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ON INEQUALITIES OF HILBERT AND WIDDER

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A short proof of Chow's generalization of Widder's inequality and generalizations of some related Hardy's results are given.

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1. Let $a=(a_0,a_1,a_2,\ldots)$ be a nonnegative sequence. We use the notation

$$A(x)=\sum a_n x^n,\ A^*(x)=\sum \frac{a_n x^n}{n!},\ a(x)=e^{-x}A^*(x),$$

where the summations run from 0 to $+\infty$.

Then

$$\sum\sum \frac{a_m a_n}{m+n+1} \leqq \pi \sum\sum \frac{(m+n)!}{m!n!}\,\frac{a_m a_n}{2^{m+n+1}}, \tag{1}$$

$$\int_0^1 A^2(x)\,dx \leqq \pi \int_0^{+\infty} a^2(x)\,dx. \tag{2}$$

These inequalities are equivalent, and they are known as Widder's inequalities. Note that (1) is stronger than the well-known Hilbert inequality (see [**4**]). G. H. Hardy [**2**] showed that (2) (and so (1)) can be obtained by using the same Hilbert inequality (see also [**3**, pp. 238–239]).

The following generalization of Widder's inequality is given by Y. C. Chow [**1**]:

Theorem 1. *Let a and b be two nonnegative sequences, $p>1$, $p'=p/(p-1)$. Then*

$$\sum\sum \frac{a_m b_n}{m+n+1} \leqq (\pi/\sin(\pi/p))\left(\int_0^{+\infty} a^p(x)\,dx\right)^{1/p}\left(\int_0^{+\infty} b^{p'}(x)\,dx\right)^{1/p'}, \tag{3}$$

where $B(x)$, $B^*(x)$ and $b(x)$ are defined like $A(x)$, $A^*(x)$ and $a(x)$.

Here we shall show that (3) can also be proved by using Hilbert's inequality. Indeed, we have

$$A(x)=\int_0^{+\infty} e^{-t}A^*(xt)\,dt=\frac{1}{x}\int_0^{+\infty} e^{-u/x}A^*(u)\,du,$$

$$B(x)=\frac{1}{x}\int_0^{+\infty} e^{-u/x}B^*(u)\,du,$$

and so

$$\int_0^1 A(x)B(x)\,dx=\int_0^1 \frac{dx}{x^2}\int_0^{+\infty} e^{-u/x}A^*(u)\,du\int_0^{+\infty} e^{-u/x}B^*(u)\,du.$$

Putting $1/x=t+1$, we have

$$\int_0^1 A(x)B(x)\,dx=\int_0^{+\infty} dt\int_0^{+\infty} e^{-tu}a(u)\,du\int_0^{+\infty} e^{-tv}b(v)\,dv$$

$$=\int_0^{+\infty}\int_0^{+\infty}\frac{a(u)b(v)}{u+v}\,du\,dv$$

$$\leqq(\pi/\sin(\pi/p))\left(\int_0^{+\infty} a^p(x)\,dx\right)^{1/p}\left(\int_0^{+\infty} b^{p'}(x)\,dx\right)^{1/p'}.$$

The last inequality is the well-known Hilbert inequality. □

2. Another generalization of (2) is given in **[2]**:
If $p>1$, then

$$\int_0^1 z^{p-2}A^p(z)\,dz\leqq\left\{\Gamma\left(\frac{1}{p}\right)\right\}^p\int_0^{+\infty} z^{p-2}a^p(z)\,dz. \tag{4}$$

The following result, similar to Theorem 1, is a simple consequence of this result:

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. Then*

$$\sum\sum \frac{a_m b_n}{m+n+1} \leqq (\pi/\sin(\pi/p))\left(\int_0^{+\infty} z^{p-2}a^p(z)\,dz\right)^{1/p}\left(\int_0^{+\infty} z^{p'-2}b^{p'}(z)\,dz\right)^{1/p'}. \tag{5}$$

Proof. Apply Hölder's inequality to $\int_0^1 A(z)B(z)\,dz$ written as $\int_0^1 z^{-2}(zA(z))(zB(z))\,dz$, and then use (4).

3. The following result is a generalization of Theorem 4 from [**2**]. (See also Theorem 353 from [**3**, p. 257].)

Theorem 3. *Suppose that* $K_0(x) \geqq 0$, *that*

$$K_1(x,y) = \int_0^{+\infty} K_0(xt)K_0(yt)\,dt$$

(*so that* K_1 *is symmetric and homogeneous of degree* -1), *and that*

$$K_2(x,y) = \int_0^{+\infty} K_1(x,t)K_1(y,t)\,dt.$$

Then, for positive sequences a and b,

$$\left(\sum\sum K_2(m,n)a_m b_n\right)^2 \leqq J^2\left(\sum\sum K_1(m,n)a_m a_n\right)\left(\sum\sum K_1(m,n)b_m b_n\right), \tag{6}$$

where the summations extend from 1 *to* $+\infty$ *and*

$$J = \int_0^{+\infty} \frac{K_1(w,1)}{\sqrt{w}}\,dw.$$

Proof. The argument is similar to that used by Hardy. It is shown that

$$\sum\sum K_2(m,n)a_m b_n = \int_0^1 K_1(w,1)\Omega(w)\,dw,$$

where

$$\Omega(w) = \sum\sum K_1(mw,n)a_m b_n = \int_0^{+\infty} \phi_a(wt)\phi_b(t)\,dt,$$

and $\phi_c(t) = \sum c_n K_0(nt)$ $(c = a, b)$. The final steps are the same as those in [**2**].

As in [**2**] we can give the following particular cases:

$$\sum\sum \frac{\log(m/n)}{m-n} a_m b_n \leqq \pi \left(\sum\sum \frac{a_m a_n}{m+n} \right)^{1/2} \left(\sum\sum \frac{b_m b_n}{m+n} \right)^{1/2}, \tag{7}$$

where the coefficient on the left-hand side is to be interpreted as $1/n$ when $m=n$;

$$\sum\sum \frac{|\log(m/n)|}{\max(m,n)} a_m b_n \leqq 2 \left(\sum\sum \frac{a_m a_n}{\max(m,n)} \right)^{1/2} \left(\sum\sum \frac{b_m b_n}{\max(m,n)} \right)^{1/2}. \tag{8}$$

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