6

An Introduction to Diagrammatic Soergel Bimodules

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6.1 Motivation

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, with a Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b}$. The Cartan subalgebra $\mathfrak{h}$ gives rise to a root system $\Phi \subset \mathfrak{h}^*$, and the choice of Borel subalgebra corresponds to a selection of simple roots Σ and positive roots Φ^+ inside Φ. The root system Φ induces a Weyl group W generated by the set S of reflections in the simple roots Σ. (We will later generalize this situation in Definition 6.3.) Inside $\mathfrak{h}^*$ we also have

$$\Lambda = \{\lambda \in \mathfrak{h}^* : \langle \alpha^\vee, \lambda \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Sigma\} \tag{6.1.1}$$

$$\cup$$

$$\Lambda^+ = \{\lambda \in \mathfrak{h}^* : \langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}_{\geqslant 0} \text{ for all } \alpha \in \Sigma\}. \tag{6.1.2}$$

Finally let $U\mathfrak{g}$ denote the universal enveloping algebra of $\mathfrak{g}$. We will consider $\mathfrak{g}$-modules and $U\mathfrak{g}$-modules interchangeably. A standard reference for all the facts about $\mathfrak{g}$-modules in this section is [11].

For each $\lambda \in \Lambda$, define the *Verma module* $M(\lambda) = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$. (Here $U\mathfrak{b}$ denotes the universal enveloping algebra of $\mathfrak{b}$, while $\mathbb{C}_\lambda$ denotes the 1-dimensional $\mathfrak{b}$-module given by $\mathfrak{b} \to \mathfrak{h} \xrightarrow{\lambda} \mathbb{C}$.) Each Verma module $M(\lambda)$ has a unique simple quotient $L(\lambda)$, which is the unique simple weight module of highest weight λ. The simple module $L(\lambda)$ is finite dimensional if and only if $\lambda \in \Lambda^+$.

The category $U\mathfrak{g}$-mod of *all* $U\mathfrak{g}$-modules is too large to be useful. Instead we restrict our attention to a smaller category which contains Verma modules and highest weight simple modules.

Definition 6.1 Let $\lambda \in \Lambda^+$, and write $U\mathfrak{g}\text{-mod}_{U\mathfrak{h}\text{-ss}}$ for the category of $\mathfrak{g}$-modules which are semisimple as $\mathfrak{h}$-modules. (In other words, $U\mathfrak{g}\text{-mod}_{U\mathfrak{h}\text{-ss}}$

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is the category of weight modules.) We define $\mathcal{O}_\lambda$ to be the minimal full subcategory of $U\mathfrak{g}\text{-mod}_{U\mathfrak{h}\text{-ss}}$ that contains $M(\lambda)$ and is closed under submodules, quotients and extensions.

It is obvious that $\mathcal{O}_\lambda$ is an abelian category. It is somewhat less obvious that $\mathcal{O}_\lambda$ is in fact a *finite* abelian category, with finite length objects, finitely many isomorphism classes of simple objects and finite-dimensional Hom-spaces.

Remark The above definition of $\mathcal{O}_\lambda$ is non-standard. Most treatments (e.g. [11]) first define the *BGG category* $\mathcal{O}$ which contains all Verma modules and all highest weight simple modules. Then $\mathcal{O}_\lambda$ is defined for arbitrary $\lambda \in \mathfrak{h}^*$ as a subcategory of $\mathcal{O}$ with a certain prescribed action of the centre $Z\mathfrak{g}$ of $U\mathfrak{g}$. In general $\mathcal{O}_\lambda$ is a union of blocks of $\mathcal{O}$, and when $\lambda \in \Lambda^+$, one can show that $\mathcal{O}_\lambda$ is the block containing $L(\lambda)$.

Example 6.2 Suppose $\mathfrak{g} = \mathfrak{sl}_2$. The corresponding root system Φ is of Dynkin type A_1, with Weyl group $W = \{1, s\}$. Within $\mathfrak{h}^*$ there are obvious identifications $\Lambda \cong \mathbb{Z}$ and $\Lambda^+ \cong \mathbb{Z}_{\geqslant 0}$. Let $n \in \mathbb{Z}_{\geqslant 0}$. The indecomposable objects in $\mathcal{O}_n$ are $L(n), L(-n-2) = M(-n-2), M(n) = P(n)$ and $P(-n-2)$. The structures of the last two modules are given by the exact sequences

$$0 \longrightarrow L(-n-2) \longrightarrow M(n) \longrightarrow L(n) \longrightarrow 0,$$

$$0 \longrightarrow M(n) \longrightarrow P(-n-2) \longrightarrow M(-n-2) \longrightarrow 0.$$

Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ be the half-sum of the positive roots. For $w \in W$ and $\lambda \in \mathfrak{h}^*$, we define the following shift

$$w \cdot \lambda = w(\lambda + \rho) - \rho \tag{6.1.3}$$

of the usual Weyl group action, called the *dot action*. The dot action parametrizes several sets of modules in $\mathcal{O}_\lambda$.

Theorem 6.1 *There are bijections*

$$\{\textit{simple modules in } \mathcal{O}_\lambda\} \ni L(w\cdot\lambda)$$

$$w \in W \longleftrightarrow \{\textit{Verma modules in } \mathcal{O}_\lambda\} \ni M(w\cdot\lambda)$$

$$\{\textit{indecomposable projective objects in } \mathcal{O}_\lambda\} \ni P(w\cdot\lambda).$$

Here $P(w\cdot\lambda)$ denotes the projective cover of $L(w\cdot\lambda)$ in $\mathcal{O}_\lambda$.

We can say a little more about the structure of the indecomposable projective objects.

Proposition 6.2 *For $w \in W$ there is a sequence of submodules*

$$0 = P_0 < P_1 < \cdots < P_n = P(w \cdot \lambda)$$

such that $P_n/P_{n-1} \cong M(w \cdot \lambda)$, and for each $1 \leqslant i < n$, there is some $w_i \in W$ with $\ell(w_i) < \ell(w)$ such that $P_i/P_{i-1} \cong M(w_i \cdot \lambda)$.

In particular, from the case $w = 1$ we conclude that $P(\lambda) = M(\lambda)$.

Since $\mathcal{O}_\lambda$ is a finite abelian category, it is equivalent to the category of finite-dimensional right modules over some finite-dimensional algebra. In fact, it can be shown that this algebra is not dependent on λ!

Theorem 6.3 *There is a finite-dimensional algebra A such that for any $\lambda \in \Lambda^+$, $\mathcal{O}_\lambda \simeq \mathrm{mod}_{\mathrm{fd}} - \mathrm{A}$.*

It is evident that the algebra A is only well defined up to Morita equivalence. A natural problem is to find a concrete presentation of A. Since A is Morita equivalent to

$$\mathrm{End}_{\mathcal{O}_\lambda}\left(\bigoplus_{w \in W} P(w \cdot \lambda)\right),$$

this problem is equivalent (in some sense) to understanding projective objects and morphisms between them. Counter-intuitively, it is more effective to investigate *functors* acting on the category of projective objects and morphisms (i.e. natural transformations) between them.

Proposition 6.4 *For each $s \in S$ there is an exact self-adjoint functor $\theta_s : \mathcal{O}_\lambda \longrightarrow \mathcal{O}_\lambda$ with the following properties:*

1 *θ_s preserves projective objects;*
2 *if $w \in W$ with $\ell(ws) > \ell(w)$ there is an exact sequence*

$$0 \to M(w \cdot \lambda) \to \theta_s(M(w \cdot \lambda)) \cong \theta_s(M(ws \cdot \lambda)) \to M(ws \cdot \lambda) \to 0;$$

3 *if $st \cdots u$ is a reduced expression for some $w \in W$ in terms of simple reflections in S, then $P(w \cdot \lambda)$ is a direct summand of $\theta_s \theta_t \cdots \theta_u(M(\lambda))$.*

Since $M(\lambda)$ is itself projective, every natural transformation

$$\theta_s \theta_t \cdots \theta_u \longrightarrow \theta_{s'} \theta_{t'} \cdots \theta_{u'}$$

for reduced expressions $st \cdots u$ and $s't' \cdots u'$ induces a homomorphism

$$\theta_s \theta_t \cdots \theta_u(M(\lambda)) \longrightarrow \theta_{s'} \theta_{t'} \cdots \theta_{u'}(M(\lambda))$$

of projective objects. In fact, it can be shown that every homomorphism between such projective objects is induced in this way [11, Theorem 10.7]. So to find a concrete presentation of the algebra A, it is enough to describe

$$\mathrm{Hom}(\theta_s\theta_t\cdots\theta_u, \theta_{s'}, \theta_{t'}\cdots\theta_{u'}),$$

the space of all natural transformations between the functors $\theta_s\theta_t\cdots\theta_u$ and $\theta_{s'}\theta_{t'}\cdots\theta_{u'}$.

Theorem 6.5 ([15, 17]) *There are $\mathbb{C}[\mathfrak{h}]$-$\mathbb{C}[\mathfrak{h}]$ bimodules $\{B_s\}_{s\in S}$ such that for any reduced expressions $st\cdots u$ and $s't'\cdots u'$, there is an isomorphism*

$$\mathrm{Hom}(\theta_s\theta_t\cdots\theta_u, \theta_{s'}, \theta_{t'}\cdots\theta_{u'}) \cong \\ C\otimes\mathrm{Hom}(B_s\otimes B_t\otimes\cdots\otimes B_u, B_{s'}\otimes B_{t'}\otimes\cdots\otimes B_{u'})\otimes\mathbb{C},$$

where all tensor products are over $\mathbb{C}[\mathfrak{h}]$ and C denotes the coinvariant algebra $\mathbb{C}[\mathfrak{h}]/\mathbb{C}[\mathfrak{h}]\mathbb{C}[\mathfrak{h}]^W_+$, i.e. the quotient of $\mathbb{C}[\mathfrak{h}]$ by the ideal generated by positive degree W-invariants. (Here we are using the fact that the space of bimodule homomorphisms between two bimodules is itself a bimodule.)

The bimodules $\{B_s\}_{s\in S}$ (and more generally any direct summand of a tensor product of such bimodules) are today called *(classical) Soergel bimodules*, and can be used to give a presentation of A as follows. Fix a reduced expression for each $w\in W$. Then A is Morita equivalent to

$$\bigoplus_{\substack{w,w'\in W\\ w=st\cdots u\\ w'=s't'\cdots u'}} C\otimes\mathrm{Hom}(B_s\otimes B_t\otimes\cdots\otimes B_u, B_{s'}\otimes B_{t'}\otimes\cdots\otimes B_{u'})\otimes\mathbb{C},$$

where $st\cdots u$ and $s't'\cdots u'$ are the fixed reduced expressions for w and w'.

6.2 The Diagrammatic Category $\mathcal{D}$ of Soergel Bimodules

It is an amazing fact that Soergel bimodules make sense for *arbitrary* Coxeter groups, not just Weyl groups. This suggests that we should define "category $\mathcal{O}_\lambda$" for arbitrary Coxeter groups in terms of Soergel bimodules.

Theorem 6.6 ([13], [5, 6, 9]) *The monoidal category of Soergel bimodules has an explicit diagrammatic presentation.*

Equivalently, the finite-dimensional algebra A above has a presentation as a *diagram algebra*. In this context, a *diagrammatic presentation* means a presentation of a (strict) monoidal category using string diagrams. The essence of this approach is summarized in Table 6.1. In short, a morphism in a monoidal

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category corresponds to a diagram or a linear combination of diagrams. The sequence of colours of the edges which meet the bottom and top of the diagram give the domain and codomain of the corresponding morphism respectively. Vertical concatenation of diagrams corresponds to composition of morphisms, while horizontal concatenation corresponds to the tensor product of morphisms.

There are several advantages of the diagrammatic approach to Soergel bimodules over classical Soergel bimodules. In general, presenting a monoidal category diagrammatically makes bifunctoriality of the tensor product visually obvious through *rectilinear isotopy* of diagrams. Informally, we say that two diagrams are equivalent up to rectilinear isotopy if we can deform one diagram into the other by continuously moving vertices and stretching or shrinking edges, without moving edges or vertices past other edges and without introducing "caps" or "cups" in any edges. (See the left-hand sides of (6.2.1) for pictures of cap/cup diagrams. For a more formal description of rectilinear isotopy, see [10, (7.5)–(7.8)].) In the specific case of Soergel bimodules, there are several other "visually intuitive" relations which we will see later. More importantly, classical Soergel bimodules sometimes behave poorly over fields of positive characteristic, while diagrammatic Soergel bimodules remain well behaved. For applications to modular representation theory it is therefore best to work in the diagrammatic category.

From now on, we generalize from the setting of semisimple Lie algebras and assume that (W,S) is an arbitrary Coxeter system. In other words, W is a group with a presentation

$$W = \langle S \mid \forall s,t \in S,\ (st)^{m_{st}} = 1 \rangle$$

for certain positive integers m_{st}, with $m_{st} = m_{ts}$ and $m_{ss} = 1$ for all $s,t \in S$. The natural replacement for the Cartan subalgebra in this setting is called a *realization*.

Definition 6.3 Let $\Bbbk$ be an integral domain. A *realization* of (W,S) over $\Bbbk$ consists of a free, finite rank $\Bbbk$-module $\mathfrak{h}$ along with subsets $\{\alpha_s^\vee : s \in S\} \subset \mathfrak{h}$ and $\{\alpha_s : s \in S\} \subset \mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h},\Bbbk)$ such that

(i) $\langle \alpha_s^\vee, \alpha_s \rangle = 2$ for all $s \in S$;

(ii) the assignment

$$s(\lambda) = \lambda - \langle \alpha_s^\vee, \lambda \rangle \alpha_s$$

for all $s \in S$ and $\lambda \in \mathfrak{h}^*$ defines a representation of W on $\mathfrak{h}^*$;

(iii) the technical condition [9, (3.3)] is satisfied.

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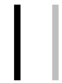
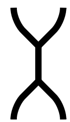
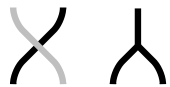
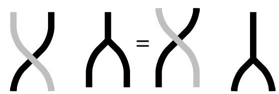
Table 6.1 *A comparison of some of the monoidal categories seen in §6.1. We note that in the diagrammatic approach, each diagram represents a morphism, so it is customary to use the diagram for the identity morphism on an object to represent an object diagrammatically.*

Category of functors on $\mathcal{O}_\lambda$	Category of (classical) Soergel bimodules	Diagrammatic category of Soergel bimodules	
functor θ_s	bimodule B_s	vertical edge	$[= \mathrm{id}_{B_s}]$
functor composition $\theta_s\theta_t$	tensor product $B_s \otimes B_t$	horizontal concatenation	 $[= \mathrm{id}_{B_s \otimes B_t}]$
natural transformation $\theta_s\theta_t \xrightarrow{\alpha} \theta_t\theta_s$	bimodule homomorphism $B_s \otimes B_t \xrightarrow{f} B_t \otimes B_s$	vertex joining edges	f 
(vertical) composition of natural transformations $\theta_s\theta_s \xrightarrow{\beta} \theta_s \xrightarrow{\gamma} \theta_s\theta_s$	composition of homomorphisms $B_s \otimes B_s \xrightarrow{g} B_s \xrightarrow{h} B_s \otimes B_s$	vertical concatenation	h g 
(horizontal) composition of natural transformations $\theta_s\theta_t\theta_s\theta_s \xrightarrow{\alpha*\beta} \theta_t\theta_s\theta_s$	tensor product of homomorphisms $B_s \otimes B_t \otimes B_s \otimes B_s \xrightarrow{f\otimes g} B_t \otimes B_s \otimes B_s$	horizontal concatenation	h g 
interchange law	bifunctoriality of $\otimes$	rectilinear isotopy	

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Example 6.4

1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{b}$ be a choice of Borel subalgebra. The Cartan subalgebra $\mathfrak{h}$ with the usual simple roots and coroots is a $\mathbb{C}$-realization of the Weyl group W.
2. Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$, and let G be a semisimple algebraic group over $\Bbbk$ with maximal torus T and cocharacter group $X(T) = \operatorname{Hom}(\mathbb{G}_m, T)$. The space $\mathfrak{h} = \Bbbk \otimes_{\mathbb{Z}} X(T)$, with the images of the usual roots and coroots, is a $\Bbbk$-realization of the Weyl group W.

We will use the data of a realization to construct the category $\widetilde{\mathcal{D}}_{\mathrm{BS}}$ below, the first step towards our goal of defining the diagrammatic category $\mathcal{D}$ of Soergel bimodules. As the construction of $\widetilde{\mathcal{D}}_{\mathrm{BS}}$ is entirely diagrammatic, it will be useful to identify the set S of simple generators with a set of colours for the purposes of drawing string diagrams. In the diagrams below, we will colour the generator s black and the generator t grey.

Definition 6.5 ($\widetilde{\mathcal{D}}_{\mathrm{BS}}$: generators) Let $\mathfrak{h}$ be a $\Bbbk$-realization of (W,S). Set $R = \operatorname{Sym}(\mathfrak{h}^*)$, the symmetric algebra of $\mathfrak{h}^*$, with $\deg \mathfrak{h}^* = 2$. The category $\widetilde{\mathcal{D}}_{\mathrm{BS}}$ is the $\Bbbk$-linear graded strict monoidal category defined as follows.

- The objects of $\widetilde{\mathcal{D}}_{\mathrm{BS}}$ are the formal (tensor) products of form $B_s \otimes B_t \otimes \cdots \otimes B_u$ for $s,t,\ldots,u \in S$.
- The morphisms in $\widetilde{\mathcal{D}}_{\mathrm{BS}}$ are generated (under $\Bbbk$-linear combinations, compositions and tensor products) by the following elementary morphisms.
  - For each homogeneous $f \in R$, there is a morphism

$$f : \mathbf{1} \longrightarrow \mathbf{1}$$

$$f$$

    of degree $\deg(f)$.
  - For each $s \in S$ there are morphisms

$$\mathrm{dot}_s : B_s \longrightarrow \mathbf{1}, \qquad \overline{\mathrm{dot}_s} : \mathbf{1} \longrightarrow B_s$$


    of degree 1 and

$$\mathrm{fork}_s : B_s \otimes B_s \longrightarrow B_s, \qquad \overline{\mathrm{fork}_s} : B_s \longrightarrow B_s \otimes B_s$$


    of degree -1.

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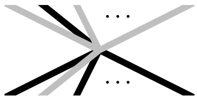
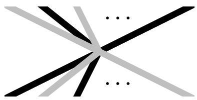
– For each pair $(s,t) \in S \times S$ with $s \neq t$ and $m_{st} < \infty$, there is a morphism

$$\mathrm{braid}_{st} : \underbrace{B_s \otimes B_t \otimes B_s \otimes \cdots \otimes B_s}_{m_{st}} \longrightarrow \underbrace{B_t \otimes B_s \otimes B_t \otimes \cdots \otimes B_t}_{m_{st}}$$



when m_{st} is odd, or

$$\mathrm{braid}_{st} : \underbrace{B_s \otimes B_t \otimes B_s \otimes \cdots \otimes B_t}_{m_{st}} \longrightarrow \underbrace{B_t \otimes B_s \otimes B_t \otimes \cdots \otimes B_s}_{m_{st}}$$

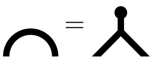


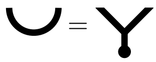
when m_{st} is even, of degree 0.

These morphisms are subject to a number of relations, which can be found in [1, §2.2], or (in a slightly different form) [9, (5.1)–(5.12)].

For convenience we will also use the following shorthand

$$\mathrm{cap}_s = \mathrm{dot}_s \circ \mathrm{fork}_s : B_s \otimes B_s \to \mathbf{1}, \quad \mathrm{cup}_s = \overline{\mathrm{fork}_s} \circ \overline{\mathrm{dot}_s} : \mathbf{1} \to B_s \otimes B_s. \tag{6.2.1}$$





In an entirely standard way, we change our point of view slightly so that we allow grade shifts of objects in $\widetilde{\mathcal{D}}_{\mathrm{BS}}$ but only consider homogeneous (i.e. degree 0) morphisms.

Definition 6.6 The *diagrammatic category of Bott–Samelson bimodules* is the $\Bbbk$-linear monoidal category $\mathcal{D}_{\mathrm{BS}}$ defined as follows.

- The objects of $\mathcal{D}_{\mathrm{BS}}$ are the formal symbols $B(m)$, for $B \in \mathrm{Obj}\,\widetilde{\mathcal{D}}_{\mathrm{BS}}$ and $m \in \mathbb{Z}$, with tensor product $B(m) \otimes B'(n) = (B \otimes B')(m+n)$.
- The morphisms in $\mathcal{D}_{\mathrm{BS}}$ are given by

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{BS}}}(B(m), B'(n)) = \mathrm{Hom}^{n-m}_{\widetilde{\mathcal{D}}_{\mathrm{BS}}}(B, B'),$$

with composition and tensor product defined via $\widetilde{\mathcal{D}}_{\mathrm{BS}}$.

Objects in $\mathcal{D}_{\mathrm{BS}}$ are called *(diagrammatic) Bott–Samelson bimodules*. As we will see below, Bott–Samelson bimodules are the prototypical Soergel bimodules, from which all others are constructed.

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Definition 6.7 The *diagrammatic category* $\mathcal{D}$ *of Soergel bimodules* is the Karoubi envelope of $\mathcal{D}_{\mathrm{BS}}$. In other words $\mathcal{D}$ is the closure of $\mathcal{D}_{\mathrm{BS}}$ with respect to all finite direct sums and all direct summands of objects and morphisms in $\mathcal{D}_{\mathrm{BS}}$.

Objects in $\mathcal{D}$ are called (diagrammatic) Soergel bimodules. It can be shown that under some mild conditions on the realization $\mathfrak{h}$, $\mathcal{D}$ is a Krull–Schmidt category, i.e. every Soergel bimodule decomposes uniquely into a direct sum of indecomposable Soergel bimodules [9, Lemma 6.25]. The indecomposable Soergel bimodules then play the same role in $\mathcal{D}$ as the indecomposable projective objects in $\mathcal{O}_\lambda$. As one might expect these objects are highly dependent on characteristic, since idempotent decompositions of the identity in the endomorphism algebra of a Bott–Samelson bimodule are usually characteristic-dependent.

6.3 Some Diagrammatic Relations

In this section we will investigate a subset of the relations which define $\widetilde{\mathcal{D}}_{\mathrm{BS}}$.

Polynomial Relations

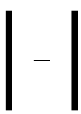
Regions labelled by polynomials add and multiply in the usual way, i.e. for any $f, g \in R$ we have

$$\left(f\right) + \left(g\right) = \left(f+g\right), \qquad \left(f\right) \otimes \left(g\right) = \left(fg\right), \qquad \left(f\right) \circ \left(g\right) = \left(fg\right). \tag{6.3.1}$$

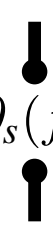
(Here we use dashed circles for emphasis around a single diagram without strings, e.g. the left-hand side of the first equation consists of a sum of two diagrams, while the right-hand side is a single diagram.)

For each $s \in S$ we also have

$$= \alpha_s, \tag{6.3.2}$$



$$f \;|\; - \;|\; s(f) = \partial_s(f), \tag{6.3.3}$$



where $\partial_s(f) = \alpha_s^{-1}(f - s(f))$.

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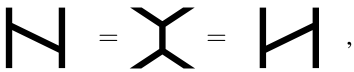
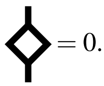
One-colour Relations

For each $s \in S$ we have

(6.3.4)



(6.3.5)



(6.3.6)

These relations give *all* the relations defining $\widetilde{\mathcal{D}}_{\mathrm{BS}}$ in a few special cases.

Definition 6.8 ($\widetilde{\mathcal{D}}_{\mathrm{BS}}$: relations (no finite dihedral parabolics)) Suppose (W,S) is a Coxeter system with no finite dihedral parabolic subgroups (i.e. $m_{st} = \infty$ whenever $s \neq t$). Then (6.3.1)–(6.3.6) is a full list of relations defining $\widetilde{\mathcal{D}}_{\mathrm{BS}}$.

Thus we have defined enough relations to understand Soergel bimodules for the smallest Lie algebra $\mathfrak{sl}_2$ $\big(W = \{1, s\}\big)$.

Other Diagrammatic Relations

In general, the definition of $\widetilde{\mathcal{D}}_{\mathrm{BS}}$ requires more diagrammatic relations than (6.3.1)–(6.3.6). Perhaps unsurprisingly, the remaining relations all involve the morphism braid_{st}, which only exists when $m_{st} < \infty$. They come in two flavours, depending on how many colours of strings appear in the diagrams.

The *two-colour relations* are defined for all distinct $s,t \in S$ such that $m_{st} < \infty$, i.e. whenever braid_{st} exists. The most important of these, the *Jones–Wenzl relation*, is closely related to the Temperley–Lieb algebra.

The *three-colour relations* are defined for all distinct $s,t,u \in S$ which generate a finite parabolic subgroup. These relations involve three different kinds of braids, but no other generating morphisms. The form of the relation also only depends on the Coxeter type of the resulting parabolic subgroup. The most complicated forms (in types A_3, B_3 and H_3) are sometimes called the *Zamolodchikov relations*.

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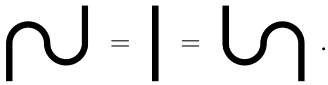
6.4 Some Consequences and Applications

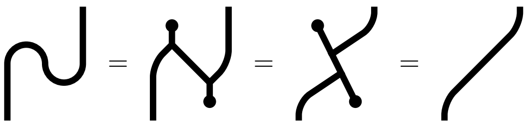
Proposition 6.7 *Any two diagrams which are isotopic correspond to equal morphisms in $\widetilde{\mathcal{D}}_{BS}$. In other words, we may freely deform the edges of any diagram without changing the morphism in $\widetilde{\mathcal{D}}_{BS}$.*

Proof (Sketch) We first must show that the zig-zag relations hold, i.e.



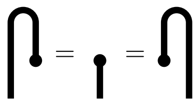
(6.4.1)

This ultimately follows by first applying (6.3.5) and then applying (6.3.4) twice:



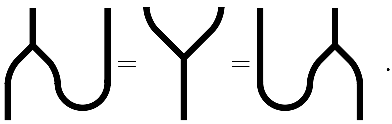
and similarly for the second equality.

Next we must show that dot_s and fork_s twist to their barred counterparts, i.e.



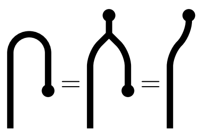
(6.4.2)

and

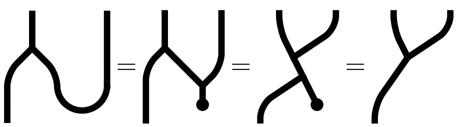


(6.4.3)

Proving (6.4.2) involves only one application of (6.3.4):



and similarly for the second equality. The proof of (6.4.3) is almost identical to that of (6.4.1):



and similarly for the second equality.

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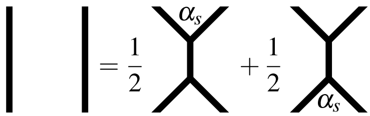
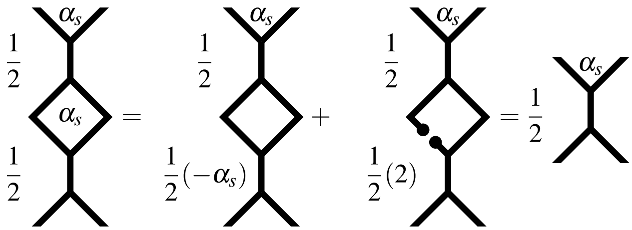
Since we already have the zig-zag relations, this also means that the barred counterparts of these two relations also hold, i.e. the vertical flips of equations (6.4.2) and (6.4.3) also hold. For $s,t \in S$ with $m_{st} < \infty$, the corresponding equation for the braid morphism braid_{st} is already a relation in $\widetilde{\mathcal{D}}_{\mathrm{BS}}$ [1, §2.2(7)]. These twisting relations are enough to ensure that *any* isotopy of string diagrams is a relation in $\widetilde{\mathcal{D}}_{\mathrm{BS}}$. For more discussion on this, see [6, Proposition 3.2]. □

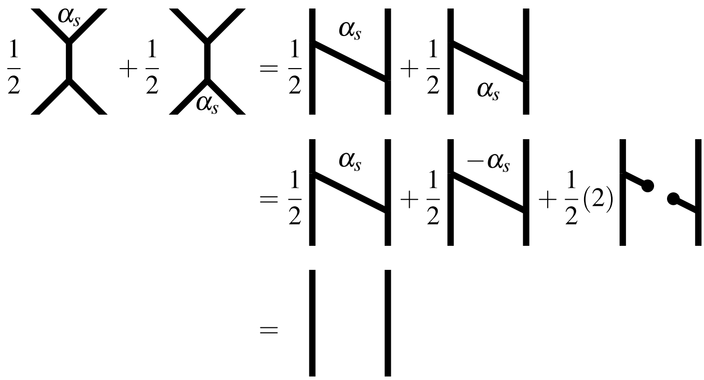
Lemma 6.8 *For $s \in S$ we have an idempotent decomposition*



Proof First, we show that each of the terms on the right-hand side are idempotents:



Next, we verify the decomposition by applying relations (6.3.3)–(6.3.5):



□

From this lemma we immediately obtain the following (cf. the natural isomorphism $\theta_s\theta_s \cong \theta_s \oplus \theta_s$).

Theorem 6.9 *Suppose $W = \{1,s\}$ and $\mathfrak{h}$ is a 1-dimensional realization of W over a field $\Bbbk$ with* $\mathrm{char}\,\Bbbk \neq 2$. *Then the split Grothendieck ring $[\mathcal{D}]$ of $\mathcal{D}$ (i.e. the*

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ring of isomorphism classes of objects of $\mathcal{D}$) is isomorphic to the following:

$$\begin{aligned} [\mathcal{D}] &\longrightarrow \mathcal{H}(S_2) = \mathbb{Z}[v^{\pm 1}][b_s]/(b_s^2 - (v+v^{-1})b_s) \\ [\mathbf{1}(1)] &\longmapsto v \\ [B_s] &\longmapsto b_s. \end{aligned}$$

Remark There is a generalization of Theorem 6.9 to all Coxeter systems known as *Soergel's categorification theorem*. It states that (under mild assumptions on the realization $\mathfrak{h}$) the split Grothendieck ring $[\mathcal{D}]$ is isomorphic to the Iwahori–Hecke algebra $\mathcal{H}(W)$. In the setting of classical Soergel bimodules, this result was proven by Soergel in [17, Satz 1.10] for suitably 'nice' realizations, and in the diagrammatic setting it was proven more generally by Elias–Williamson [9, Corollary 6.27].

We conclude with some applications and references.

1 The original motivating application for Soergel was the Kazhdan–Lusztig conjectures, which describe the characters of the simple modules of $\mathcal{O}_\lambda$ in terms of Kazhdan–Lusztig polynomials. This was originally proven in the 1980s by Beilinson–Bernstein [2] (and independently by Brylinski–Kashiwara [4]) using highly geometric techniques. In the 1990s Soergel suggested an alternative proof based on decomposing $B_s \otimes B_t \otimes \cdots \otimes B_u$ into a direct sum of indecomposable Soergel bimodules [15]. Soergel's proof was substantially more algebraic, but relied crucially on an important geometric result called the Decomposition Theorem. In [8] Elias–Williamson removed this dependence to produce an entirely algebraic proof (for a more readable introduction, see also [7, 18]).

2 A similar character-theoretic conjecture in modular representation theory is Lusztig's conjecture, which describes the characters of simple modules for a semisimple algebraic group G over a field of characteristic $p > 0$. Soergel showed that Soergel bimodules for the Weyl group in characteristic p give an analogous description of "modular category $\mathcal{O}$" [16], a subquotient of the category of rational G-modules. In the celebrated paper [19] Williamson used this framework to show that Lusztig's conjecture is in fact false, except when p is extremely large!

3 Soergel's categorification theorem provides another way to think about the above results wholly within the context of Soergel bimodules. To be more precise, Soergel showed in [15] that the Kazhdan–Lusztig conjectures hold if and only if a statement known as Soergel's conjecture holds. Soergel's conjecture states that the indecomposable Soergel bimodules correspond to the Kazhdan–Lusztig basis of the corresponding Hecke algebra. This is

difficult to prove because the Kazhdan–Lusztig basis is defined 'combinatorially' with no reference to the morphisms in $\mathcal{D}$. Elias–Williamson [8] proved Soergel's conjecture algebraically in characteristic 0, while Williamson [19] found counterexamples to Soergel's conjecture in positive characteristic. These counterexamples suggest defining the *p-canonical basis* or *p-Kazhdan–Lusztig basis* to be the basis of the Hecke algebra corresponding to the indecomposable Soergel bimodules in characteristic p [12]. Unlike the ordinary Kazhdan–Lusztig basis, the p-Kazhdan–Lusztig basis is *not* combinatorial and requires understanding of the morphisms in $\mathcal{D}$ in general.

4 Achar *et al*. have shown that the p-Kazhdan–Lusztig basis for the corresponding affine Weyl group in characteristic p gives the characters of tilting modules (another class of G-modules parametrized by highest weight) [1]. This fits in with a conjectured categorical equivalence involving the functors $\{\theta_s\}$ in characteristic p [14], similar to Theorem 6.5. In type A these decompositions also give the simple characters of the symmetric group. More recently the author (together with Chris Bowman and Anton Cox) has given an alternative, more direct proof of the symmetric group result [3].

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