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ON A CLASS OF ELLIPTIC SYSTEM OF SCHRÖDINGER–POISSON TYPE

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Abstract

In this paper we prove existence and qualitative properties of solutions for a nonlinear elliptic system arising from the coupling of the nonlinear Schrödinger equation with the Poisson equation. We use a contraction map approach together with estimates of the Bessel potential used to rewrite the system in an integral form.

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1. Introduction

We are concerned with existence and qualitative properties of solutions for the system

$$\begin{cases} -\Delta u + Vu + \omega K(x)\varphi u = a(x)|u|^{p-1}u + f(x) & \text{in } \mathbb{R}^n, \\ -\Delta \varphi = K(x)u^2 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $p > n/(n-2)$, $p \geq 2$, $n \geq 3$, the potential $V \geq 0$ is a constant, $\omega \in \mathbb{R}$ and K, a, f are given functions in some appropriate Lebesgue space. Throughout this paper, the weight functions K and a satisfy the following assumptions:

(H1) $K \in L^q(\mathbb{R}^n)$ for $q = n(p-1)/(2p-4)$ ($q = \infty$ if $p = 2$), $K(x) \geq 0$ for almost every $x \in \mathbb{R}^n$ and $K \not\equiv 0$;
(H2) $a \in L^\infty(\mathbb{R}^n)$ and $V = 1$.

Equations similar to (1.1) have been considered in [8, 12, 25, 26], where the authors studied the Thomas–Fermi–von Weizsäcker model in quantum mechanics theory. In this model $p = 5/3$ and u^p is replaced by $-u^p$. If one drops the nonlinear term u^p,

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then problem (1.1) is the Schrödinger–Poisson equation (also called the Schrödinger–Maxwell equation), which has been studied in connection with semiconductor theory; see [6, 7, 27, 28] and references therein. Taking $n = 3$, $K \equiv 1$, $a \equiv 1$ and $f \equiv 0$, system (1.1) reduces to

$$\begin{cases} -\Delta u + Vu + \omega\varphi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta\varphi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

Recent works dealing with (1.2) have addressed existence and nonexistence of solutions, multiplicity of solutions, ground states, radially and nonradially symmetric solutions, the semiclassical limit and concentration of solutions; see [1–3, 5, 13, 15–18, 24, 30–32, 34, 36]. In [15], there is proved the existence of a nontrivial radial solution of (1.2) when $3 < p < 5$ and V is a positive constant. The same result was established in [17] for $3 \le p < 5$. In [16], by using a Pohozaev-type identity, D'Aprile and Mugnai proved that (1.2) has no nontrivial H^1-solution for $p \le 1$ or $p \ge 6$. This result was completed in [31], where Ruiz showed that if $p \le 2$, then the problem (1.2) does not admit any nontrivial solution in $H^1 \times \mathcal{D}^{1,2}$ and, if $2 < p < 5$, there exists a nontrivial radial solution for (1.2). To the best of our knowledge, the first result on the existence of $H^1 \times \mathcal{D}^{1,2}$ ground state solutions to the problem (1.2) was obtained by Azzollini and Pomponio in [5] when $2 < p < 5$ and V is a positive constant. The nonconstant-potential case was also treated in [5] for $3 < p < 5$ and V being a function bounded from above. In [2], Azzollini dealt with the case $V = 0$ by means of the concentration–compactness principle and proved existence of a nonradial solution in $H^1 \times \mathcal{D}^{1,2}$ by considering a nonlinearity of Berestycki–Lions type in place of $|u|^{p-1}u$. The nonhomogeneous case, that is, $f \not\equiv 0$, has been treated in [9, 14, 33] and existence of multiple radially symmetric solutions for problem (1.1) was obtained. If $K = 1$, $a = 1$ and $n = 3$, problem (1.1) can be regarded as a perturbation problem of the homogeneous one:

$$\begin{cases} -\Delta u + u + \omega\varphi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta\varphi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

Candela and Salvatore [9] proved that if $p \ge 5$ and $u \in H^1(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ is a solution of (1.3) for some $q > 1$, then $u = \varphi = 0$. Thus, it is natural to wonder about existence of nontrivial solutions for the perturbation problem (1.1) in a L^q-framework. System (1.1) is different from commonly studied elliptic problems due to its nonlocal nonlinearity

$$\omega K(x)\varphi u = \omega K(x)\Big[\frac{c_n}{|x|^{n-2}} * (K(x)u^2)\Big]u,$$

where c_n is a positive constant. Nonlocal nonlinearities arise naturally in a wide variety of physical phenomena and their mathematical analysis is at the crossroads of a number of mathematical approaches; see for example [10] for minimization and flow-gradient techniques combined with probability ideas. Models like (1.1) have been treated in the light of variational methods, mainly by a suitable critical point theory as observed in [29]. Unlike this, we will use a nonvariational approach based on contraction arguments and scaling invariance. This general strategy has already

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been used in [19–22] to treat elliptic equations with nonlinearities depending on local operators and with singular anisotropic potentials in suitable spaces; namely, weighted singular $L^{\infty}(\mathbb{R}^n, |x|^k\,dx)$, anisotropic Lebesgue and pseudo-measure spaces. We show existence of a solution $(u,\varphi) \in L^{r_0}(\mathbb{R}^n) \times L^{r_0}(\mathbb{R}^n)$ for p in the range $n/(n-2) < p < \infty$ (with $p \geq 2$), which covers critical and supercritical cases for variational approaches. We use the Bessel potential to state the integral formulation of (1.1), where $r_0 = n(p-1)/2$ is the unique exponent such that the norm of $L^{r_0} \times L^{r_0}$ is invariant by the scaling $(u,\varphi) \to (u_\lambda, \varphi_\lambda) := \lambda^{2/p-1}(u(\lambda x), \varphi(\lambda x))$ defined in Section 2. In fact, we also show that $|\nabla u| \in L^{r_0}(\mathbb{R}^n)$ and then u is also a W^{1,r_0}-weak solution for (1.1). The existence of solutions for (1.1) in the general setting of the present paper, as well as their properties we proceed to describe, have not been addressed before. The solutions are unique in a suitable ball of L^{r_0} and depend continuously on the given data K and f (see Theorem 3.1). Also, we show qualitative properties of the obtained solutions like positivity, radial symmetry and parity, depending on the data K, a, f. For instance, solutions are even when $K(x), a(x), f(x)$ are also even functions, and they are positive if $K(x), a(x), f(x)$ are nonnegative functions, $f \not\equiv 0$ and $\omega < 0$. We refer the reader to [29] for further positivity results for (1.1) in the case $\omega < 0$. Finally, we remark that the proofs performed here work well for arbitrary $V \geq 0$ (including $V = 0$) and the hypothesis $V = 1$ above is prescribed only for the sake of simplicity. Also, our approach can be used to study the nonlinear Klein–Gordon equations coupled with Maxwell equations

$$\begin{cases} -\Delta u + [m^2 - (\omega+\varphi)^2]u = a(x)|u|^{p-1}u + f & \text{in } \mathbb{R}^3, \\ -\Delta\varphi + (\omega+\varphi)u^2 = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

which have been the object of study of many authors; see [4, 9, 11] and their references.

This paper is organized as follows. In the next section, we state and prove our existence result for (1.1). The continuous dependence of solutions with respect to given data is analyzed in Section 3. In the last section, we deal with symmetry and positivity properties of solutions.

2. Scaling and existence result

In this section, we are concerned with the existence of solutions for problem (1.1). From now on we assume that $\omega \neq 0$ and for simplicity take $V = 1$ in (1.1). Before stating our results, we perform a scaling analysis in order to find the right space setting to investigate existence of solutions for (1.1). For that matter, just for a moment, we assume that $K_\lambda(x) = \lambda^{\alpha} K(\lambda x)$ for some $\alpha \geq 0$. Let the pair (u,φ) be a classical solution for (1.1). We look for the values of k and l so that the rescaled pair $(u_\lambda, \varphi_\lambda)$, defined by

$$u_\lambda(x) = \lambda^k u(\lambda x) \quad \text{and} \quad \varphi_\lambda(x) = \lambda^l \varphi(\lambda x), \quad \lambda > 0,$$

is also a solution for (1.1) with $V = 0$. Inserting $u_\lambda(x)$ and $\varphi_\lambda(x)$ into (1.1) with $f \equiv 0$,

$$-\lambda^{l+2}\Delta\varphi(\lambda x) = \lambda^{2k+\alpha} K(\lambda x)(u(\lambda x))^2$$

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for all $\lambda > 0$ and $x \in \mathbb{R}^n$. Since (u, φ) is a solution,

$$l + 2 = \alpha + 2k. \tag{2.1}$$

Taking into account that

$$-\lambda^{k+2}\Delta u(\lambda x) + \lambda^k V u(\lambda x) + \omega\lambda^{k+l+\alpha} K(\lambda x)u(\lambda x)\varphi(\lambda x) = \lambda^{kp}|u(\lambda x)|^{p-1}u(\lambda x)$$

and ignoring the linear term,

$$k + l + \alpha = k + 2 = kp. \tag{2.2}$$

It follows from relations (2.1)–(2.2) that

$$k = l = \frac{2}{p-1} \quad \text{and} \quad \alpha = \frac{2p-4}{p-1}. \tag{2.3}$$

Motivated by this informal analysis, we define the scaling map

$$(u, \varphi) \to (u_\lambda, \varphi_\lambda) \tag{2.4}$$

with $k = l$ given in (2.3). Notice that in fact $(u_\lambda, \varphi_\lambda)$ is a solution of (1.1) for $V = 0$ whenever (u, φ) is, and then (2.4) works like an intrinsic scaling of (1.1) inherited from this latest case. For brevity, we simply call (2.4) *the scaling map* of (1.1). We shall study existence of solutions in Lebesgue spaces, whose norms are invariant by (2.4). Indeed, looking for invariant norms, the map (2.4) indicates the correct index in order to perform a contraction argument for (1.1). Set r_0 and r_1 by

$$r_0 = \frac{n}{k} = \frac{n(p-1)}{2} \quad \text{and} \quad r_1 = \frac{n}{k+1} = \frac{n(p-1)}{p+1}. \tag{2.5}$$

The index r_0 is the unique one that makes the $L^{r_0} \times L^{r_0}$-norm scaling invariant. Also, the norm $\|\nabla(\cdot)\|_{r_1}$ is invariant by (2.4) and it will be useful to reach regularity of solutions. Let $G_1(x - y) > 0$ be the Bessel kernel associated to the operator $\mathcal{L} = -\Delta + I$. We recall that the Bessel kernel G_α, $\alpha > 0$, is defined by the Fourier transform

$$\widehat{G_\alpha}(\xi) = (1 + |\xi|^2)^{-\alpha} \quad \text{for all } \xi \in \mathbb{R}^n. \tag{2.6}$$

The system (1.1) is formally equivalent to the following system of integral equations:

$$u(x) = \int_{\mathbb{R}^n} G_1(x - y)(a|u|^{p-1}u - \omega K u\varphi + f)(y)\,dy, \tag{2.7}$$

$$\varphi(x) = \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2}} (Ku^2)(y)\,dy, \tag{2.8}$$

where $\omega_n = \sigma(\mathbb{S}^{n-1})$ stands for the area of $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. We rewrite the integral system (2.7)–(2.8) in the functional form

$$\begin{cases} u = B_1(u) - \omega B_2(u, \varphi) + F(f), \\ \varphi = B_3(u), \end{cases} \tag{2.9}$$

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where

$$F(f)(x) = \int_{\mathbb{R}^n} G_1(x-y)f(y)\,dy, \tag{2.10}$$

$$B_1(u)(x) = \int_{\mathbb{R}^n} G_1(x-y)(a|u|^{p-1}u)(y)\,dy, \tag{2.11}$$

$$B_2(u,\varphi)(x) = \int_{\mathbb{R}^n} G_1(x-y)(Ku\varphi)(y)\,dy, \tag{2.12}$$

$$B_3(u)(x) = \frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}(Ku^2)(y)\,dy. \tag{2.13}$$

Now we are ready to state our existence result.

THEOREM 2.1. *Let $p > n/(n-2)$, $p \geq 2$, and let r_0, r_1 be as in (2.5). Suppose that $f \in L^\theta(\mathbb{R}^n)$ and $K \in L^q(\mathbb{R}^n)$, where*

$$\theta = \frac{n}{k+2} = \frac{n(p-1)}{2p} \quad \text{and} \quad q = \frac{n}{\alpha} = \frac{n(p-1)}{2p-4} \quad (q = \infty \text{ if } p = 2).$$

(A) *There exists $\varepsilon > 0$ such that if $\|f\|_\theta \leq \varepsilon/2C_1$, then the integral system (2.9) has a unique solution*

$$(u,\varphi) \in L^{r_0}(\mathbb{R}^n) \times L^{r_0}(\mathbb{R}^n)$$

satisfying $\|u\|_{r_0} \leq \varepsilon$ and $\|\varphi\|_{r_0} \leq \varepsilon$, where C_1 is as in Lemma 2.3.

(B) *Moreover, the pair (u,φ) is a solution in the sense of distributions and satisfies*

$$|\nabla u| \in L^{r_1}(\mathbb{R}^n) \quad \text{and} \quad |\nabla \varphi| \in L^{r_1}(\mathbb{R}^n).$$

In what follows, we establish some technical lemmas that will be needed in the proof of the main theorems. First, we recall the Hardy–Littlewood–Sobolev inequality in L^r spaces (see for example [23, Theorem 6.1.3, page 415]).

LEMMA 2.2 (Hardy–Littlewood–Sobolev). *Let r and z be such that $1 < r < z < \infty$, $1/z = 1/r - \beta/n$, where $0 < \beta < n$. Then there exists $C = C(r,\beta)$ such that*

$$\||x|^{-(n-\beta)} * f\|_z \leq C\|f\|_r$$

for all $f \in L^r(\mathbb{R}^n)$.

Before proceeding, we recall that (see Stein [35]) there exists $C > 0$ such that

$$0 \leq G_1(x) \leq C|x|^{2-n} \quad \text{for all } x \in \mathbb{R}^n, \tag{2.14}$$

$$|\nabla G_1(x)| \leq C|x|^{1-n} \quad \text{for all } x \in \mathbb{R}^n. \tag{2.15}$$

As a consequence of the Hardy–Littlewood–Sobolev inequality, our first task is to establish some useful estimates in our functional setting.

Lemma 2.3. *Under the hypotheses of Theorem 2.1, consider the operator defined by*

$$H(h)(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} h(y)\, dy \quad \text{and} \quad \widetilde{H}(h)(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} h(y)\, dy. \tag{2.16}$$

Then there exist constants $M_1, M_2 > 0$ (independent of h) such that

$$\|H(h)\|_{r_0} \le M_1 \|h\|_\theta, \tag{2.17}$$

$$\|\widetilde{H}(h)\|_{r_1}, \|\nabla H(h)\|_{r_1} \le M_2 \|h\|_\theta. \tag{2.18}$$

Furthermore, if m is a multi-index and $1 < t_1 < b_1 < \infty$ with $1/b_1 = 1/t_1 - 2/n$, then there exists a constant $M_3 > 0$ (independent of h and m) such that

$$\|\nabla^m H(h)\|_{b_1} \le M_3 \|\nabla^m h\|_{t_1}. \tag{2.19}$$

Proof. In view of (2.5), we have that $1/r_0 = 1/\theta - 2/n$. According to Lemma 2.2 with $z = r_0$, $r = \theta$ and $\beta = 2$,

$$\|H(h)\|_{r_0} = \left\| \frac{1}{|x|^{n-2}} * h \right\|_{r_0} \le C \|h\|_\theta,$$

which yields estimate lemma 2.3. To prove lemma 2.3, first we observe that

$$\nabla H(h)(x) = \int_{\mathbb{R}^n} \nabla_x \Big(\frac{1}{|x-y|^{n-2}} \Big) h(y)\, dy$$

and

$$\left| \nabla_x \Big(\frac{1}{|x-y|^{n-2}} \Big) \right| \le \frac{C}{|x-y|^{n-1}}.$$

Taking into account the fact that

$$\frac{1}{r_1} = \frac{k}{n} + \frac{1}{n} = \frac{k+2}{n} - \frac{1}{n} = \frac{1}{\theta} - \frac{1}{n},$$

Lemma 2.2 with $z = r_1$, $r = \theta$ and $\beta = 1$ yields

$$\begin{aligned} \|\nabla H(h)\|_{r_1} &\le C \left\| \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} |h(y)|\, dy \right\|_{r_1} \\ &\le C \|h\|_\theta, \end{aligned}$$

which is the estimate (2.18). Since $H(h)$ is a convolution, we can compute the weak derivative $\nabla^m H(h)$ as

$$\nabla^m H(h)(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} (\nabla^m h)(y)\, dy.$$

In view of the hypothesis $1/b_1 = 1/t_1 - 2/n$, Lemma 2.2 with $z = b_1$, $r = t_1$ and $\beta = 2$ yields

$$\begin{aligned} \|\nabla^m H(h)\|_{b_1} &= \left\| \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} (\nabla^m h)(y)\, dy \right\|_{b_1} \\ &\le M_3 \|\nabla^m h\|_{t_1}, \end{aligned}$$

and the estimate (2.19) follows. □

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REMARK 2.4. Invoking (2.14)–(2.15), we conclude that there exists a constant $C > 0$ such that

$$|\nabla^m F(f)(x)| \leq CH(|\nabla^m f|)(x) \quad \text{and} \quad |\nabla F(f)(x)| \leq C\widetilde{H}(|f|)(x) \tag{2.20}$$

for all $x \in \mathbb{R}^n$ and $m \in \mathbb{N}$. Thus, by Lemma 2.3, we obtain $C_1, C_2 > 0$ (independent of f) such that

$$\|F(f)\|_{r_0} \leq C_1 \|f\|_\theta, \tag{2.21}$$

$$\|\nabla F(f)\|_{r_1} \leq C_2 \|f\|_\theta. \tag{2.22}$$

Furthermore, if m is a multi-index and $1/b_1 = 1/t_1 - 2/n$, then there exists a constant $C_3 > 0$ (independent of f) such that

$$\|\nabla^m F(f)\|_{b_1} \leq C_3 \|\nabla^m f\|_{t_1}.$$

In the next three lemmas we estimate the nonlinear operators in (2.11)–(2.13).

LEMMA 2.5 (Estimate of B_1). *Under the hypotheses of Theorem 2.1, there are positive constants L_1, K_1 such that*

$$\|B_1(u) - B_1(v)\|_{r_0} \leq L_1 \|u - v\|_{r_0} (\|u\|_{r_0}^{p-1} + \|v\|_{r_0}^{p-1}), \tag{2.23}$$

$$\|\nabla [B_1(u) - B_1(v)]\|_{r_1} \leq K_1 \|u - v\|_{r_0} (\|u\|_{r_0}^{p-1} + \|v\|_{r_0}^{p-1}) \tag{2.24}$$

for all $u, v \in L^{r_0}(\mathbb{R}^n)$.

PROOF. We first observe that $B_1(u) - B_1(v) = F(au|u|^{p-1} - av|v|^{p-1})$. Invoking the pointwise estimate

$$|s|s|^{p-1} - t|t|^{p-1}| \leq p|s - t|(|s|^{p-1} + |t|^{p-1}) \quad \text{for all } s, t \in \mathbb{R}, \tag{2.25}$$

and using the Hölder inequality with

$$\frac{1}{\theta} = \frac{k+2}{n} = \frac{1}{r_0} + \frac{p-1}{r_0},$$

we infer from (2.21) that

$$\begin{aligned} \|B_1(u) - B_1(v)\|_{r_0} &\leq C_1 p \|a\|_\infty \| \, |u - v|(|u|^{p-1} + |v|^{p-1})\|_\theta \\ &\leq L_1 \|u - v\|_{r_0} (\|u\|_{r_0}^{p-1} + \|v\|_{r_0}^{p-1}), \end{aligned}$$

which proves estimate (2.23). To prove (2.24), first we observe that

$$\nabla B_1(u)(x) = \int_{\mathbb{R}^n} \nabla_x (G_1(x - y))(a|u|^{p-1}u)(y)\, dy.$$

Then we can invoke (2.15) and (2.20) to derive that

$$\begin{aligned} |\nabla B_1(u)(x) - \nabla B_1(v)(x)| &= |\nabla F[(a|u|^{p-1}u - a|v|^{p-1}v)](x)| \\ &\leq C\|a\|_\infty \widetilde{H}[| \, |u|^{p-1}u - |v|^{p-1}v|](x). \end{aligned}$$

Using estimates (2.25) and (2.18), we easily conclude that estimate (2.24) holds. □

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Lemma 2.6 (Estimate of B_2). *Under the hypotheses of Theorem 2.1, there are positive constants* L_2, K_2 *such that*

$$\|B_2(u,\varphi)\|_{r_0} \le L_2\|u\|_{r_0}\|\varphi\|_{r_0}, \tag{2.26}$$
$$\|\nabla B_2(u,\varphi)\|_{r_1} \le K_2\|u\|_{r_0}\|\varphi\|_{r_0} \tag{2.27}$$

for all $u, \varphi \in L^{r_0}(\mathbb{R}^n)$.

Proof. Clearly, we have $B_2(u,\varphi) = F(Ku\varphi)$. Using inequality (2.21) and the Hölder inequality with $1/\theta = n/\alpha = 1/r_0 + 1/r_0 + 1/q$, we conclude that

$$\|B_2(u,\varphi)\|_{r_0} \le C_1\|Ku\varphi\|_\theta \le C_1\|K\|_q\|u\|_{r_0}\|\varphi\|_{r_0},$$

which is the required inequality with $L_2 = C_1\|K\|_q$. To prove (2.27), we observe that $\nabla B_2(u,\varphi) = \nabla F(Ku\varphi)$. Therefore, we can use (2.22) to obtain

$$\|\nabla B_2(u,\varphi)\|_{r_1} = \|\nabla F(Ku\varphi)\|_{r_1} \le C_2\|Ku\varphi\|_\theta,$$

and the result follows by the Hölder inequality. □

Lemma 2.7 (Estimate of B_3). *Under the hypotheses of Theorem 2.1, there are constants* $L_3, K_3 > 0$ *such that*

$$\|B_3(u) - B_3(v)\|_{r_0} \le L_3\|u-v\|_{r_0}(\|u\|_{r_0} + \|v\|_{r_0}), \tag{2.28}$$
$$\|\nabla[B_3(u) - B_3(v)]\|_{r_1} \le K_3\|u-v\|_{r_0}(\|u\|_{r_0} + \|v\|_{r_0}) \tag{2.29}$$

for all $u, v \in L^{r_0}(\mathbb{R}^n)$.

Proof. Clearly, $B_3(u) - B_3(\varphi) = H(K(u^2 - v^2))$. Since

$$\frac{1}{\theta} = \frac{n}{\alpha} = \frac{1}{r_0} + \frac{1}{r_0} + \frac{1}{q},$$

using inequality (2.17) together with the Hölder inequality we conclude that

$$\begin{aligned}\|B_3(u) - B_3(v)\|_{r_0} &\le M_1\|K(u^2 - v^2)\|_\theta \\ &\le M_1\|K\|_q\|u-v\|_{r_0}(\|u\|_{r_0} + \|v\|_{r_0}),\end{aligned}$$

which gives the desired inequality with $L_3 = M_1\|K\|_q$. To prove (2.29), in view of (2.18),

$$\begin{aligned}\|\nabla[B_3(u) - B_3(v)]\|_{r_1} &= \|\nabla H(K(u^2 - v^2))\|_{r_1} \\ &\le M_2\|K(u^2 - v^2)\|_\theta,\end{aligned}$$

and the result follows by the Hölder inequality. □

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Now we are ready to present the proof of Theorem 2.1.

2.1. Proof of Theorem 2.1. *Part A*. Notice that the system (2.9) can be rewritten as follows:

$$u = B_1(u) - \omega B_2(u, B_3(u)) + F(f). \tag{2.30}$$

Consider the map $T : L^{r_0}(\mathbb{R}^n) \to L^{r_0}(\mathbb{R}^n)$ defined by

$$T(u) = B_1(u) - \omega B_2(u, B_3(u)) + F(f).$$

In view of the previous estimates, the map T is well defined. Furthermore, if $\|f\|_\theta \le \varepsilon/2C_1$ and $\|u\|_{r_0} \le \varepsilon$,

$$\begin{aligned}\|T(u)\|_{r_0} &\le \|B_1(u)\|_{r_0} + |\omega|\,\|B_2(u, B_3(u))\|_{r_0} + \|F(f)\|_{r_0}\\ &\le L_1\|u\|_{r_0}^p + L_2|\omega|\,\|u\|_{r_0}\|B_3(u)\|_{r_0} + C_1\|f\|_\theta\\ &\le L_1\varepsilon^p + L_2L_3|\omega|\,\|u\|_{r_0}\|u\|_{r_0}^2 + \frac{\varepsilon}{2}\\ &\le L_1\varepsilon^p + L_2L_3|\omega|\varepsilon^3 + \frac{\varepsilon}{2} < \varepsilon,\end{aligned}$$

provided that

$$2L_1\varepsilon^{p-1} + 2L_2L_3|\omega|\varepsilon^2 < 1. \tag{2.31}$$

Thus, if $\mathcal{B}_\varepsilon = \{u \in L^{r_0} : \|u\|_{r_0} \le \varepsilon\}$, then we conclude that $T(\mathcal{B}_\varepsilon) \subset \mathcal{B}_\varepsilon$ for $\varepsilon > 0$ as in (2.31). We claim that indeed $T : \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$ is a contraction. Taking $u, v \in \mathcal{B}_\varepsilon$ and using estimates (2.23), (2.26) and (2.28),

$$\begin{aligned}&\|T(u) - T(v)\|_{r_0}\\ &\quad\le \|B_1(u) - B_1(v)\|_{r_0} + |\omega|\,\|B_2(u, B_3(u)) - B_2(v, B_3(v))\|_{r_0}\\ &\quad\le \|B_1(u) - B_1(v)\|_{r_0} + |\omega|\,\|B_2(u - v, B_3(u))\|_{r_0} + |\omega|\,\|B_2(v, B_3(u) - B_3(v))\|_{r_0}.\end{aligned} \tag{2.32}$$

Now observe that

$$\begin{aligned}\|B_2(u - v, B_3(u))\|_{r_0} &\le L_2\|u - v\|_{r_0}\|B_3(u)\|_{r_0}\\ &\le L_2L_3\|u - v\|_{r_0}\|u\|_{r_0}^2\\ &\le L_2L_3\varepsilon^2\|u - v\|_{r_0}\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\|B_2(v, B_3(u) - B_3(v))\|_{r_0} &\le L_2\|v\|_{r_0}\|B_3(u) - B_3(v)\|_{r_0}\\ &\le L_2L_3\|v\|_{r_0}\|u - v\|_{r_0}(\|u\|_{r_0} + \|v\|_{r_0})\\ &\le 2L_2L_3\varepsilon^2\|u - v\|_{r_0}.\end{aligned} \tag{2.34}$$

It follows from estimates (2.23), (2.32), (2.33) and (2.34) that

$$\|T(u) - T(v)\|_{r_0} \le [2L_1\varepsilon^{p-1} + |\omega|(3L_2L_3)\varepsilon^2]\|u - v\|_{r_0}.$$

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Choosing $\varepsilon > 0$ sufficiently small in such a way that $2L_1\varepsilon^{p-1} + |\omega|(3L_2L_3)\varepsilon^2 < 1$, our claim is proved. For each $\varepsilon > 0$ fixed, the ball $\mathcal{B}_\varepsilon = \{u \in L^{r_0} : \|u\|_{r_0} \leq \varepsilon\}$ endowed with the metric $\mathcal{Z}(u, v) = \|u - v\|_{r_0}$ is complete. Hence, T has a fixed point in $\mathcal{B}_\varepsilon$, which is the unique solution u for (2.30) satisfying $\|u\|_{r_0} \leq \varepsilon$. Now, going back to the second equation in (2.9) and reducing $\varepsilon > 0$ if necessary,

$$\|\varphi\|_{r_0} = \|B_3(u)\|_{r_0} \leq L_3\|u\|_{r_0}^2 \leq (L_3\varepsilon)\varepsilon \leq \varepsilon,$$

which concludes the proof of Part A.

Part B. Since $f \in L^\theta(\mathbb{R}^n)$ and $(u, \varphi) \in L^{r_0}(\mathbb{R}^n) \times L^{r_0}(\mathbb{R}^n)$ satisfies (2.9), it follows from (2.29) with $\varphi = 0$ that

$$\|\nabla\varphi\|_{r_1} = \|\nabla B_3(u)\|_{r_1}^2 \leq K_3\|u\|_{r_0}^2 < \infty.$$

On the other hand, by (2.9) and the estimates (2.24) and (2.29),

$$\begin{aligned}\|\nabla u\|_{r_1} &\leq \|\nabla F(Ku\varphi)\|_{r_1} + \|\nabla F(f)\|_{r_1} \\ &\leq C_1\|Ku\varphi\|_\theta + C_1\|f\|_\theta \\ &\leq C_1\|K\|_q\|u\|_{r_0}\|\varphi\|_{r_0} + C_1\|f\|_\theta < \infty,\end{aligned}$$

and this concludes the proof. □

3. Continuous dependence

In this section, we show the continuous dependence of solutions for the given data.

THEOREM 3.1. *Under the assumptions of Theorem 2.1, let u and $\tilde{u}$ be solutions as in Part A of Theorem 2.1 corresponding to (f, K, ε) and $(\tilde{f}, \tilde{K}, \tilde{\varepsilon})$, respectively. Then*

$$\|u - \tilde{u}\|_{r_0} \leq \frac{C_1[|\omega|\varepsilon^2(1 + M_1\tilde{\varepsilon}\|\tilde{K}\|_q) + 1]}{1 - \psi(\varepsilon, \tilde{\varepsilon})}(\|K - \tilde{K}\|_q + \|f - \tilde{f}\|_\theta), \tag{3.1}$$

provided that

$$\psi(\varepsilon, \tilde{\varepsilon}) = C_1[(\varepsilon^{p-1} + \tilde{\varepsilon}^{p-1})\|a\|_\infty + \varepsilon\|\tilde{K}\|_q + \tilde{\varepsilon}(\varepsilon + \tilde{\varepsilon})M_1|\omega|\|\tilde{K}\|_q^2] < 1. \tag{3.2}$$

In particular, the data-solution map $(f, K) \to (u, \varphi)$ is continuous.

PROOF. We have that $\|(u, \varphi)\|_{r_0} \leq \varepsilon$ and $\|(\tilde{u}, \tilde{\varphi})\|_{r_0} \leq \tilde{\varepsilon}$ with

$$u = B_1(u) - \omega F(Ku\varphi) + F(f)$$

and

$$\tilde{u} = B_1(\tilde{u}) - \omega F(\tilde{K}\tilde{u}\tilde{\varphi}) + F(\tilde{f}).$$

Subtracting the last two inequalities,

$$\begin{aligned}\|u - \tilde{u}\|_{r_0} \leq{}& \|a\|_\infty C_1\|u - \tilde{u}\|_{r_0}(\|u\|^{p-1} + \|\tilde{u}\|_{r_0}^{p-1}) \\ &+ |\omega|\, \|F(Ku\varphi - \tilde{K}\tilde{u}\tilde{\varphi})\|_{r_0} + C_1\|f - \tilde{f}\|_\theta.\end{aligned} \tag{3.3}$$

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Since

$$K u\varphi - \tilde{K}\tilde{u}\tilde{\varphi} = (K - \tilde{K})u\varphi + \tilde{K}(u - \tilde{u})\varphi + \tilde{K}\tilde{u}(\varphi - \tilde{\varphi})$$

and $1/\theta = n/\alpha = 1/r_0 + 1/r_0 + 1/q$, one can use the Hölder inequality together with Lemma 2.2 to infer that

$$\begin{aligned}
&\|F(Ku\varphi - \tilde{K}\tilde{u}\tilde{\varphi})\|_{r_0} \\
&\quad \le C_1(\|K - \tilde{K}\|_q\|u\|_{r_0}\|\varphi\|_{r_0} + \|\tilde{K}\|_q\|u - \tilde{u}\|_{r_0}\|\varphi\|_{r_0} + \|\tilde{K}\|_q\|\tilde{u}\|_{r_0}\|\varphi - \tilde{\varphi}\|_{r_0}) \\
&\quad \le C_1(\varepsilon^2\|K - \tilde{K}\|_q + \varepsilon\|\tilde{K}\|_q\|u - \tilde{u}\|_{r_0} + \tilde{\varepsilon}\|\tilde{K}\|_q\|\varphi - \tilde{\varphi}\|_{r_0}). \qquad (3.4)
\end{aligned}$$

On the other hand, using again the Hölder inequality together with Lemma 2.2,

$$\begin{aligned}
\|\varphi - \tilde{\varphi}\|_{r_0} &= \|H((K - \tilde{K})u^2 + \tilde{K}(u^2 - \tilde{u}^2))\|_{r_0} \\
&\le M_1[(\|K - \tilde{K}\|_q\|u\|_{r_0}^2 + \|\tilde{K}\|_q\|u - \tilde{u}\|_{r_0}(\|u\|_{r_0} + \|\tilde{u}\|_{r_0}))] \\
&\le M_1(\varepsilon^2\|K - \tilde{K}\|_q + (\varepsilon + \tilde{\varepsilon})\|\tilde{K}\|_q\|u - \tilde{u}\|_{r_0}). \qquad (3.5)
\end{aligned}$$

From estimates (3.3), (3.4) and (3.5),

$$\begin{aligned}
\|u - \tilde{u}\|_{r_0} &\le C_1|\omega|\varepsilon^2(1 + M_1\tilde{\varepsilon}\|\tilde{K}\|_q)\|K - \tilde{K}\|_q + C_1\|f - \tilde{f}\|_\theta \\
&\quad + C_1[(\varepsilon^{p-1} + \tilde{\varepsilon}^{p-1})\|a\|_\infty + \varepsilon\|\tilde{K}\|_q + \tilde{\varepsilon}(\varepsilon + \tilde{\varepsilon})M_1|\omega|\,\|\tilde{K}\|_q^2]\|u - \tilde{u}\|_{r_0} \\
&\le C_1|\omega|\varepsilon^2(1 + M_1\tilde{\varepsilon}\|\tilde{K}\|_q)\|K - \tilde{K}\|_q + C_1\|f - \tilde{f}\|_\theta + \psi(\varepsilon, \tilde{\varepsilon})\|u - \tilde{u}\|_{r_0},
\end{aligned}$$

which, together with (3.2), gives (3.1).

The last assertion in the statement follows at once from (3.1) and (3.5). □

4. Qualitative properties

In this section, we deal with symmetry and positivity properties of the solution depending on K, a, f.

THEOREM 4.1. *Assume the hypotheses of Theorem 2.1. Then we have the following statements.*

(A) *(Radial symmetry) If $K(x), a(x), f(x)$ are radial, then the solution (u, φ) is radial.*

(B) *(Parity) If $K(x), a(x), f(x)$ are even functions, then u and φ are even functions. If $K(x), a(x)$ are even functions and $f(x)$ is odd, then u is odd and φ is even.*

(C) *(Positivity) Let $f \not\equiv 0$ be nonnegative. The solution (u, φ) is positive, that is, $\varphi > 0$ and $u > 0$, when $a(x), K(x)$ are nonnegative functions and $\omega < 0$.*

PROOF. *Part A*. Recall that f is radial if and only if $f(x) = f(Ax)$ for every orthogonal transformation A. Denoting $f_A(x) = f(Ax)$, for each such A,

$$\begin{aligned}
F(f)(A(x)) &= \int_{\mathbb{R}^n} G_1(A(x) - y)f(y)\,dy \\
&= \int_{\mathbb{R}^n} G_1(A(x - A^{-1}(y)))f(y)\,dy \\
&= \int_{\mathbb{R}^n} G_1(x - A^{-1}(y))f(y)\,dy,
\end{aligned}$$

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because G_1 is radially symmetric (see (2.6)). The change of variables $A^{-1}(y) = z$ yields

$$F(f)(A(x)) = \int_{\mathbb{R}^n} G_1(x-z)[f(A(z))]\,dz = \int_{\mathbb{R}^n} G_1(x-z)f(z)\,dz = F(f)(x).$$

Therefore, $F(f)$ is radial whenever f is radial. Similarly, if h is radial, then $H(h)$ is also, where the operator $H(\cdot)$ is defined in (2.16). Rewriting (2.7)–(2.8) as

$$u = F(a|u|^{p-1}u) - \omega F(Ku\varphi) + F(f) \quad \text{and} \quad \varphi = \frac{1}{(n-2)\omega_n} H(Ku^2), \tag{4.1}$$

one can see that the pair (u_A, φ_A) is also a solution for (4.1). In view of L^p-norms being invariant by the operator $f \to f_A$, it follows that $\|(u_A, \varphi_A)\|_{L^{r_0}\times L^{r_0}} = \|(u, \varphi)\|_{L^{r_0}\times L^{r_0}}$. So, for each rotation A, $(u, \varphi) = (u_A, \varphi_A)$ because solutions are unique in the ball $\|(u, \varphi)\|_{L^{r_0}\times L^{r_0}} \le \varepsilon$.

Part B. Let (u, φ) be the solution of (4.1) and denote $(\tilde{u}, \tilde{\varphi}) = (u(-x), \varphi(-x))$. Let f be even, that is, $f(z) = f(-z)$. Then, since G_1 is even (see (2.6)),

$$F(f)(-x) = \int_{\mathbb{R}^n} G_1(-x-z)[f(z)]\,dz = \int_{\mathbb{R}^n} G_1(x-z)f(-z)\,dz = F(f)(x),$$

which implies that $F(f)$ is also even. Similarly, $H(h)$ is even provided that h is also. So, we can check that $(\tilde{u}, \tilde{\varphi})$ also verifies (4.1) and $\|(\tilde{u}, \tilde{\varphi})\|_{L^{r_0}\times L^{r_0}} = \|(u, \varphi)\|_{L^{r_0}\times L^{r_0}} \le \varepsilon$. Again, by uniqueness, we obtain $(u, \varphi) = (\tilde{u}, \tilde{\varphi})$, as required.

The second part of the statement follows similarly by considering $(\tilde{u}, \tilde{\varphi}) = (-u(-x), \varphi(-x))$ and f odd instead of $(\tilde{u}, \tilde{\varphi}) = (u(-x), \varphi(-x))$ and f even, respectively.

Part C. From the fixed point argument in the proof of Theorem 2.1(A), we deduce that the solution (u, φ) can be obtained as the limit in $L^{r_0} \times L^{r_0}$ of the Picard interaction

$$u_{k+1} = B_1(u_k) - \omega B_2(u_k, \varphi_k) + F(f) \quad \text{and} \quad \varphi_k = B_3(u_k), \quad k \in \mathbb{N},$$

where $u_1 = F(f)$. We have that $f(x) > 0$ in a positive measure set Q. In view of (2.10), it follows that $u_1 > 0$ almost everywhere in $\mathbb{R}^n$, because the Bessel kernel $G_1(x-y)$ is positive. By an induction argument, one can see that u_k and φ_k are positive if $a, K \ge 0$ and $\omega < 0$. Since $(u_k, \varphi_k) \to (u_k, \varphi_k)$ in $L^{r_0} \times L^{r_0}$, it follows that $u \ge 0$ almost everywhere in $\mathbb{R}^n$. Returning to the integral equation (2.30),

$$u = B_1(u) - \omega F(KuB_3(u)) + u_1 \ge 0 + 0 + u_1 > 0 \quad \text{almost everywhere in } \mathbb{R}^n.$$

Since $\varphi = B_3(u)$, we also get that $\varphi > 0$ almost everywhere in $\mathbb{R}^n$. This concludes the proof. □

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