Bull. Aust. Math. Soc. **87** (2013), 503–513
doi:10.1017/S0004972712000573

THE PROBABILITY THAT x^m AND y^n COMMUTE IN A COMPACT GROUP

KARL H. HOFMANN✉ and FRANCESCO G. RUSSO

(Received 16 May 2012; accepted 7 June 2012; first published online 2 August 2012)

Abstract

In a recent article [K. H. Hofmann and F. G. Russo, 'The probability that x and y commute in a compact group', *Math. Proc. Cambridge Phil Soc.*, to appear] we calculated for a compact group G the probability $d(G)$ that two randomly selected elements $x, y \in G$ satisfy $xy = yx$, and we discussed the remarkable consequences on the structure of G which follow from the assumption that $d(G)$ is positive. In this note we consider two natural numbers m and n and the probability $d_{m,n}(G)$ that for two randomly selected elements $x, y \in G$ the relation $x^m y^n = y^n x^m$ holds. The situation is more complicated whenever $n, m > 1$. If G is a compact Lie group and if its identity component G_0 is abelian, then it follows readily that $d_{m,n}(G)$ is positive. We show here that the following condition suffices for the converse to hold in an arbitrary compact group G: for any nonopen closed subgroup H of G, the sets $\{g \in G : g^k \in H\}$ for both $k = m$ and $k = n$ have Haar measure 0. Indeed, we show that if a compact group G satisfies this condition and if $d_{m,n}(G) > 0$, then the identity component of G is abelian.

2010 *Mathematics subject classification*: primary 20C05, 20P05; secondary 43A05.
Keywords and phrases: probability of commuting pairs, commutativity degree, *FC*-groups, compact groups, Haar measure.

1. Introducing the problem

We fix a compact group G and numbers $m, n \in \{1, 2, 3, \ldots\}$, and we let ν denote the Haar measure of G. Define

$$\mathcal{D}_{m,n}(G) = \{(x, y) \in G \times G : [x^m, y^n] = 1\} \quad \text{and} \quad d_{m,n}(G) = (\nu \times \nu)(D_{m,n}(G)).$$

So let us say that G is *k-straight* for a natural number k whenever for a closed nowhere dense subgroup H of G the set $\{g \in G : g^k \in H\}$ has Haar measure 0. Every compact group is 1-straight, and every finite group is n-straight for every n, but the profinite group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is not 2-straight since the squaring map is constant, nor is the Lie group $\mathbb{R}/\mathbb{Z} \rtimes \{1, -1\}$ (the 'continuous dihedral group') 2-straight, since all elements of $\mathbb{R}/\mathbb{Z} \times \{-1\}$ have order two.

We shall prove the following theorem.

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https://doi.org/10.1017/S0004972712000573 Published online by Cambridge University Press

THEOREM 1.1 (The structure of compact groups in which x^m and y^n commute often). *Let G denote a compact group and assume that G is m- and n-straight. If $d_{m,n}(G) > 0$, then the identity component G_0 of G is abelian. Moreover, $d_{m,n}(G) > 0$ and G_0 abelian are equivalent conditions when G is a Lie group.*

Our strategy to prove this theorem will be as follows. We define $p_n \colon G \to G$ by $p_n(x) = x^n$. The measures $\mu_1 = p_m(\nu)$ and $\mu_2 = p_n(\nu)$ are defined by $\mu_1(B) = p_m(\nu)(B) = \nu(p_m^{-1}(B))$ for each Borel subset B of G, and analogously for μ_2. Now write $D(G) = D_{1,1}(G)$ and note that

$$\begin{aligned} D_{m,n}(G) &= \{(x, y) : [p_m(x), p_n(y)] = 1\} \\ &= (p_m \times p_n)^{-1}(\{(u, v) : [u, v] = 1\}) = (p_m \times p_n)^{-1}(D(G)), \end{aligned}$$

and thus

$$\begin{aligned} 0 < (\nu \times \nu)(D_{m,n}(G)) &= (p_m \times p_n)(\nu \times \nu)(D(G)) \\ &= (p_m(\nu) \times p_n(\nu))(D(G)) = (\mu_1 \times \mu_2)(D(G)). \end{aligned}$$

We propose a proof based, in the end, on this formula, but we shall have to draw from Lie group theory and real analytic function theory among other things.

In the historical remarks of [8] we remind the reader that for a compact connected nonabelian Lie group G the set Σ of pairs $(x, y) \in G \times G$ for which $\langle x, y\rangle$ is free and dense in G has Haar measure 1 in $G \times G$ and is dense (see, for example, [7, pp. 282–283]). The set Σ, on the other hand, does not meet any of the sets $D_{m,n}(G)$ above, which therefore cannot have positive measure. That is, $d_{m,n}(G) = 0$ for all positive natural numbers m and n in this case. It is at this point that one might find a point of contact of the topic of this paper with the topic of the 'ubiquity' of free subgroups in compact groups, which has attracted considerable attention in the past (see, for example, [1, 2, 5, 9]).

2. Actions and product measures

We recall some background material from [8]. Let $(g, x) \mapsto g \cdot x : G \times X \to X$ be a continuous action α of a compact group G on a compact space X. All spaces in sight are assumed to be Hausdorff. We specify a Borel probability measure P on $G \times X$ and discuss the probability that a group element $g \in G$ fixes a phase space element $x \in X$ for a pair (g, x), randomly selected from $G \times X$, that is, that $g \cdot x = x$. We define

$$E = \{(g, x) \in G \times X : g \cdot x = x\},$$

that is, E is the equaliser of the two functions $\alpha, \mathrm{pr}_X : G \times X \to X$ and is therefore a closed subset of $G \times X$.

Let $G_x = \{g \in G : g \cdot x = x\}$ be the isotropy (or stability) group at x and let $X_g = \{x \in X : g \cdot x = x\}$ be the set of points fixed under the action of g. We note that $G_{g\cdot x} = gG_xg^{-1}$. The function $g \mapsto g \cdot x : G \to G \cdot x$ induces a continuous equivariant bijection $G/G_x \to G \cdot x$ which, due to the compactness of G, is a homeomorphism.

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Definition 2.1. *We shall say that a Borel probability measure σ on G respects closed subgroups if every closed subgroup H with $\sigma(H) > 0$ is open.*

Recall that an open subgroup H of a topological group G, being the complement of all the cosets gH for $g \notin H$, is closed and that it contains the *identity component* G_0 of G. If G is compact, then H has finite index in G. Haar measure μ on a compact group G respects closed subgroups.

We shall say that the group G *acts automorphically* on X if X is a compact group and $x \mapsto g \cdot x : X \to X$ is an automorphism for all $g \in G$.

In [8] the following result was established.

Proposition 2.2 (See [8, Proposition 2.6]). *Let G and X be compact groups and assume that G acts automorphically on X. Let μ and ν be normalised positive Borel measures on G and X, respectively. Define*

$$E = \{(g, x) \in G \times X : g \cdot x = x\} \subseteq G \times X.$$

Assume that μ and ν respect closed subgroups and that X is a Lie group. Then the following statements are equivalent.

(1) $(\mu \times \nu)(E) > 0$.
(2) *The subgroup $F \leq X$ of all elements with finite G-orbits is open and thus has finite index in X.*

The main application of this general situation will be the case of a compact group G and the automorphic action of G on $X = G$ via inner automorphisms:

$$(g, x) \mapsto g \cdot x = gxg^{-1} : G \times X \to X.$$

The orbit $G \cdot x$ of x is the conjugacy class $C(x)$ of x, and the isotropy group G_x of the action at x is the centraliser $Z(x, G) = \{g \in G : gx = xg\}$ of x in G. The set E is the set $D(G) = \{(x, y) \in G \times G : [x, y] = 1\}$, and F is the union of all finite conjugacy classes, the so-called *FC-centre*. In particular, F is a characteristic subgroup of G whose elements have finite conjugacy classes. If G is a Lie group, then F is closed in G. In passing we recall that a group agreeing with its FC-centre is called an FC-group.

In this setting, Proposition 2.2 has the following consequence.

Corollary 2.3 (See [8, Theorem 3.10]). *Let G be a compact Lie group and F its FC-centre. Further, let μ_1 and μ_2 be two Borel probability measures on G and set $P = \mu_1 \times \mu_2$ and $D = \{(g, h) \in G \times G : [g, h] = 1\}$. Assume that μ_1 and μ_2 respect closed subgroups. Then the following conditions are equivalent.*

(1) $P(D(G)) > 0$.
(2) *F is open in G.*
(3) *The characteristic abelian subgroup $Z(F)$ is open in G.*

Under these conditions, the centraliser $Z(F, G)$ of F in G is open, and the finite group $\Gamma = G/Z(F, G)$ is finite and acts effectively on F with the same orbits as G under the

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well-defined action $\gamma \cdot x = gxg^{-1}$ for $(\gamma, x) \in \Gamma \times F$, $g \in \gamma$. The isotropy group Γ_x at $x \in F$ is $Z(x, G)/Z(F, G)$, and the set F_γ of fixed points under the action of γ is $Z(g, F)$ for any $g \in \gamma$.

3. Compact Lie groups and measures respecting closed subgroups

Recall that for a continuous function $f : Y \to Z$ between compact spaces and a Borel measure λ on Y, the image measure $f(\lambda)$ on Z is given by $f(\lambda)(B) = \lambda(f^{-1}(B))$ for any Borel subset $B \subseteq Z$.

On a smooth manifold Y we denote by $T_p(Y)$ the tangent space at the point $p \in Y$. We say that a smooth function $f : Y \to Z$ between two real smooth manifolds Y and Z of the same dimension is *regular* at a point p if its derivative $D_p f : T_p(Y) \to T_{f(p)}(Z)$ is invertible, that is, if $\det(D_p f) \neq 0$. The complement of the set of points at which f is regular is called the *singular set* of f.

The singular set is the zero set of the smooth function $\Delta : Y \to \mathbb{R}$, $\Delta(p) = \det(D_p f)$ and thus is closed. If f is a real analytic function, then Δ is a real-valued real analytic function. Accordingly, it vanishes on a neighbourhood of a point p if and only if it vanishes on the entire connected component of p in Y. Thus the singular set of a real analytic function f is closed and nowhere dense except for possibly containing entire components of Y.

Lemma 3.1. *Let H be a closed subgroup of a compact infinite Lie group G and let $f : G \to G$ be a real analytic function whose singular set does not contain a connected component of G. Assume that $f(\nu)(H) > 0$ for the normalised Haar measure ν of G. Then H is open. In particular, the measure $\mu = f(\nu)$ respects closed subgroups.*

Proof. We know that $0 < \mu(H) = \nu(f^{-1}(H)) = \nu(\{g \in G : f(g) \in H\})$. We claim that this implies that $\nu(H) > 0$ which in turn will imply that H is open, as asserted.

Suppose that H fails to be open. Then H is a closed proper real analytic submanifold of the real analytic manifold G. Since the singular set of f is nowhere dense, it follows that $f^{-1}(H)$ is a closed nowhere dense real analytic subset of G. Haar measure on G is a real analytic $\dim(G)$-form on G. Therefore $f(\nu)(H) = \nu(f^{-1}(H)) = 0$. This contradiction proves the claim. □

As a next step we specialise f_1 and f_2 to power functions. For a number $n \in \{1, 2, 3, \ldots\}$ we define the power function $p_n : G \to G$ by $p_n(g) = g^n$ and f_j to be p_{n_j}. Therefore we need a precise understanding of the singularities of p_n on a Lie group. Clearly p_n is a real analytic self-map of the not necessarily connected compact real manifold G.

Firstly, let us say that a real analytic function $f : G \to G$ is *totally singular* at $g \in G$ if f takes a constant value on a neighbourhood of g and therefore on the entire connected component gG_0 of g in G. This happens for $f = p_2$ on $G = \mathbb{T} \rtimes \{1, -1\}$, where $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$ is the torus group (see Example 5.2 below). Here p_2 is totally singular at each point of $\mathbb{T} \times \{-1\}$.

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Let $\exp : \mathfrak{g} \to G$ be the exponential function of G and U an open neighbourhood of 0 in $\mathfrak{g}$ so small that $\exp | U : U \to \exp U$ is a diffeomorphism onto an open neighbourhood of 1 in G. The function p_n is singular (respectively, totally singular) at g if and only if the function $x \mapsto ((\exp x)g)^n : U \to G$ is singular (respectively, totally singular) at 0. For the following we recall that each element $g \in G$ yields a canonical Lie algebra automorphism $\mathrm{Ad}\, g : \mathfrak{g} \to \mathfrak{g}$ such that $g(\exp x)g^{-1} = \exp(\mathrm{Ad}\, g)x$.

Proposition 3.2 (The singularities of the power function). *Let G be a compact Lie group and n a natural number. Then the following statements are equivalent for an element $g \in G$.*

(i) *p_n is singular at $g \in G$.*

(ii) *At least one eigenvalue of* Ad g *is an nth root of unity different from* 1.

Moreover, the following conditions are also equivalent.

(a) *p_n is totally singular in $g \in G$.*

(b) *$g^n = 1$ and every eigenvalue of* Ad g *is an nth root of unity different from* 1.

Proof. From $g(\exp y)g^{-1} = \exp(\mathrm{Ad}\, g)y$ for all $y \in \mathfrak{g}$ it follows by induction that

$$((\exp x)g)^n = (\exp x)(\exp(\mathrm{Ad}\, g)x)(\exp(\mathrm{Ad}\, g)^2 x) \cdots (\exp(\mathrm{Ad}\, g)^{n-1} x)g^n. \qquad (\dagger)$$

For fixed g and n we may assume that U is so small that the Campbell–Hausdorff multiplication $*$ is defined where needed for $x \in U$ such that

$$(\exp x)(\exp(\mathrm{Ad}\, g)x)(\exp(\mathrm{Ad}\, g)^2 x) \cdots (\exp(\mathrm{Ad}\, g)^{n-1} x)$$
$$= \exp(x * (\mathrm{Ad}\, g)x * (\mathrm{Ad}\, g)^2 x * \cdots * (\mathrm{Ad}\, g)^{n-1} x).$$

We note that

$$x * (\mathrm{Ad}\, g)x * (\mathrm{Ad}\, g)^2 x * \cdots * (\mathrm{Ad}\, g)^{n-1} x$$
$$= (\mathbf{1} + \mathrm{Ad}\, g + (\mathrm{Ad}\, g)^2 + \cdots + (\mathrm{Ad}\, g)^{n-1})x + r(x)$$

where $r(x) \in \mathfrak{g}$ is a function $r : U \to \mathfrak{g}$ satisfying $\lim_{x \to 0} \|x\|^{-1} \cdot r(x) = 0$ for one norm, hence all norms, on $\mathfrak{g}$. Such a function we shall call a *remainder function*. Thus for $x \in U$,

$$p_n((\exp x)g)) = (\exp(\mathbf{1} + \mathrm{Ad}\, g + (\mathrm{Ad}\, g)^2 + \cdots + (\mathrm{Ad}\, g)^{n-1})x + r(x))g^n.$$

Since exp is regular on U, and since right translation by g^n is a diffeomorphism on G, the function $x \mapsto ((\exp x)g)^n : U \to G$ is regular at 0 if and only if

$$\alpha_g = \sum_{m=0}^{n-1} (\mathrm{Ad}\, g)^m : \mathfrak{g} \to \mathfrak{g}$$

is an isomorphism. Let λ be an eigenvalue of $\mathrm{Ad}\, g$. Then $\lambda \neq 0$ since $\mathrm{Ad}\, g$ is an automorphism of $\mathfrak{g}$. Using the semisimplicity of $\mathrm{Ad}\, g$, which, due to the fact that G is

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compact,

$$\rho = \sum_{m=0}^{n-1} \lambda^m = \begin{cases} n & \text{if } \lambda = 1 \\ (\lambda^n - 1)/(\lambda - 1) & \text{if } \lambda \neq 1, \end{cases}$$

is an eigenvalue of α_g, and all eigenvalues ρ of α_g are so obtained. Thus p_n is singular in g if and only if one of the eigenvalues ρ vanishes, and that is the case if and only if $\lambda^n = 1$, $\lambda \neq 1$. This completes the proof of the equivalence of (i) and (ii).

Next we observe that (a) happens if and only if $((\exp x)g)^n = 1$ for all $x \in \mathfrak{g}$, including, of course, $x = 0$. By (†) this is equivalent to $g^n = 1$ and

$$(\exp x) \exp((\mathrm{Ad}\, g)x) \exp((\mathrm{Ad}\, g)^2 x) \cdots \exp((\mathrm{Ad}\, g)^{n-1} x) = 1, \quad \forall x \in \mathfrak{g}.$$

In particular, due to analyticity, this is equivalent to the fact that for all sufficiently small x for which the required Campbell–Hausdorff products exist,

$$x * (\mathrm{Ad}\, g)x * (\mathrm{Ad}\, g)^2 x * \cdots * (\mathrm{Ad}\, g)^{n-1} x = 0. \tag{u}$$

By the Campbell–Hausdorff formalism, there is a zero-neighbourhood of $\mathfrak{g}$ and a remainder function r such that, for $\alpha_g = \mathbf{1} + \mathrm{Ad}\, g + (\mathrm{Ad}\, g)^2 + \cdots + (\mathrm{Ad}\, g)^{n-1}$,

$$x * (\mathrm{Ad}\, g)x * (\mathrm{Ad}\, g)^2 x * \cdots * (\mathrm{Ad}\, g)^{n-1} x = \alpha_g(x) + r(x), \quad \forall x \in U.$$

Thus by (u) above, condition (a) is equivalent to

$$\begin{aligned} &g^n = 1 \text{ and there is a sufficiently small neighbourhood of } 0 \text{ in } \mathfrak{g} \\ &\qquad \text{and a remainder function } r \text{ such that } \alpha_g(x) + r(x) = 0, \quad \forall x \in U. \end{aligned} \tag{a$'$}$$

Let $0 \neq y \in \mathfrak{g}$. Setting $x = t \cdot y$ with $t > 0$, we have $\|x\| = t \cdot \|y\|$ and $x \in U$ if t is sufficiently small. Then (a′) implies $0 = \alpha(t \cdot x) + r(t \cdot y)$ and thus $0 = \alpha_g(y) + (1/t) \cdot r(t \cdot y) \to \alpha_g(y)$ for $t \to 0$ by the definition of a remainder function. Hence $\alpha_g = 0$. In view of the semisimplicity of $\mathrm{Ad}\, g$, no eigenvalue of $\mathrm{Ad}\, g$ can then be 1, while $g^n - 1$ implies that $(\mathrm{Ad}\, g)^n = \mathbf{1}$. Thus all eigenvalues of $\mathrm{Ad}\, g$ are nth roots of unity different from 1. This establishes that (a) implies (b).

Conversely, assume (b) and let λ be an eigenvalue of $\mathrm{Ad}\, g$. Then λ is an nth root of unity and $\lambda \neq 1$. Then

$$1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1} = \frac{\lambda^n - 1}{\lambda - 1} = 0.$$

Since $\mathrm{Ad}\, g$ is semisimple (and thus diagonalisable over $\mathbb{C}$) we conclude that $\alpha_g = 0$. Then (a′) holds with $r \equiv 0$. Hence (a) and (b) are equivalent. □

Assume that the equivalent conditions (a) and (b) of Proposition 3.2 are satisfied and assume, without essential loss of generality, that n is the order of the element g; then $\langle G \cup \{g\} \rangle = G_0 \langle g \rangle$ is an open subgroup of G which is isomorphic to the semi-

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direct product $G_g = G_0 \rtimes_\iota \mathbb{Z}/n\mathbb{Z}$ where the morphism $\iota : \mathbb{Z}/n\mathbb{Z} \to \operatorname{Aut} G_0$ is defined by $\iota(m + n\mathbb{Z})(h) = g^m h g^{-m}$. Recall that the multiplication on G_g is given by

$$(h, m + n\mathbb{Z})(h', m' + n\mathbb{Z}) = (hg^m h' g^{-m}, m + m' + n\mathbb{Z}).$$

The power function p_n is strongly singular in G_g in each point of $G_0 \times \{1 + n\mathbb{Z}\}$. In this sense these semidirect products (such as the continuous dihedral group $\mathbb{T} \rtimes \{1, -1\}$) are typical of the presence of totally singular points.

Lemma 3.3. *Let G be an infinite compact Lie group with normalised Haar measure ν, and let $n \in \{2, 3, \dots\}$. Then the following conditions are equivalent.*

(i) *$p_n(\nu)$ respects closed subgroups of G.*

(ii) *p_n is nowhere totally singular.*

Proof. Proposition 3.2 shows that p_n is a real analytic self-map of G whose set of singular points is contained in the union of the sets

$$S_m = \{g \in G : \det(\operatorname{Ad} g - e^{2\pi i m/n} \cdot \mathrm{id}) = 0\}, \quad m = 1, 2, \dots, n-1.$$

On each of the finitely many connected components $G_0 c$, $c \in G$, the set $S_m \cap G_0 c$ is a real algebraic variety and thus is a closed nowhere dense analytic subset or else contains all of $G_0 c$. Hence (ii) is equivalent to

$$\begin{aligned} &p_n \text{ is a real analytic function whose singular set does not} \\ &\quad \text{contain a connected component of } G. \end{aligned} \tag{ii$'$}$$

Then by Lemma 3.1, (ii′) implies (i).

Conversely, assume (i) and suppose that (ii) is false. Then we have a component $G_0 c$ such that $p_n(G_0 c) = \{1\}$ and so $p_n(\nu)(\{1\}) = \nu(p_n^{-1}\{1\}) \geq \nu(G_0 c) = \nu(G_0) = |G/G_0| > 0$. Since $p_n(\nu)$ respects closed subgroups by (ii), we conclude that $\{1\}$ is open. Hence the compact group G is discrete and thus finite. This is in contradiction to the hypothesis that G is infinite. Hence (i) implies (ii) and the proof is complete. □

In the Introduction, we called a compact group k-straight for a natural number k if the measure $p_k(\nu)$ respects closed subgroups. For a compact Lie group G, by Proposition 3.2 this is the same as saying that the power function p_k is totally singular at no $g \in G$, and by Proposition 3.2, for a compact Lie group G, this is also equivalent to the statement that no element $g \in G$ satisfies $g^k = 1$ and every eigenvalue of $\operatorname{Ad} g$ is a kth root of unity different from 1. With this notation we have the following theorem.

Theorem 3.4. *Let n_1 and n_2 be natural numbers and G a compact Lie group. Assume that G is n_j-straight for both $j = 1$ and $j = 2$. Then the following statements are equivalent.*

(1) *The probability that for a randomly selected pair of elements $x, y \in G$, the powers x^{n_1} and y^{n_2} commute is positive.*

(2) *G_0 is abelian.*

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Proof. By Lemmas 3.1 and 3.3 we know that the measures $\mu_1 = p_{n_1}(\nu)$ and $\mu_2 = p_{n_2}(\nu)$ respect closed subgroups of the Lie group G. Then Proposition 3.2(i) says that

$$(\mu_1 \times \mu_2)(\{(x, z) \in G \times G : [x, y] = 1\}) > 0.$$

Now by Corollary 2.3, this is equivalent to

$$Z(F) \text{ is open.} \tag{2'}$$

Then G_0, being contained in the abelian group $Z(H)$, is commutative. Now assume (2). Since G/G_0 is finite for a compact Lie group and G_0 is abelian each conjugacy class of G_0 is finite, that is, $G_0 \subseteq F$, where F is the FC-centre of G. Hence F is open and so, by Corollary 2.3, this implies (1). □

4. Consequences for arbitrary compact groups

For a compact group G we write $\mathcal{N}(G)$ for the set of all closed normal subgroups N of G for which G/N is a Lie group. Then $G \cong \lim_{N\in\mathcal{N}(G)} G/N$ (see [7, Lemma 9.1, p. 448]). We shall now keep n_1 and n_2 fixed throughout the remainder of the section and show that Theorem 3.4(1) implies the commutativity of the identity component G_o of an arbitrary compact group G provided that all Lie group quotients are n_j-straight for $j = 1, 2$.

First a general remark. We have $G \cong \lim_{N\in\mathcal{N}(G)} G/N$. For $N \in \mathcal{N}(G)$ and for a fixed $n = 1, 2, \ldots$ we set

$$D_{n_1,n_2}(G) = \{(x, y) \in G \times G : [x^{n_1}, y^{n_2}] = 1\},$$
$$D_{n_1,n_2,N}(G) = \{(x, y) \in G \times G : [x^{n_1}, y^{n_2}] \in N\}.$$

Obviously $N \subseteq M$ implies that

$$D_{n_1,n_2,N}(G) \subseteq D_{n_1,n_2,M}(G); \quad \text{also } D_{n_1,n_2}(G) = \bigcap_{P\in\mathcal{N}(G)} D_{n_1,n_2,P}(G). \tag{\#}$$

Now we define the generalised commutativity degree (depending on n_1 and n_2):

$$d_{n_1,n_2}(G) = (\nu_G \times \nu_G)(D_{n_1,n_2}(G)).$$

Note that with the quotient morphism $\pi_N : G \to G_N$,

$$D_{n_1,n_2}(G/N) = \{(xN, yN) \in G/N \times G/N : [x^{n_1}, y^{n_2}] \in N\} = (\pi_N \times \pi_N)(D_{n_1,n_2,N}(G)).$$

For a quotient map $\rho : \Gamma \to \Omega$ of compact groups and a Borel set $B \subseteq \Omega$ we have $\nu_\Gamma(\rho^{-1}(B)) = \nu_\Omega(B)$. Hence

$$d_{n_1,n_2}(G/N) = (\nu_G \times \nu_G)(D_{n_1,n_2,N}(G)) \geq (\nu_G \times \nu_G)(D_{n_1,n_2}(G)) = d_{n_1,n_2}(G).$$

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In particular, $\{d_{n_1,n_2}(G/N) : N \in \mathcal{N}(G)\}$ is a decreasing family of numbers in [0, 1] with $d_{n_1,n_2}(G)$ as a lower bound. So by the monotone convergence theorem of nets of real numbers,

$$\inf\{d_{n_1,n_2}(G/N) : N \in \mathcal{N}(G)\} = \lim_{N\in\mathcal{N}(G)} d_{n_1,n_2}(G/N).$$

Proposition 4.1. *We have* $d_{n_1,n_2}(G) = \lim_{N\in\mathcal{N}(G)} d_{n_1,n_2}(G/N)$.

Proof. In view of observation (#), this is a consequence of the outer regularity of Haar measure. □

Lemma 4.2. *Let k be a natural number and G a k-straight compact group. If N is a closed normal subgroup of G, then the quotient group G/N is k-straight.*

Proof. Let H/N be an arbitrary closed subgroup of G/N with a closed subgroup H of G and assume that $\nu_{G/N}(p_k^{-1}(H/N)) > 0$ for Haar measure $\nu_{G/N}$ of G/N. Now

$$\begin{aligned} p_k^{-1}(H/N) &= \{gN \in H/N : (gN)^k \in H/N\} \\ &= \{g \in G : g^k \in H\}N/N = p_k^{-1}(H)N/N \quad \text{in } G/N. \end{aligned}$$

If $g^k \in H$ and $n \in N$, then $(gn)^k \in g^k N \subseteq H$. Thus $p_k^{-1}(H)N = p^{-1}(H)$. Now $\nu_{G/N}(p_k^{-1}(H)/N) = \nu_G(p_k^{-1}(H))$. Thus our assumption yields $\nu_G(p_n^{-1}(H)) > 0$. Since G is k-straight, we conclude that H is open. But then H/N is open in G/N and so G/N is k straight. □

We conclude with the proof of Theorem 1.1.

Theorem 4.3 (Reformulation of Theorem 1.1). *Let n_1 and n_2 be natural numbers and G a compact group which is both n_1- and n_2-straight. Assume that the probability that $g_1^{n_1}$ and $g_2^{n_2}$ commute for two randomly selected elements $g_1, g_2 \in G$ is positive. Then G_0 is abelian.*

Proof. Let $N \in \mathcal{N}(G)$. By Lemma 4.2, G/N is both n_1- and n_2-straight. We know $0 < d_{n_1,n_2}(G) \leq d_{n_1,n_2}(G/N)$. Thus from Theorem 3.4 for Lie groups we know that $(G/N)_0$ is abelian. On the other hand, $(G/N)_0 = G_0N/N$, and so $[G_0, G_0] \subseteq N$. Since the intersection of all normal subgroups N such that G/N is a Lie group is singleton, we obtain $[G_0, G_0] = \{1\}$. □

It may be useful to recall that $G = G_0D$ for some profinite group D such that $G_0 \cap D$ is normal in G. This follows from Dong Hoon Lee's supplement theorem for compact groups (see [7, Theorem 9.41]). However, the issue of commuting powers in the case of a profinite group G is not discussed in this paper.

5. An example and some final comments

Finally, we record a relevant example even though it treats the simplest case that $n_1 = n_2 = 1$. This example was not mentioned in [8]. First we prove the following lemma.

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LEMMA 5.1. *Let G_n, $n = 1, 2, \ldots$, be a sequence of finite groups with the commutativity degrees $d(G_n) = d_n$. Then the product $G = \prod_{n=1}^{\infty} G_n$ has commutativity degree $\prod_{n=1}^{\infty} d_n = \lim_{n\to\infty} d_1 d_2 \cdots d_n$.*

PROOF. Define

$$E_n = D(G_1) \times \cdots \times D(G_n) \times G_{n+1}^2 \times \cdots \subseteq \prod_{n=1}^{\infty} (G_n \times G_n) \cong G \times G.$$

Let $P = \nu \times \nu$ for Haar measure on G. Then $P(E_n) = d_1 \cdots d_n$. If $\prod_{n=1}^{\infty} G_n \times G_n$ is identified with $G \times G$, then $\bigcap_{n=1}^{\infty} E_n = D(G)$. Since ν is σ-additive, so is P and therefore $P(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} d_1 \cdots d_n$. The assertion follows. □

The following example, among other things, illustrates Proposition 4.1.

EXAMPLE 5.2. Let $G = (\mathbb{T} \rtimes \{1, -1\})^{\mathbb{N}}$. Then $d(G) = 0$ while G_0 is abelian and not open. Indeed, $d(\mathbb{T} \rtimes \{1, -1\}) = 1/4$ by [8, Example 5.4]. Then, by Lemma 5.1 above, $d(G) = \lim_{n\to\infty} 1/4^n = 0$.

For the history of the general combinatorial issue of commuting elements in finite and compact groups, see [3, 6, 10, 11]; for a more extensive list of references we refer to [8, Section 6] where the history is discussed explicitly. Some recent developments on the topic can also be found in [4, 12, 13].

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https://doi.org/10.1017/S0004972712000573 Published online by Cambridge University Press

KARL H. HOFMANN, Fachbereich Mathematik,
Technische Universität Darmstadt, Schlossgartenstr. 7,
64289 Darmstadt, Germany
e-mail: hofmann@mathematik.tu-darmstadt.de

FRANCESCO G. RUSSO, Dieetcam,
Universitá degli Studi di Palermo, Viale delle Scienze,
Edificio 8, 90128, Palermo, Italy
e-mail: francescog.russo@yahoo.com

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