Proof that every rational algebraic equation has a root

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The following arrangement of the proof of this theorem could, I think, be given at a comparatively early stage, even if the necessary case of De Moivre's theorem had to be proved as an introductory lemma.

Let u, v be two rational integral algebraic functions of x, y with real coefficients, and let c be a simple closed contour in the plane. As the point (x, y) travels round c let those changes in the sign of u that take place when v is positive be marked and let $(u, v\,;\,c)$ denote the excess in number among these of changes from + to − over changes from − to + *.

If c is deformed continuously, $(u, v\,;\,c)$ will not be changed except (1), when c passes over a point where u, v both vanish, (2), when there is a change in the number of points where c meets one of the curves $u=0$, $v=0$. In case (1) there will generally be a change in the value of $(u, v\,;\,c)$ since a change of sign in u on c will pass from the part of c where v is negative to that where v is positive, or conversely.

In case (2) suppose c to be deformed so that the number of its intersections with the curve $u=0$ is increased. The increase must be an even number since both curves are continuous and endless. The sign of v is constant in the neighbourhood unless we are dealing with a case under (1) and since the new changes in sign of u are alternately + − and − + there is no effect on $(u, v\,;\,c)$: similarly if the number of intersections with $u=0$ is decreased.

If the number of intersections of c with the curve $v=0$ is altered, it must again be by an even number and there will be no change in $(u, v\,;\,c)$ unless $u=0$ at the same place as v when the case falls under (1).

Hence the deformation of c produces no effect on $(u, v\,;\,c)$ unless c passes over a point where $u=0=v$. In particular, if c does not contain such a point, it can be made to shrink up to a small contour in a neighbourhood where u, v are of constant signs and $(u, v\,;\,c)$ being unaffected by this process must be 0 throughout.

* It is not hard to see that $(v,u\,;\,c)=(-u,v\,;\,c)=-(u,v\,;\,c)$ etc.

Now take u, v to be given by the equation $f(z) = u + \iota v$ where $z = x + \iota y$ and $f(z) = z^n + (a_1 + \iota b_1) z^{n-1} + (a_2 + \iota b_2) z^{n-2} + \ldots$: let c be a circle with centre at the origin and radius R, so that on c $z = \mathrm{R}(\cos\theta + \iota\sin\theta)$, and θ runs from 0 to 2π.

We have by De Moivre's theorem

$$u = \mathrm{R}^n\cos n\theta + \mathrm{R}^{n-1}(a_1\cos\overline{n-1}\theta - b_1\sin\overline{n-1}\theta) + \text{lower powers of R},$$
$$v = \mathrm{R}^n\sin n\theta + \mathrm{R}^{n-1}(a_1\sin\overline{n-1}\theta + b_1\cos\overline{n-1}\theta) + \text{lower powers of R}.$$

The sum of all the terms after the first, either in u or v, is not greater than the sum of $a_1\mathrm{R}^{n-1}$, $b_1\mathrm{R}^{n-1}$, $a_2\mathrm{R}^{n-2}\ldots$ all taken positively and a value of R may be chosen so great that this sum does not exceed $k\mathrm{R}^n$ where k is any finite quantity. Thus the sign of u will be that of its first term if $\cos^2 n\theta > k^2$ and similarly for v: we shall take $k = \frac{1}{2}$.

Divide c into $4n$ parts at the points where

$$\theta = (2r+1)\pi/4n \qquad (r = 0, 1, \ldots 4n-1)$$

In the part of c when θ rises from $(8m-1)\pi/4n$ to $(8m+1)\pi/4n$ $\cos n\theta \not< 1/\sqrt{2}$ and thus u is positive while the sign of v is at first $-$ and at last $+$.

When θ rises from $(8m+1)\pi/4n$ to $(8m+3)\pi/4n$, v is positive, but u begins $+$ and ends $-$.

When θ rises from $(8m+3)\pi/4n$ to $(8m+5)\pi/4n$, u is always $-$.

When θ rises from $(8m+5)\pi/4n$ to $(8m+7)\pi/4n$, v is always $-$.

Only in the second case is there any contribution to $(u, v; c)$ and as this case occurs n times contributing 1 each time we have

$$(u, v;\ c) = n.$$

Hence c must contain a point where u, v vanish together, and the equation $f(z) = 0$ must have a root.

The proof that there are exactly n roots is now easy.