ON INVERSES OF PRODUCTS OF IDEMPOTENTS IN REGULAR SEMIGROUPS

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Let E be the set of idempotents of a regular semigroup; we prove that $V(E^n) = E^{n+1}$ (see below for the meaning of this notation). This generalizes a result of Miller and Clifford ([3], theorem 4, quoted as exercise 3(b), p. 61, of Clifford and Preston [1]) and the converse, proved by Howie and Lallement ([2], lemma 1.1), which together establish the case $n = 1$. As a corollary, we deduce that the subsemigroup generated by the idempotents of a regular semigroup is itself regular.

Let N denote the set of natural numbers; let $n \in N$ and S be any semigroup. We denote by E the set of idempotents of S and by E^n the set of all products of n idempotents of S. Further, if E is not empty, let $\langle E \rangle$ denote the subsemigroup of S generated by E; then $\langle E \rangle = \bigcup_{i \in N} E^i$.

For any $x \in S$, we set

$$V(x) = \{y \in S : xyx = x \text{ and } yxy = y\},$$

the set of all inverses of x; and for any subset X of S we set $V(X) = \bigcup_{x \in X} V(x)$. For $m \in N$, we define $V^m(X)$ inductively, thus: $V^{m+1}(X) = V(V^m(X))$.

LEMMA 1. *Let $y \in S$. Then $y \in yE^n y$ implies $y \in E^{n+1}$.*

PROOF. Suppose $y = yxy$ for some $x \in E^n$, say $x = e_1 \cdots e_n$, where $e_i \in E$ for $i = 1, \cdots, n$.

For $i = 1, \cdots, n$, set $t_i = e_1 \cdots e_i$ and $u_i = e_i \cdots e_n$, so that $t_i u_i = x$. If $n \geqq 2$ set, for $j = 2, \cdots, n$, $f_j = u_j y t_{j-1}$, so that

$$\begin{aligned} f_j^2 = u_j y t_{j-1} u_j y t_{j-1} &= u_j y x y t_{j-1} \\ &= u_j y t_{j-1} \\ &= f_j, \end{aligned}$$

that is, $f_j \in E$. But

$$\begin{aligned} y = yxy &= y(xy)^n \\ &= y \cdot t_n u_n y \cdot t_{n-1} u_{n-1} y \cdot \cdots \cdot t_1 u_1 y \\ &= y t_n \cdot u_n y t_{n-1} \cdot \cdots u_2 y t_1 \cdot u_1 y \\ &= yx \cdot f_n \cdots f_2 \cdot xy. \end{aligned}$$

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(In the above, an undefined symbol is to be understood as the empty symbol.) Since $xy, yx \in E$, it follows that $y \in E^{n+1}$.

COROLLARY. $V(E^n) \subseteq E^{n+1}$.

LEMMA 2. *Let S be regular. Then $E^{n+1} \subseteq V(E^n)$.*

PROOF. Let $x = e_1 \cdots e_{n+1}$, where $e_i \in E$ for $i = 1, \cdots, n+1$. Since S is regular, there exists some $y \in S$ such that $xyx = x$ and $yxy = y$.

For $i = 1, \cdots, n+1$, set $t_i = e_1 \cdots e_i$ and $u_i = e_i \cdots e_{n+1}$, so that $t_i u_i = x$. Further, for $j = 1, \cdots, n$, set $f_j = u_{j+1} y t_j$, so that

$$f_j^2 = u_{j+1} y t_j u_{j+1} y t_j = f_j.$$

Then $z = f_n \cdots f_1 \in E^n$. But

$$\begin{aligned} x &= xyx = x(yx)^n \\ &= xe_{n+1} \cdot yt_n u_n \cdot \cdots yt_1 u_1 \\ &= x \cdot u_{n+1} yt_n \cdot \cdots u_2 yt_1 \cdot u_1 \\ &= xf_n \cdots f_1 x \\ &= xzx, \end{aligned}$$

and

$$\begin{aligned} zxz &= f_n \cdots f_1 x f_n \cdots f_1 \\ &= f_n \cdots f_2 \cdot u_2 yt_1 \cdot x \cdot u_{n+1} yt_n \cdot \cdots u_2 yt_1 \\ &= f_n \cdots f_2 \cdot u_2 y \cdot t_1 x u_{n+1} \cdot yt_n u_n \cdot \cdots yt_2 u_2 \cdot yt_1 \\ &= f_n \cdots f_2 \cdot u_2 y \cdot e_1 x e_{n+1} \cdot (yx)^{n-1} yt_1 \\ &= f_n \cdots f_2 u_2 yx \cdot yxyt_1 \\ &= f_n \cdots f_2 u_2 yt_1 \\ &= z. \end{aligned}$$

Thus $x \in V(z) \subseteq V(E^n)$, and the lemma is proved.

Lemmas 1 and 2 together establish the

THEOREM. *Let S be regular. Then $V(E^n) = E^{n+1}$.*

COROLLARY. *Let S be regular. Then $\langle E \rangle$ is regular.*

REMARK. *Moreover, $\langle E \rangle$ is then a complete regular subsemigroup of S in the sense that each inverse in S of an element of $\langle E \rangle$ is a member of $\langle E \rangle$. Indeed,*

$$\langle E \rangle = E \cup \Big(\bigcup_{m \in N} E^m\Big) = \bigcup_{m \in N} V^m(E).$$

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References

[1] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups, Volume* 1, (Math. Surveys, No. 7, Amer. Math. Soc., 1961).

[2] J. M. Howie and G. Lallement, 'Certain fundamental congruences on a regular semigroup', *Proc. Glasgow Math. Assoc.* 7 (1966), 145—159.

[3] D. D. Miller and A. H. Clifford, 'Regular $\mathscr{D}$-classes in semigroups', *Trans. Amer. Math. Soc.* 82 (1956), 270—280.

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