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A NOTE ON THE GEOMETRIC ERGODICITY OF A MARKOV CHAIN

K. S. CHAN,* *University of Chicago*

Abstract

It is known that if an irreducible and aperiodic Markov chain satisfies a 'drift' condition in terms of a non-negative measurable function $g(x)$, it is geometrically ergodic. See, e.g. Nummelin (1984), p. 90. We extend the analysis to show that the distance between the nth-step transition probability and the invariant probability measure is bounded above by $\rho^n(a + bg(x))$ for some constants $a, b > 0$ and $\rho < 1$. The result is then applied to obtain convergence rates to the invariant probability measures for an autoregressive process and a random walk on a half line.

AUTOREGRESSIVE PROCESS; DRIFT CONDITION; RANDOM WALK

1. Introduction

Let $(R, \mathscr{E})$ be a measurable space, $\mathscr{E}$ being assumed countably generated. Let ϕ be a non-trivial σ-finite measure on $(E, \mathscr{E})$. An E-valued Markov chain $(X_n)_{n\in\mathbb{N}}$ is said to be ϕ-irreducible if

(1.1) $$\Sigma_{0\leqq n<\infty} P^n(x, A) > 0 \text{ for all } A \in \mathscr{E} \text{ with } \phi(A) > 0,$$

where $P(.,.)$ denotes the transition probability. In Sections 1 and 2, (X_n) is assumed to be a ϕ-irreducible Markov chain. (X_n) is said to be geometrically ergodic if it admits an invariant probability measure π, a π-integrable function M and a constant $\rho < 1$ such that

(1.2) $$\|P^n(x, .) - \pi\| \leqq M(x)\rho^n, \qquad \forall x \in E, \quad n \geqq 0,$$

where $\|.\|$ denotes the total variation norm.

It is known that if (X_n) is aperiodic and satisfies a 'drift' condition in terms of a well-behaved non-negative measurable function $g(.)$, then it is geometrically ergodic. We extend the analysis to show that $M(x)$ in (1.2) can be taken as $a + bg(x)$ for some positive constants a and b. We follow the notation adopted in Nummelin (1984) and refer the reader to it for any unexplained notation.

2. Main result

We now state the main theorem.

Theorem 1. Suppose that (X_n) is aperiodic and that for some small set C, a non-negative measurable function g, a constant $r > 1$ such that

(2.1*a*) $$\sup_{x\in C^c} \mathbb{E}(rg(X_{n+1}) - g(X_n) \mid X_n = x) \equiv \gamma < 0;$$

(2.1*b*) $$\sup_{x\in C} \mathbb{E}(g(X_{n+1}); X_{n+1} \in C^c \mid X_n = x) \equiv B < \infty;$$

(2.1*c*) $g(x)$ is bounded away from 0 and $+\infty$ on C.

Then (X_n) is geometrically ergodic and $M(x)$ in (1.2) can be taken as $a + bg(x)$ for some constants a and b.

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* Postal address: Department of Statistics, The University of Chicago, Chicago, IL 60637, USA.

Before we prove the theorem, first some remarks. Suppose that condition (2.1) hold. Then Proposition 5.21 in Nummelin (1984) shows that (X_n) is geometrically recurrent. Thus if (X_n) is aperiodic, then it is geometrically ergodic. It also follows from Theorem 4 in Tweedie (1983) that $g(.)$ is π-integrable. Here, we extend the analysis to show that actually $M(x)$ in (1.2) can be taken as $a + bg(x)$ for some positive constants a and b.

Condition (2.1) is referred to as a kind of 'drift' condition in the literature.

Proof of Theorem 1. It follows from the smallness of C that there exist an integer m_0, a constant $1 \geqq \beta > 0$, a probability measure ν such that

$$P^{m_0}(x, A) \geqq \beta 1_C(x)\nu(A), \qquad \forall x \in E, \quad A \in \mathscr{E}. \tag{2.2}$$

Let $\beta 1_C(x)$ be denoted by $s(x)$. The pair (s, ν) is called an atom and is denoted by α. Since $\|P^n(x, .) - \pi\|$ is non-increasing in n, with no loss of generality, we can assume that $m_0 = 1$ lest we work with (X_{nm_0}).

Arguing as in the proof of Theorem 6.14 in Nummelin (1984), we have

$$\begin{aligned} \Sigma_n r^n \|P^n(x, .) - \pi\| \leqq G_\alpha^{(r)}1(x) + rG_\alpha^{(r)}s(x)\Sigma_m r^m |u_m - M_b^{-1}| \nu G_\alpha^{(r)}1 \\ + r(r-1)^{-1}\pi(s)\nu G_\alpha^{(r)}1(G_\alpha^{(r)}s(x) + 1), \end{aligned} \tag{2.3}$$

all summations being from 0 to ∞.

It follows from condition (2.1) and the proof of Proposition 5.21 in Nummelin (1984) that, after some arrangement,

$$\mathbb{E}_x(r^{S_C}) \leqq a_1 + b_1 g(x) \tag{2.4}$$

where a_1 and b_1 can be chosen as $r((r-1)B/\gamma + 1) + 1 + r$ and $(r-1)/\gamma$ respectively.

From (2.2), we see that there is a probability $\beta > 0$ that $X_n \in \alpha$ given that $X_n \in C$. Thus, in view of the smallness of C, it follows from Lemma 5.6 in Nummelin that there exists a constant a_2 such that

$$\mathbb{E}_x(r^{S_\alpha}) \leqq a_2\mathbb{E}_x(r^{S_C}) = a_1 a_2 + b_1 a_2 g(x). \tag{2.5}$$

Now,

$$G_\alpha^{(r)}s(x) = \mathbb{E}_x(r^{T_\alpha}) \leqq \mathbb{E}_x(r^{S_\alpha}). \tag{2.6}$$

Applying Lemma 6.2 in Nummelin (1984) with λ there chosen as ε_x, the probability measure with all its mass at x, we have

$$G_\alpha^{(r)}1(x) < (r-1)^{-1}G_\alpha^{(r)}s(x). \tag{2.7}$$

Now arguing again as in the proof of Theorem 6.14 in Nummelin (1984), by decreasing $r > 1$ if necessary, we have both $\Sigma_m r^m |u_m - M_b^{-1}|$ and $\nu G_\alpha^{(r)}1$ being finite. Combining (2.3), (2.5), (2.6) and (2.7), it is readily seen that there exist positive constants a and b such that $\Sigma_n r^n \|P^n(x, .) - \pi\| \leqq a + bg(x)$. So, by taking $\rho = r^{-1}$, the proof of the theorem is completed.

3. Examples

Theorem 1 provides an upper bound on the convergence rate to the invariant probability measure in the form of $\rho^n(a + bg(x))$ with $\rho < 1$. We now consider two examples in which $g(x)$ may be chosen as linear in $|x|$ and exponential in x respectively. It is also noted that the convergence rate in x thus obtained is exact for some special cases.

Example 1. Let (X_n) be the stable first-order autoregressive process, i.e.,

$$X_n = \phi X_{n-1} + a_n, \qquad n = 1, 2, 3, \cdots \tag{3.1}$$

where $|\phi| < 1$, (a_n) i.i.d. with finite first absolute moment and a_n independent of $X_{n-1}, X_{n-2}, \cdots, X_0$. It is assumed that a_1 has an absolutely continuous component which admits a density positive over some open interval about 0. Then (X_n) is irreducible and

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aperiodic. Let $g(x) = |x| + 1$. Then condition (2.1) holds with C chosen as $[-c, c]$ for some $c > 0$. It follows from Theorem 1 that an upper bound on the rate of convergence to the invariant probability measure is linear in $|x|$. In the case of a_n being Gaussian, it can be directly verified that the rate of convergence is indeed linear in $|x|$.

Example 2. Let (X_n) be a random walk on $\mathbb{R}^+$, i.e.,

$$X_{n+1} = (X_n + a_n)^+, \qquad n = 1, 2, 3, \cdots \tag{3.2}$$

where (a_n) is i.i.d.; a_n independent of $X_{n-1}, X_{n-2}, \cdots, X_0$; $E(a_1) < 0$ and, for some $M < \infty$ and $\beta > 0$, $\Pr(a_1 > y) \leqq M \exp(-\beta y)$, $\forall y > 0$. Then (X_n) is aperiodic and irreducible. It is shown in Nummelin and Tuominen (1982) that condition (2.1) holds with $g(x) = \exp(tx) + 1$ for some positive t and C chosen as $[0, c]$ for some $c > 0$. It follows from Theorem 1 that an upper bound on the convergence rate to the invariant probability measure is exponential in x. In the special case when

$$a_1 = \begin{cases} -1 & \text{with probability } p \\ 0 & \text{with probability } q \end{cases} \tag{3.3}$$

with $p + q = 1$ and $0 < p < 1$, it is readily seen that the rate of convergence is indeed exponential in x.

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