end choose ε and δ small but positive. Let $T(n,\varepsilon) = \inf\{t : P[\tau^n \leq t] \geq \varepsilon\}$. It follows from property 1 and 2 of the sequences $\{a(n)\}$, $\{b(n)\}$ that for all n sufficiently large, we can find $W \in [T(n,\varepsilon)/2, T(n,\varepsilon)]$, so that $P[\zeta^1_{W+a(n)} \neq \zeta^{W,1}_{a(n)}] \leq \sup_{\zeta_0 \in \{0,1\}^{[1,n]^2}} P^{\zeta_0}[\zeta^1_{a(n)} \neq \zeta_{a(n)}$ and $\tau^n \geq a(n)] + P^1\Big[\tau \in [W, W+a(n)]\Big] < \varepsilon e^{-\frac{1}{\delta^2}}$. Note that by Lemma 1.1, $\nu = P^1[\tau \leq W] \in [1-\sqrt{1-\varepsilon-\varepsilon e^{-1/\delta^2}}, \varepsilon]$. We assume that ε is sufficiently small to ensure $1-\sqrt{1-\varepsilon} > \varepsilon/3$.

Lemma 1.1 implies that

$$P[\tau^n \geq mW] \leq (1-\nu)^m.$$

For an inequality in the opposite direction observe that

$$\{\tau^n \geq mW\} \subset \bigcap_{i=1}^m \{\tau^{n,1,(i-1)W} \geq W\} \Big\backslash \bigcup_{i=1}^{m-1} \{\zeta^{(i-1)W,1}_{iW+a(n)} \neq \zeta^{iW,1}_{a(n)}\},$$

where $\tau^{n,1,s} = \inf\{t : \tau_t^{s,1} = \emptyset\}$.

The events $\{\tau^{n,1,(i-1)W} \geq W\}$ are independent, while our choice of W ensures that for each i $P[\zeta^{(i-1)W,1}_{iW+a(n)} \neq \zeta^{iW}_{a(n)}] \leq \varepsilon e^{-\frac{1}{\delta^2}}$. Therefore

$$P[\tau^n \geq mW] \geq (1-\nu)^m - m\varepsilon e^{-\frac{1}{\delta^2}}.$$

Thus

$$E^1[\tau^n] \leq W \sum_{m=0}^{\infty} P^1[\tau^n \geq mW] \leq \frac{W}{\nu}$$

and

$$E^1[\tau^n] \geq W \sum_{m=1}^{\infty} P^1[\tau^n \geq mW] \geq W \sum_{m=1}^{\frac{1}{\varepsilon\delta}} (1-\nu)^m - m\varepsilon e^{-\frac{1}{\delta^2}} \geq \frac{W}{\nu}(1-2\nu)$$

for ε and δ small enough. To conclude, for ε and δ small enough and n large enough $E[\tau^n] \in [\frac{W}{\nu}(1-2\nu), \frac{W}{\nu}]$. Given these inequalities and Lemma 1.1, we have for n large

$$P\Big[\tau^n \geq sE[\tau^n]\Big] \leq P\Big[\tau^n \geq s\frac{W}{\nu}(1-2\nu)\Big] \leq (1-\nu)^{\frac{s(1-2\nu)-\nu}{\nu}} \leq \frac{e^{-s(1-2\nu)}}{(1-\nu)}$$

and

$$P\Big[\tau^n \geq sE[\tau^n]\Big] \geq P\Big[\tau^n \geq s\frac{W}{\nu}\Big] \geq (1-\nu)^{\frac{s}{\nu}+1} - s\frac{\varepsilon}{\nu}e^{-\frac{1}{\delta^2}} \geq (1-\nu)^{\frac{s}{\nu}+1} - 3se^{-\frac{1}{\delta^2}}.$$

Since ε (and hence ν) and δ may be taken arbitrarily small, the proof of Proposition 1.2 is completed. ∎

Section Two. The breakthrough of Bezuidenhout and Grimmett (1990) enables one to compare (renormalized) supercritical contact processes with supercritical oriented percolations. Therefore we begin by assembling some facts from oriented percolation. These facts are culled from Durrett (1984), to which the reader is referred for a complete account. Let V_m be the set $\{(x,y) \subset [1,m] \times Z_+ : x+y \equiv 0 \pmod 2\}$. Let there be directed edges from sites (x,y) in V_m to sites $(x-1,y+1)$ and $(x+1,y+1)$, with probability p and no other edges. In addition, suppose that bonds with no endpoints or beginning points in common are independent; then the resulting random graph G is a 1-dependent oriented percolation system with bond probability p. For $A \subset [1,m] \times \{0\} \cap V_m$, let $\psi_n^{m,A} = \{x :$ there is a path along edges of G from a point in A to $(x,n)\}$. If $A = (x,0)$, we abuse notation and write $\psi_n^{m,x}$ for the above. If $A = [1,m] \times \{0\} \cap V_m$, then we suppress the superscript A. Define the random quantity

$$T^{m,A} = \inf\{n : \psi_n^{m,A} = \emptyset\};$$

again we suppress the superscript A if $A = [1,m] \times \{0\} \cap V_m$.

The following facts all follow via the dual-contour methods of Durrett (1984).

There exists $\delta, \delta(k), k = 1,2,\ldots$ all strictly positive and $p_0 < 1$ so that for any $d > 0$:

FACT 2.1. If $p > p_0$, then $P[T^m \leq e^m] \leq e^{-m}$ for all m large enough.

FACT 2.2. If $p > p_0$, then for m large enough, $\inf_{x \in [1,m], x \text{ even}} P[T^{m,x} \geq e^m] \geq \delta$.

FACT 2.3. If $p > p_0$, then for m large enough and all non-empty $A \subset [1,m] \times \{0\} \cap V_m$, $P[dm \leq T^{m,A} \leq e^m] \leq e^{-dm}$.

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FACT 2.4. If $p > p_0$, then for any positive k, m large enough and $n \in [m^2/k, km^2]$, $\inf_{\substack{x\in[1,m],x \text{ even} \\ y\in[1,m],y+n \text{ even}}} P[(x,0) \text{ is connected to } (y,n)] \geq \delta(k)$.

Naturally the above facts hold for percolations on intervals of length m.

We recall the result of Bezuidenhout and Grimmett (1990) in a form that will suit our purposes. For fixed L let $S_i = [1,n] \times [4L(i-1)+1, 4iL]$ for $i \leq M$, where M is the integer part of $n/4L$. Define the regions

$$R_i^{\pm} = \left\{(x,y,t) : t \in [0,(2k+2)S], x \in \left[-5L \pm \frac{Lt}{2S}, 5L \pm \frac{Lt}{2S}\right] y \in [4L(i-1)+1, 4iL]\right\}$$

and

$$V^i = [-2L, 2L] \times [(4i-3)L, (4i-1)L] \times [0, 2S].$$

Let $R_i^{\pm}(p,q) = R_i^{\pm} + (pkL, 0, 2qkS)$ and $V^i(p,q) = V^i + (pkL, 0, 2qkS)$.

We say that $(p,q,+)$ is *good* if $R_i^+(p,q)$ is a subset of S_i. Similarly for $(p,q,-)$. Note that the particular i is irrelevant. We write G^n for the set $\{p :$ for some q $(p,q,+)$ or $(p,q,-)$ is good$\}$. Clearly there is a constant $d \in (0,1)$ so that for n large enough G^n is an interval of length at least dn. Let $D^n = \{(p,q) : (p,q,+)$ or $(p,q,-)$ is good$\}$. We write

$$R^i = \bigcup_{(p,q,+) \text{ is good}} R_i^+(p,q) \bigcup_{(p,q,-) \text{ is good}} R_i^-(p,q).$$

Bezuidenhout and Grimmett (1990) showed that given $\varepsilon > 0$, a disc D about the origin, k, L and S may be chosen so that if ζ_t^i is the contact process on $[1,n]^2$ restricted to R_i and

$$A^i = \{(p,0) \in G^n : \exists x \in V^i(p,0) \cap t = 0 \text{ s.t. } \zeta_0^i \equiv 1 \text{ on } x + D\},$$

there is a 1-dependent oriented percolation system ψ on G^n, independent of ζ_0, of bond connection probability $> 1 - \varepsilon$, so that $\forall n$

$$\psi_n^{A^i} \subset \{(p,n) \in G^n : V^i(p,n) \text{ contains } (x,t) \text{ with } \zeta_t \equiv 1 \text{ on } x + D\}.$$

In the following we will suppose that k, L, D, S have been chosen so that the bond probability is $> p_0$.

DEFINITION. Given a contact process ζ, we say that $(p,q) \in G^n$ is i-occupied, if $\exists (x,t) \in V^i(p,q)$ with $\zeta_t \equiv 1$ on $x + D$. We say (p,q) is i-*well occupied* if $\exists x$ so that $(x, 2qkS) \in V^i(p,q)$ and $\zeta_t \equiv 1$ on $x + D$.

The coupling of Bezuidenhout and Grimmet (1990) enables us to compare oriented percolations on an interval of length dn with the contact process on $[1,n]^2$. This and Fact 2.1 immediately yield the result below.

PROPOSITION 2.1. *For n large $P^1[\tau^n \leq e^{dn}] < e^{-dn}$.*

The result will enable us to apply Proposition 1.2 with $b(n) = e^{nd}$. If we can find a suitable $\{a(n)\}$ we will be done. The result below states that we may take $a(n) = 2(n^4 + n^2)2kS$.

PROPOSITION 3.1. *Let ζ_t be a contact process on $[1,n]^2$. Let $a(n) = 2(n^4+n^2)2kS$.*

$$\sup_{\zeta_0} P^{\zeta_0} P[\zeta_{a(n)} \neq \zeta^1_{a(n)} \text{ and } \zeta_{a(n)} \neq \emptyset] \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

We will prove this proposition (and therefore the Theorem) in the last section; for the present we only make the following remarks: it follows from the self-duality of the contact process that $\zeta^1_{a(n)}$ consists of all those points x in $[1,n]^2$ from which there is a time reversed path from $\big(x, a(n)\big)$ to some point $(z,0)$. Likewise $\zeta_{a(n)}$ consists of all those points x, in $[1,n]^2$ from which there is a time reversed path from $\big(x, a(n)\big)$ to some point $(z,0)$ with $z \in \zeta_0$. Therefore (since obviously $\zeta \subset \zeta^1$), Proposition 3.1 will be established by showing

$$\sum_{x\in[1,n]^2} P\Big[\exists \text{ a dual path from } \big(x, a(n)\big) \text{ to } [1,n]^2 \times \{0\}$$
$$\text{but not to } \zeta_0 \times \{0\}, \zeta_{a(n)} \neq \emptyset\Big] \longrightarrow 0.$$

This will of course be implied by showing that there exists some $c < 1$, not depending on n, so that for n large

$$\sup_{x\in[1,n]^2} P\Big[\exists \text{ a dual path from } \big(x, a(n)\big) \text{ to } [1,n]^2 \times \{0\}$$
$$\text{but not to } \zeta_0 \times \{0\}, \zeta_{a(n)} \neq \emptyset\Big] \leq c^n.$$

The rest of this section is devoted to proving

LEMMA 2.2. *Let $A(m,i)$ be the event that there does not exist $z \in S_i$ with $\zeta_{2kSm}(z) = 1$. Then for some $c < 1$,*

$$\sup_{\zeta_0} P^{\zeta_0}\Big[\{\zeta_{n^4 2kS} \neq \emptyset\} \cap \Big\{\bigcup_i A(n^4, i)\Big\}\Big] \leq c^n.$$

This lemma states that provided the contact process has survived until time $n^4 2kS$; then, outside of a set of exponentially small probability, at this time the process will have occupied sites in every slice S_i. The exponential bound is obtained uniformly over initial configurations. For $x \in [1,n]^2$, let $\hat{\zeta}^x_t$ be the dual process starting from $\big(x, 2(n^4 + n^2)\big)2kS$, so that $\zeta^A_{2(n^4+n^2)2kS}(x) = 1$ if and only if $\zeta^A_s \cap \hat{\zeta}^x_{2(n^4+n^2)2kS-s} \neq \emptyset$ for some (all) $s \in [0, 2(n^4+n^2)2kS]$. Therefore we obtain as an immediate corollary:

COROLLARY 2.3. *Let $B(m,i,x)$ $\big(m \leq 2(n^4+n^2)\big)$ be the event that there does not exist $z \in S_i$ with $\hat{\zeta}^{x,a(n)}_{2kSm}(z) = 1$. Then*

$$P\Big[\{\hat{\zeta}^{x,a(n)}_{n^4 2kS} \neq \emptyset\} \cap \Big\{\bigcup_i B(n^4, i, x)\Big\}\Big] \leq c^n.$$

The heuristic behind our proof can be split up into four thoughts:

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A) The contact process, starting from a non-zero configuration and restricted to a strip S_i has a non-zero chance of becoming "established," that is surviving for an exponential amount of time.
B) If we disregard events of exponentially small probability, the contact process, restricted to a strip, becomes "established" if it survives for time of order n.
C) If the contact process survives for time of order n^2, then, outside of exponentially small probability, it will be "established" in some strip S_i.
D) We again neglect sets of exponentially small probability. If the contact process becomes established in a strip S_i at time t, then it will be established in strips $S_{i\pm 1}$ by time $t+$ order n^2, and, continuing, in all strips by time $t+$ order n^3.

Let ζ_t be a contact process on $[1,n]^2$. For a (possibly random time) W and $i \in \{1,2,3,\ldots, M\}$, let $\zeta_t^{W,i}$ be the contact process obtained by suppressing all particles and particle births outside S_i after time W. These sub-processes are introduced for the following properties: for W a stopping time, the processes $\zeta_{W+t}^{W,i}$, $i = 1,2,\ldots$ are conditionally independent, given ζ_W, and for each time t are contained in the original process ζ_{W+t}.

Given that the contact process is Markov, the following is a simple consequence of Bezuidenhout and Grimmett (1990).

LEMMA 2.4. *Let W be a stopping time so that*
(i) $W \in \{4kS, 8kS, 12kS, \ldots\}$ a.s.,
(ii) There exists a.s. F_W measurable $p(W)$ and $i(W)$ so that $(p, W/2kS)$ is i-well occupied.

Then there exists a 1-dependent oriented percolation system ψ with bond probabilities $> p_0$ on G^n, independent of F_W, so that

$$\{x : x \in \psi_n^p\} \subset \left\{x : \left(x, \frac{W}{2kS} + n\right) \text{ is } i \text{ occupied by } \zeta^{W,i}\right\}.$$

The following corollary follows from this coupling and the contour methods used to prove Facts 2.1–2.4 and so the proof is not explicitly written out.

COROLLARY 2.5. *Let W be as in Lemma 2.4 and let $W' = \inf\{2kSq : 2kSq > W$, there is no x with (x,q) i-occupied by $\zeta^{W,i}\}$. Then for δ and $\delta(k)$ as in Facts 2.1–2.4 and all n large,*
a) $P[n \leq \frac{W'-W}{2kS} \leq e^{dn}] \leq e^{-dn}$.
b) $P[\frac{W'-W}{2kS} \geq n] \geq \delta$.
c) For $k \in [n^2/2, 2n^2]$, $(y,k) \in D^n$, $P[(y - \frac{W}{2kS} + n)$ is i-occupied$] > \delta(\frac{2}{d^2})$.

Corollary 2.5 states that (1) once a slice S_i possesses a renormalized well-occupied site then there is a chance, bounded away from zero, that the contact process restricted to S_i becomes established and (2) that being established is essentially the same as the contact process, restricted to the slice S_i, surviving for $2kSn$ units of time. This is the content of parts a) and b) of Corollary 2.5 and essentially corresponds to steps A and B of the heuristic.

Let $\varepsilon > 0$ be such that

$$\sup_{x\in[1,n]^2} P[\exists i,p, \text{ s.t. } (p,q) \text{ is } i\text{-well occupied } | \eta_{2kSq-1} = \{x\}] > \varepsilon.$$

This quantity is bounded away from 0 and may be treated as being independent of n.

Let random times W_j, W_j' and W_j'' be defined by

$$W_1 = \inf\{4kSq : \exists i,p \text{ (denoted } I(W_1), p(W_1)) \text{ such that } (p,2q) \text{ is } i\text{-well occupied}\}$$
$$W_j'' = \inf\{2kSq > W_j : \nexists x \text{ with } (x,q)i(W_j) \text{ occupied by } \zeta^{W_j, i(W_j)}\}$$
$$W_j' = \min(W_j'', W_j + 2kSn)$$
$$W_j = \inf\{4kSq > W_{j-1}' : \exists i(W_j), p(W_j) \text{ such that } (p,2q) \text{ is } i\text{-well-occupied}\}.$$

The W_i represent stopping times at which we may apply Corollary 2.5. Thus at each W_i there is a probability bounded away from 0, that the contact process becomes established in some slice S_i. The W_j'' are, loosely, the first times after the W_j that the contact process, restricted to the relevant S_i, no longer has "enough" occupied sites. We will be interested in whether the contact process becomes established in a slice S_i. Since, by Corollary 2.5 part (a), if $W_j'' - W_j \geq 2kSn$, then with very large probability the contact process is established in $S_{i(W_j)}$, we also introduce the random time W_j'. At this time we essentially know whether the contact process became "established" in a slice after W_j.

(NOTE. The W_j are stopping times while the W_j' are not. This difference comes from the difference between the definitions of *occupied* and *well occupied*. However $V = \inf\{W_j' : W_j' - W_j = 2kSn\}$ is a stopping time.)

LEMMA 2.6. *The random times W_j, W_j', W_j'' satisfy:*

(i) $P[\zeta_{4kSn^2} \neq \emptyset, W_n \geq 4kSn^2] \leq \nu^n$ *for some* $\nu < 1$ *and all* n *large enough.*

(ii) $P[W_n < \infty, \nexists j \leq n\ W_j' - W_j = 2kSn] \leq \nu^n$ *for some* $\nu < 1$ *and all* n *large enough.*

(iii) $P[\zeta_{4kSn^2} \neq \emptyset; V = \inf\{W_j' : W_j' - W_j = 2kSn\} > 4kSn^2] \leq \nu^n$ *for some* $\nu < 1$.

(iv) $P[\zeta_{4kSn^2} \neq \emptyset;$ *for some* $j, V = W_j', W_j'' - W_j \leq e^{dn}\}] \leq \nu^n$ *for some* $\nu < 1$.

PROOF. (i) By our choice of ε above and the Markov property, we have

$$P[W_1 = 4kSq \mid W_1 \geq 4kSq, \zeta_{4kSq-1}] \geq \varepsilon.$$

Therefore $P[\tau^n \wedge W_1 \geq 2kSn] \leq (1-\varepsilon)^{n/2-1}$. (Here $\tau^n \wedge W_1$ denotes the minimum of the two stopping times.) By the strong Markov property this implies that $P[\tau^n \wedge W_j - \tau^n \wedge W_{j-1}' \geq 2kSn] \leq (1-\varepsilon)^{n/2-1}$. By definition, $W_j' - W_j \leq 2kSn$, so if $\tau^n \wedge W_n \geq 4kSn^2$, then for some $j \leq n$, $\tau^n \wedge W_j - \tau^n \wedge W_{j-1}' \geq 2kSn$. (Here we take $W_0' = 0$.) But this latter event has probability less than or equal to $n(1-\varepsilon)^n$. To conclude,

$$P[\zeta_{4kSn^2} \neq \emptyset, W_n \geq 4kSn^2] \leq P[\tau^n \wedge W_n \geq 4kSn^2] < n(1-\varepsilon)^{n/2-1} \leq (\sqrt{1-\varepsilon/3}\,)^n$$

for n large enough.

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(ii) Fact 2.2 and the Markov property imply that for each j, $P[W_j' - W_j = 2kSn \mid F_{W_j}] > \delta$. So (ii) follows with $\nu = 1 - \delta$.

(iii) This is simply (i) and (ii) put together.

(iv) This follows from (ii), (iii) and Corollary 2.5 part (a). ∎

Part (iv) of the above lemma corresponds to step C of the heuristic outlined before Lemma 2.4.

LEMMA 2.7. *Fix i and $j \in \{1, 2, \ldots, M\}$ with $|i - j| = 1$. For integer $q \leq n^4 - n^2$, let $C(i, j, q)$ be the event*

$$\{\zeta_t \cap S_i \neq \emptyset \text{ for } 4kSq \leq t \leq 4kSq + 4kSn^2\} \\ \cap \{\exists t \in (4kSq + 4kSn^2, 4kSn^4) \text{ with } \zeta_t \equiv 0 \text{ on } S_j\}.$$

Then there exists $\nu < 1$ with $P[C(i, j, q)] < \nu^n$ for n large.

PROOF. Define the stopping times

$$G_1 = \inf\{4kSr > q : \exists p \text{ such that } (p, 2r) \text{ is } j\text{-well occupied}\}$$
$$G_l'' = \inf\{2kSr > G_l : \nexists x \text{ with } (x, r)\ j\text{-occupied by } \zeta^{G_l, j}\}$$
$$G_l' = \min(G_l'', G_l + 2kSn)$$
$$G_l = \inf\{4kSr > G_{l-1}' : \exists p \text{ such that } (p, 2r) \text{ is } j\text{-well occupied}\}.$$

It follows in the same way as with the previous lemma that there exists $\nu < 1$ such that for n large $P[\text{there is no } l \in [1, n] \text{ with } G_l' - G_l = 2kSn] < \nu^n$. Also by Corollary 2.5, part (c), $P[\exists l \leq n \text{ with } G_l'' - G_l \in [2kSn, 2kSe^{nd}] \leq ne^{-nd}$. The lemma follows. ∎

PROOF OF LEMMA 2.2. If $\bigcup_i A(n^4, i)$ does not occur then we must have either

1) For stopping times W_j, W_j' of Lemma 2.6, $V = \inf\{W_j + 2kSn : W_j' - W_j \geq 2kSn\} > 4kSn^2$.
2) For j such that $V = W_j + 2kSn$, $W_j' - W_j \in [2kSn, 2kSe^{nd}]$.
3) For some $i, j \in \{1, 2, \ldots, M\}$ with $|i - j| = 1$ and integer $r \in [n^2, n^4 - n^2]$, the event $C(i, j, r)$ occurs.

Lemma 2.3 states that the probability of events 1 and 2 above is bounded by ν^n for some $\nu < 1$ and all n large enough. Lemma 2.7 gives the crude bound $P[\bigcup C(i, j, r)] \leq M^2 n^4 \nu^n$ for some $\nu < 1$. It is clear that these bounds give the desired conclusion. ∎

Section Three. In this section we complete the proof of the Theorem. In Section Two this was reduced to proving Proposition 3.1. The remarks following the statement of Proposition 3.1 showed that a sufficient condition for the proposition to hold is that for some $\nu < 1$,

$$\sup_{\zeta_0} P^{\zeta_0}[\zeta_{a(n)} \neq \emptyset, \hat{\zeta}_{a(n)}^{x, a(n)} \neq \emptyset, \zeta_{a(n)/2} \cap \hat{\zeta}_{a(n)/2}^{x, a(n)} = \emptyset] \leq \nu^n$$

for n sufficiently large and all $x \in [1, n]^2$. (Recall that $a(n) = 2(n^4 + n^2)2kS$). We will achieve this by combining Lemma 2.2, Corollary 2.3 and the lemma below.

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A vital component in the completion of the proof of Proposition 3.1 is the fact that the two dimensional contact process can survive in a strip $Z \times [1,4L]$, where L is as in Section Two. Corollary 2.3 and Lemma 2.2 enable us to say that if $\zeta_{2kSn^4}, \hat{\zeta}^{a(n),x}_{2kSn^4} \neq \emptyset$, then outside of events of exponentially small probability, both configurations contain infected sites in every slice S_i. We can view the contact processes restricted to the respective strips as conditionally independent. Lemma 3.2 below says that in each slice S_i, the chance that at time $2kS(n^4+n^2)$ the processes ζ and $\hat{\zeta}^{a(n),x}$ restricted to S_i have a common infected site is bounded away from zero. Hence the chance that $\zeta_{2kS(n^4+n^2)}$ and $\hat{\zeta}^{a(n),x}_{2kS(n^4+n^2)}$ do not have a common infected site is exponentially small in n.

LEMMA 3.2. *Let ζ^i be a contact process restricted to S^i, with $\zeta^i_0 \neq \emptyset$. For arbitrary $z \in S_i$, $P[\zeta^i_{2kSn^2}(z) = 1] \geq \delta(L,S,k,D) > 0$.*

PROOF. Let $\varepsilon_2 = \varepsilon_2(L,D,k,S)$ be so that

$$0 < \varepsilon_2 < \inf_{\zeta^i_0} P^{\zeta^i_0}[\exists p : (p,2) \text{ is } i\text{-well occupied}].$$

Let p_1 be chosen so that $(p_1, n^2-2) \in G_n$ and $|z - 2p_1L| < 10L$. By Corollary 2.5 and the Markov property, $P[(p_1, n^2-2)$ is i-well occupied $\mid \exists p$ with $(p,2)$ i-well occupied$] > \gamma > 0$, for some γ not depending on n. Clearly we can find ε_3, not depending on n, so that

$$0 < \varepsilon < \inf_{|w-kp_1L|<5L} P[\zeta^i_{4kS}(z) = 1 \mid \zeta^i_0 = w].$$

By the Markov property, our result follows with $\delta = \varepsilon_2\gamma\varepsilon_3$. ■

PROOF OF PROPOSITION 3.1. For notational brevity we define, for $s \in [2kSn^4, 2kS(n^4+n^2)]$, $\hat{\zeta}^{i,x}_s$ to be the process obtained from $\hat{\zeta}^{a(n),x}$ by suppressing all particles and births of particles occurring outside S_i on and after time $2kSn^4$.

It is important to note that for $s \in [0, 2kSn^2]$, the processes $\hat{\zeta}^{i,x}_{2kSn^4+s}$ and $\zeta^{i,2kSn^4}_{2kSn^4+s}$ are conditionally independent given $\hat{\zeta}^{x,a(n)}_{2kSn^4}$ and ζ_{2kSn^4}. The probability

$$\begin{aligned}
&P[\zeta_{a(n)} \neq \emptyset, \hat{\zeta}^{a(n),x}_{a(n)} \neq \emptyset, \zeta_{a(n)/2} \cap \hat{\zeta}^{a(n),x}_{a(n)/2} = \emptyset] \\
&\quad \leq P[\zeta_{2kSn^4} \neq \emptyset, \hat{\zeta}^{a(n),x}_{2kSn^4} \neq \emptyset, \zeta_{a(n)/2} \cap \hat{\zeta}^{a(n),x}_{a(n)/2} = \emptyset] \\
&\quad \leq P\Big[\{\zeta_{n^4 2kS} \neq \emptyset\} \cap \Big\{\bigcup_i A(n^4,i)\Big\}\Big] + P\Big[\{\hat{\zeta}^{a(n),x}_{n^4 2kS} \neq \emptyset\} \cap \Big\{\bigcup_i B(n^4,i,x)\Big\}\Big] \\
&\qquad + P\Big[\hat{\zeta}^{i,x}_{2kS(n^4+n^2)} \cap \zeta^{i,2kSn^4}_{2kS(n^4+n^2)} = \emptyset, \forall i, \Big(\bigcup A(n^4,i) \cup (B(n^4,i,x)\Big)^c\Big].
\end{aligned}$$

But, by Lemma 2.2 and Corollary 2.3, the first two of these probabilities are $\leq \nu^n$ for some $\nu < 1$ and all n large enough, while the third term is bounded by $P\Big[\hat{\zeta}^{i,x}_{2kS(n^4+n^2)} \cap \zeta^{i,2kSn^4}_{2kS(n^4+n^2)} = \emptyset, \forall i, \mid \Big(\bigcup A(n^4,i) \cup B(n^4,i,x)\Big)^c\Big]$.

As has been previously noted, the events $\{\hat{\zeta}^{i,x}_{2kS(n^4+n^2)} \cap \zeta^{i,2kSn^4}_{2kS(n^4+n^2)} = \emptyset\}$ are conditionally independent given $\hat{\zeta}^{x,a(n)}_{2kSn^4}$ and ζ_{2kSn^4}. It follows from Lemma 3.2 that on $\Big(\bigcup A(n^4,i) \cup B(n^4,i,x)\Big)^c$, these events all have conditional probability less than $1 - \delta^2$. Thus

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$P\Big[\hat{\zeta}^{i,x}_{2kS(n^4+n^2)} \cap \zeta^{i,2kSn^4}_{2kS(n^4+n^2)} = \emptyset, \forall i, \mid \Big(\bigcup A(n^4,i) \cup B(n^4,i,x)\Big)^c\Big] \leq (1-\delta^2)^n$ and we are done. ∎

References

C. Bezuidenhout and G. Grimmet (1990), *The critical contact process dies out*, Annals of Probability **18**, 1462–1482.

M. Cassandro, A. Galves, E. Olivieri and M. Vares (1984), *Metastable behaviour of stochastic dynamics: a pathwise approach*, Journal Stat. Phys. **35**, 603–634.

R. Durrett (1984), *Oriented percolation in two dimensions*, Annals of Probability **12**, 999–1040.

______ **(1988),** *Lecture Notes on Particle Systems and Percolation*, Wadsworth and Brooks/Cole, Pacific Grove, California.

R. Durrett and R. Schonmann (1988), *The contact provess on a finite set II*, Annals of Probability **16**, 1570–1583.

T. Liggett (1985), *Interacting Particle Systems*, Springer, New York.

R. Schonmann (1985), *Metastability for the contact process*, Journal Stat. Phys. **41**, 445-464.

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