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MITTAG-LEFFLER THEOREMS ON RIEMANN SURFACES AND RIEMANNIAN MANIFOLDS

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ABSTRACT. Cauchy and Poisson integrals over *unbounded* sets are employed to prove Mittag-Leffler type theorems with massive singularities as well as approximation theorems for holomorphic and harmonic functions.

1. **Introduction.** Let Ω be a connected complex (respectively, Riemannian) manifold. In the Riemannian case, we assume that Ω is C^∞ and oriented. Our results will clearly hold also on an open subset Ω of the complex plane $\mathbf{C}$ (respectively, of Euclidean space $\mathbf{R}^n$), by considering components, although such an Ω need not be connected. For $F \subset \Omega$, we write $f \in H(F)$, if f is holomorphic (respectively, harmonic) on a neighbourhood of F.

THEOREM 1 (MITTAG-LEFFLER, MASSIVE SINGULARITIES). *Let $e \subset \omega \subset \Omega$, where Ω is a Stein or noncompact Riemannian manifold, e is closed and ω is open. For any $f_e \in H(\omega \setminus e)$ there exists a function $f \in H(\Omega \setminus e)$ such that $f - f_e$ extends to a a function in $H(\omega)$.*

In particular: let Ω be the complex plane $\mathbf{C}$; let e be a discrete subset of $\mathbf{C}$, which we may think of as a sequence of distinct points $e_1, e_2, \ldots$; let ω be a sequence of disjoint discs $\omega_j, j = 1, 2, \ldots$, centered respectively at the e_j's; let $p_j, j = 1, 2, \ldots$, be polynomials; let

$$f_j(x) = p_j\Big(\frac{1}{z - e_j}\Big), \quad j = 1, 2, \ldots$$

and define f_e on $\omega \setminus e$ by setting $f_e = f_j$ on each $\omega_j \setminus \{e_j\}$. Theorem 1, in this case, says that, given a sequence of distinct points e_j tending to ∞, there exists a meromorphic function f on $\mathbf{C}$, having prescribed principal parts $p_j\big(1/(z - e_j)\big)$ at the points e_j and having no other singularities. This is the classical Mittag-Leffler theorem.

If we consider a singular point of a function to be a point where the function is not holomorphic, then, since holomorphic functions are defined on open sets, it is natural to look at closed sets as possible singularity sets of functions. Given a closed set $e \subset \mathbf{C}$, we say that f_e is a singularity function at e if there is an open set ω containing e such that f_e is holomorphic on $\omega \setminus e$. The singularity function f_e prescribes the *nature* of the singularity at e. Two singularity functions f_e and g_e at e are said to have singularities of

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the same nature at e if $f_e - g_e$ extends holomorphically to (a neighbourhood of) e. In the classical Mittag-Leffler theorem, the nature of the prescribed singularities is given by prescribing the principal parts at the prescribed sequence of points. Theorem 1 allows us to replace the discrete set of singularities in the classical Mittag-Leffler theorem by an arbitrary singularity set and, moreover, the nature of the singularity can be arbitrary instead of being poles. That is, there is *no* restriction on the singularity function f_e at e.

This theorem was stated by Saakian [20] for the case that $\Omega = \mathbf{C}$. In fact, Theorem 1 is a particular case of the following.

LEMMA 1 (COUSIN LINKING). *Let a Stein or noncompact Riemannian manifold* Ω *be written as the union of two open sets* Ω_-, Ω_+ *and let* $g \in H(\Omega_- \cap \Omega_+)$*. Then, there are functions* $g_- \in H(\Omega_-)$ *and* $g_+ \in H(\Omega_+)$ *such that*

$$g = g_- + g_+.$$

For the case that Ω is an arbitrary open set in $\mathbf{C}$, this linking lemma is the well known theorem of Aronszajn [1] on the separation of singularities, for which Havin [15] gave a short proof using duality. Another proof (*cf.* Example 3.2.5 in the book by Berenstein and Gay [3]) follows immediately from the solvability of the additive Cousin problem, and hence clearly holds on any Stein manifold Ω, and, in particular, on any open set of holomorphy $\Omega \subset \mathbf{C}^n$.

Since the additive Cousin problem can also be solved for harmonic functions, in fact, for elliptic systems on manifolds (*cf.* Tarkhanov [23]), the proof also works for harmonic functions on Riemannian manifolds.

Let e_- and e_+ be two disjoint closed subsets of a manifold Ω. By the condenser (or capacitor) $C(e_-, e_+)$, we understand the open set $C(e_-, e_+) = \Omega \setminus (e_- \cup e_+)$. Let $H(e_-, e_+)$ denote the family of holomorphic (if Ω is Stein) or harmonic (if Ω is Riemannian) functions on the condenser $C(e_-, e_+)$. We say that a condenser $C(e_-, e_+)$ on a manifold Ω admits linking if, for each $g \in H(e_-, e_+)$, there are $g_- \in H(\Omega \setminus e_-)$ and $g_+ \in H(\Omega \setminus e_+)$ such that

$$g = g_- + g_+.$$

The following weaker form of the Cousin linking lemma is clearly equivalent to Theorem 1 and brings out the symmetry of the problem.

LEMMA 2 (WEAK LINKING). *On a Stein manifold or a noncompact Riemannian manifold, every condenser admits linking.*

Theorem 1, for the simple case that $\Omega \setminus e$ is an annulus in the complex plane, is an immediate consequence of the Cauchy formula (which in this case yields a more explicit representation, namely, the Laurent expansion). Although the general proof for elliptic systems [23] is in the sophisticated language of cohomology, we shall show that, at least for holomorphic functions on Riemann surfaces and harmonic functions on Riemannian manifolds, we can imitate the proof for the annulus, by using Cauchy and (respectively) Poisson integrals. In the next section, we start with the holomorphic case on Riemann surfaces although (or because) it is better known and simpler. In Section 3, we prove the harmonic case. Then, in Section 4, we look at these questions on compact surfaces. Finally, in the last section, we take a closer look at the more traditional point singularities.

In fact, this paper was at one time a short note on the Mittag-Leffler theorem for harmonic functions with prescribed *point* singularities. I wish to thank Ashot Nersessian for suggesting that such a theorem might also hold with *massive* singularities and Thomas Ransford for pointing out that the holomorphic case is a consequence of the Cousin linking lemma. Our approach finesses the language of cohomology by exploiting the properties of Cauchy and Poisson integrals over *unbounded* sets.

This paper was completed in the fragrance of lotus pools and surrounded by the enchanting musical strains of Chinese opera emanating from the island temple on the campus of Peking University. Xiexie Beida!

2. **Open Riemann surfaces.** In this section, Ω will denote a Riemann surface and, for $F \subset \Omega$, we write $f \in \mathrm{Hol}(F)$ if f is a function holomorphic on (a neighbourhood of) F. We denote the one-point compactification of Ω by $\Omega^* = \Omega \cup \{*\}$.

In 1927, Carleman established the following remarkable result. If f is any continuous function on the real axis $\mathbf{R}$ and ϵ is any positive continuous function on $\mathbf{R}$, then, there is an entire function g such that $|f-g| < \epsilon$. In attempting to perform similar approximations on more general unbounded subsets of the complex plane $\mathbf{C}$ than the real axis, the author introduced the so-called "long islands" condition on a closed subset F of $\mathbf{C}$, which he showed to be a necessary condition for such Carleman type approximation. Nersessian completed the characterization of sets of Carleman approximation by showing that this condition was, in fact, also sufficient. This result was extended to Riemann surfaces by Boivin [4]. For our purposes, we do not require the full strength of Boivin's theorem; the following is adequate.

THEOREM 2 (CARLEMAN TYPE). *Suppose γ is a closed subset of an open Riemann surface Ω. The following are equivalent.*

i) For each $f \in C(\gamma)$ and positive $\epsilon \in C(\gamma)$, there is a $g \in \mathrm{Hol}(\Omega)$ such that $|f-g| < \epsilon$.

ii) $\gamma^0 = \emptyset$ and $\Omega^* \setminus \gamma$ is both connected and locally connected.

The above theorem gives a complete answer to the problem of (better than) uniform approximation of *continuous* functions on closed subsets of Riemann surfaces.

The analogous problem of uniform approximation of *holomorphic* functions on a closed subset $e \subset \Omega$ was solved by Alice Roth for the case that Ω is a plane domain and partially extended to Riemann surfaces by the author [10]. The basic tool exploited was that of Cauchy integrals on (possibly) unbounded curves. The main objective of this section will be to point out that the Cauchy integral also (in fact, more easily) yields the Mittag-Leffler theorem with massive singularities on Riemann surfaces.

Let φ be a global local uniformizer on Ω, shown to exist by Gunning and Narasimhan [14]. This is just a function $\varphi \in \mathrm{Hol}(\Omega)$ which is locally injective. Thus, in the neighbourhood of each point $p_0 \in \Omega$, we may take $z = \varphi(p)$ as a local parameter. By a Cauchy kernel C on the Riemann surface Ω, we mean a meromorphic function $C(p, q)$ on $\Omega \times \Omega$,

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whose only singularities are along the diagonal, and if (D, φ) is a parametric disc and $(z, \zeta) = \big(\varphi(p), \varphi(q)\big)$, then, in the local coordinates (z, ζ),

$$C(z, \zeta) = \frac{1}{\zeta - z} + h(z, \zeta),$$

where h is holomorphic. Every open Riemann surface admits a Cauchy kernel. A short proof [10] can be based on the fact that the product $\Omega \times \Omega$ of an open Riemann surface with itself is a Stein manifold. Indeed, we have only to solve the first Cousin problem on $\Omega \times \Omega$, with Cousin data equal to 0 off of the diagonal and equal to $1/[\varphi(p) - \varphi(q)]$ on every set $D \times D$, where D is an open subset of Ω on which the global parameter function φ is injective. Of course, by invoking Stein theory, we are making an implicit appeal to cohomology theory. However, the Cauchy kernel is such a natural entity that it seems of interest to present an exposition based on the Cauchy kernel as an alternative to that based on the linking lemma. Moreover, Magnus [17] has recently given an elementary construction of the Cauchy kernel, which is free of cohomology.

Let σ be a smooth curve on Ω. By this we mean that σ is a closed set in Ω and, for each $z \in \sigma$, there is an open neighbourhood $z \in U \subset \Omega$ and a C^1-diffeomorphism $s \colon \Delta \longrightarrow U$ of the unit disc Δ onto U such that the image of the interval $(-1, +1)$ is the set $\sigma \cap U$. We fix an orientation on σ. If ψ is a continuous function on σ, we consider, for $p \in \Omega - \sigma$, the Cauchy integral

$$\Psi(p) = \Psi(z) = \frac{1}{2\pi i} \int_\sigma \psi(\zeta) C(z, \zeta)\, d\zeta,$$

which is well defined if $(z, \zeta) = \big(\varphi(p), \varphi(q)\big)$, φ is the previously defined holomorphic function on Ω, and ψ decays sufficiently rapidly at the ideal boundary.

LEMMA 3. *Given a smooth curve σ in Ω, there exists a positive continuous function $\eta \in C(\sigma)$ such that, if $|\psi| < \eta$, then the Cauchy integral Ψ of ψ converges uniformly on compact subsets of $\Omega \setminus \sigma$ and hence $\Psi \in \mathrm{Hol}(\Omega \setminus \sigma)$.*

We shall say that a Cauchy kernel C on an open Riemann surface Ω is bounded at infinity if, for each compact $K \subset \Omega$, and each neighbourhood U of K, the Cauchy kernel is bounded on $K \times (\Omega \setminus U)$ and on $(\Omega \setminus U) \times K$. For example, if Ω is a plane domain, then the standard Cauchy kernel $1/(\zeta - z)$, is bounded at infinity. On the other hand, there exist open Riemann surfaces, for example, the so-called Myrberg surfaces, for which no Cauchy kernel is bounded at infinity (*cf.* [12]).

LEMMA 4. *If, under the hypotheses of the previous lemma, the Cauchy kernel is bounded at infinity, then, given a closed set $e \subset \Omega \setminus \sigma$ and a positive constant $\epsilon > 0$, we may so choose the function η that $|\Psi| < \epsilon$ on e.*

Thus far, we have considered the Cauchy integral Ψ of a general continuous (or even locally integrable) function ψ on σ which decays rapidly at infinity. We have noted that $\Psi \in \mathrm{Hol}(\Omega \setminus \sigma)$. The behaviour of Ψ as we approach σ has been studied in depth in the literature. From the definition of σ, each point of σ has an open neighbourhood V which

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is separated by $\sigma \cap V$ into two disjoint domains V_- and V_+, where $\sigma \cap V$ is oriented positively with respect to V_+ and negatively with respect to V_-. Let us denote by Ψ_- and Ψ_+ the restrictions of Ψ to V_- and V_+ respectively. The following jump lemma is easily verified.

LEMMA 5 (JUMP). *If $\psi \in \mathrm{Hol}(V_-)$, then Ψ_+ extends holomorphically across $\sigma \cap V$. If we retain the notation Ψ_+ for this extension, then*

$$\Psi_+ - \Psi_- = \psi, \quad \textit{on } V_-.$$

We now have all of the ingredients to prove a Runge type approximation theorem, that is, a theorem on the uniform approximation of holomorphic functions on closed subsets of Riemann surfaces.

THEOREM 3 (RUNGE TYPE). *Let Ω be an open Riemann surface having a Cauchy kernel bounded at infinity. Let e be a closed subset of Ω. The following are equivalent.*

1) For each constant $\epsilon > 0$ and each $f_e \in \mathrm{Hol}(e)$, there is a $g \in \mathrm{Hol}(\Omega)$ such that $|f_e - g| < \epsilon$.

2) $\Omega^* \setminus e$ is connected and locally connected.

If Ω is a plane domain, then, for e compact, this is the classical theorem of Runge (1885); for e closed, the implication 2) $\longrightarrow$ 1) is due to Alice Roth (1938); and the implication 1) $\longrightarrow$ 2) is due to Arakelian (see [8] and [12]). If Ω is an open Riemann surface and e is compact, the assumption on the Cauchy kernel is superfluous and the result is the famous theorem of Behnke-Stein (*cf.* [18]). The general version was proved by the author (see [10]). If the assumption on the Cauchy kernel is dropped, the theorem no longer holds [12].

PROOF OF THEOREM 1 FOR RIEMANN SURFACES. In fact, we shall notice that the proof also yields the implication 2) $\longrightarrow$ 1) in Theorem 3. Suppose, then, that ω is an open neighbourhood of e and $f_e \in \mathrm{Hol}(\omega \setminus e)$ (respectively, $f_e \in \mathrm{Hol}(\omega)$.) We may construct a smoothly bordered neighbourhood U of e whose closure is in ω and such that $\omega^* \setminus \overline{U}$ is connected and also *locally* connected. We decompose the boundary of U into two disjoint parts $\partial U = \gamma + \sigma$. The part γ consists of certain components of ∂U which lie on the boundary of *unbounded* components of U: namely, all unbounded boundary components and also those bounded boundary components, which lie on the boundary of unbounded components of U and which do not separate Ω. The part σ consists of all other boundary components of U. From the construction of γ, we have that $\Omega^* \setminus \gamma$ is connected and locally connected. Let η be a positive continuous function on γ. By Theorem 2, there is a function $g_\eta \in \mathrm{Hol}(\Omega)$ such that $|f_e - g_\eta| < \eta$ on γ. By Lemma 3, we may choose η such that the Cauchy integral Ψ_γ of $f_e - g_\eta$ on γ is in $\mathrm{Hol}(\Omega \setminus \gamma)$.

Let $\Psi_\gamma^+ = \Psi_\gamma | U$ and $\Psi_\gamma^- = \Psi_\gamma | \Omega \setminus \overline{U}$. Let σ_0 be a component of σ. Since Ψ_γ is well defined in a neighbourhood of σ_0, we may extend Ψ_γ^+ and Ψ_γ^- across σ_0 and $\Psi_\gamma^+ = \Psi_\gamma = \Psi_\gamma^-$ near σ_0. This continuation, starting near γ, cannot lead to other values than

$$\Psi_\gamma^+ - \Psi_\gamma^- = f_e - g_\eta$$

because by continuing through components of σ it is impossible to travel from one side of γ to the other side. This is because a path starting on the inner side of a component γ_0 of γ is in an unbounded component of U. As it traverses a σ_0, it can never get to the other side of γ, because σ_0 separates Ω. Thus, Ψ_γ^+ extends to a function in $\mathrm{Hol}(\omega)$ and Ψ_γ^- extends to a function in $\mathrm{Hol}(\Omega \setminus e)$ (respectively, to a function in $\mathrm{Hol}(\Omega)$ in the case of Theorem 3) and

(1) $$f_e = (\Psi_\gamma^+ + g_\eta) - \Psi_\gamma^- \quad \text{on } \omega \setminus e.$$

Now write $\sigma = \sum \sigma(j)$, where each $\sigma(j)$ is either the entire boundary of a bounded component of U or a component of σ which separates Ω. Choose a sequence $\{\delta(j)\}$ of positive numbers such that $\sum \delta(j) < \infty$ and the function

$$\Psi_\sigma(z) = \sum_j \Psi_{\sigma(j)}(z) = \sum_j \frac{\delta(j)}{2\pi i} \int_{\sigma(j)} f_e(\zeta) C(z,\zeta)\, d\zeta$$

is in $\mathrm{Hol}(\Omega \setminus \sigma)$. By Lemma 5,

$$\Psi_{\sigma(j)}^+ - \Psi_{\sigma(j)}^- = \delta(j) f_e$$

near $\sigma(j)$. Since $\sigma(j)$ separates Ω, this allows us to extend $\Psi_{\sigma(j)}^+$ and $\Psi_{\sigma(j)}^-$ so $\Psi_{\sigma(j)}^+$ is in $\mathrm{Hol}(\omega)$ and $\Psi_{\sigma(j)}^-$ is in $\mathrm{Hol}(\Omega \setminus e)$ (respectively, in $\mathrm{Hol}(\Omega)$ in the case of Theorem 3.) Hence,

(2) $$\Psi_\sigma^+ - \Psi_\sigma^- = \sum \delta(j) f_e \quad \text{on } \omega \setminus e.$$

Set $\delta = \sum \delta(j)$. Combining (1) and (2), we have

$$(1+\delta) f_e = (\Psi_\gamma^+ + \Psi_\sigma^+ + g_\eta) - (\Psi_\gamma^- + \Psi_\sigma^-),$$

which concludes the proof of Theorem 1.

To prove the implication 2) $\longrightarrow$ 1) in Theorem 3, we write

$$(1+\delta) f_e = (g_\eta - \Psi_\gamma^- - \Psi_\sigma^-) + (\Psi_\gamma^+ + \Psi_\sigma^+) = g + \Psi^+$$

on ω, where $g \in \mathrm{Hol}(\Omega)$. If ϵ is any positive continuous function on e, we may, by Lemma 4, choose η and $\{\delta(j)\}$ to decrease so rapidly that $|\Psi^+| < \epsilon/(1+\delta)$ on e. This proves the implication 2) $\longrightarrow$ 1).

3**. Riemannian manifolds.** The proof of the *harmonic* Mittag-Leffler theorem with massive singularities is completely analogous to the proof we have just given for the holomorphic case on Riemann surfaces. Of course, we must replace the Cauchy integral of a function by a Poisson-Green integral. Since the Poisson-Green integral of a function involves, not only the function and the "Poisson" kernel, but also their normal derivatives, we shall need a Carleman type lemma which allows one to approximate, not only a function, but also its derivatives. On a manifold, derivatives do not have definite values, but depend on the local coordinates. As Narasimhan notes [18, p. 176], since, locally, any vector bundle is isomorphic to the trivial bundle, we may speak of the *local* uniform

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convergence, together with all partial derivatives, of a sequence of smooth sections. It is not so clear what would be meant by *global* uniform convergence of a sequence and its derivatives. Thus, although the notion of uniform approximation of a *function* on an unbounded set makes sense, we do not see how to make sense of the notion of uniform approximation of the *derivative* of a function on an unbounded set. Nevertheless, we can give a sense to the stronger notion of *Carleman* approximation of the derivative. We shall use Carleman approximation of functions and their derivatives on very particular sets to obtain uniform approximation of functions on rather general sets and linking for completely general condensers.

Let F be a closed subset of a manifold Ω. By a Carleman gauge for F, we mean a family $\{B_j,\varphi_j,\epsilon_j\}_{j\in J}$, where $\{B_j,\varphi_j\}_{j\in J}$ is a locally finite cover of F by parametric balls and $\{\epsilon_j\}_{j\in J}$ is a set of positive numbers.

The notations $f\in C^1(S)$ and $f\in \mathrm{Har}(S)$ mean that f is C^1 and harmonic respectively on (a neighbourhood of) the set S. If $x=\varphi(p)$ is a local coordinate in a parametric ball (B,φ), we use the abusive notation $f(x)$ and $(\nabla f)(x)$ for $f\circ\varphi^{-1}$ and $\nabla(f\circ\varphi^{-1})$ respectively evaluated at x. Of course, the former is independent of the chart while the latter is not. With this understanding, we introduce the notation $\|f\|_1=|f|+\|\nabla f\|$ and the notation $\|f\|_1(x)$ stands for the function $\|f\|_1$ evaluated at the point x.

By a Newtonian kernel on Ω, we mean a function $e\colon \Omega\times\Omega\longrightarrow(-\infty,+\infty]$, such that, if R is a smoothly bounded subregion of Ω, then we may write $e(p,q)=G_R(p,q)+v_R(p,q)$, for p and q in R, where G_R is the Green function for R and v_R is harmonic in each variable. For the existence and properties of the Newtonian kernel e, see [2]. In particular, we have the usual Poisson-Green formula

$$h(q)=\int_{\partial R}\Big[e(p,q)\frac{\partial h}{\partial\nu}(p)-h(p)\frac{\partial e}{\partial\nu}(p,q)\Big]\,dA,$$

which follows from the Green formula II [6, p. 144], where $q\in R$, $h\in\mathrm{Har}(\overline{R})$, ν is the outward unit vector field along ∂R which is pointwise orthogonal to ∂R, $\partial f/\partial\nu=\langle \mathrm{grad} f,\nu\rangle$, and dA is the Riemannian "area" measure on ∂R.

LEMMA 6 (RUNGE TYPE). *Let $f\in\mathrm{Har}(K)$, where $K\subset\Omega$ is compact and $\Omega^*\setminus K$ is connected. Let $\{B_j,\varphi_j\}_{j\in J}$ be a finite cover of K by parametric balls. For each $\epsilon>0$, there exists $g\in\mathrm{Har}(\Omega)$ such that, for each $j\in J$,*

$$\|f-g\|_1(x)<\epsilon,\quad x\in\varphi_j(K\cap\overline{B}_j).$$

More generally, we wish to perform such an approximation in certain situations, where the hypothesis that $f\in\mathrm{Har}(K)$ is relaxed (see the Saturn lemma below).

PROOF. We may construct a compact set Q such that $L\subset Q^o$, $\Omega^*\setminus Q$ is connected and $f\in\mathrm{Har}(Q)$. By [2, Theorem 9.3], there is a sequence $g_k\in\mathrm{Har}(\Omega)$ which converges uniformly to f in Q. It follows that in any parametric ball $\overline{B}\subset Q^o$, the convergence also holds for derivatives. Since any $K\cap\overline{B}_j$ can be covered by finitely many such parametric balls B, the lemma follows.

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Let us say that two sets K_1 and K_2 are *properly linked* if $K_1 \setminus K_2$ and $K_2 \setminus K_1$ have disjoint closures. In the following lemma, we shall, in some sense, "fuse" two functions, g_1 and g_2, by a single function g.

LEMMA 7 (FUSION). *Let $K = K_1 \cup K_2$, where K_1 and K_2 are properly linked compact sets and let $\{B_j, j \in J\}$ be a finite cover of K by parametric balls. There is a constant $a > 0$ such that, if $g_i \in \mathrm{Har}(K)$, $i = 1, 2$, then there exists $g \in \mathrm{Har}(K)$ such that, for $i = 1, 2$ and $j \in J$,*

$$\|g_i - g\|_1(x) < a \cdot M_j(g_1 - g_2), \quad x \in \varphi_j(K_i \cap \overline{B}_j),$$

with

$$M_j(g_1 - g_2) = \max\{\|g_1 - g_2\|_1(y) : y \in \varphi_j(K_1 \cap K_2 \cap \overline{B}_j)\}.$$

PROOF. Let $h \in C_o^\infty(\Omega)$ such that $0 \le h \le 1$, $h = 1$ on a neighbourhood of $K_1 \setminus K_2$ and $h = 0$ on a neighbourhood of $K_2 \setminus K_1$. The constant a will depend on h only. We may assume that $g_2 = 0$ and that $g_1 \in C_o^\infty(\Omega)$. Thus, we seek a constant a and a $g \in \mathrm{Har}(K)$ such that g fuses g_1 and 0 in the sense that, for $j \in J$,

$$\|g_1 - g\|_1(x) < a \cdot M_j(g_1), \quad x \in \varphi_j(K_1 \cap \overline{B}_j)$$
$$\|g\|_1(x) < a \cdot M_j(g_1), \quad x \in \varphi_j(K_2 \cap \overline{B}_j).$$

Set $\psi = h \cdot g_1$. Then, ψ does indeed fuse g_1 and 0 with some constant a_o depending only on h. However, ψ is merely a C^∞-fusion whereas we are seeking a fusion $g \in \mathrm{Har}(K)$.

Fix a bounded neighbourhood V of $K_1 \cap K_2$ such that $g_1 \in \mathrm{Har}(\overline{V})$ and for $j \in J$,

$$\max\{\|g_1\|_1(y) : y \in \varphi_j(v \cap \overline{B}_j)\} \le 2M_j(g_1).$$

If $e(p, q)$ is the the Newtonian kernel introduced above, then following [2, Theorem 4.8],

$$\begin{aligned}\psi(q) &= -\int e(p, q)(\Delta\psi)(p)\, d\lambda(p) \\ &= -\int_{\Omega\setminus V} e(p, q)(\Delta\psi)(p)\, d\lambda(p) - \int_V e(p, q)(\Delta\psi)(p)\, d\lambda(p).\end{aligned}$$

Set

$$g(q) = -\int_{\Omega\setminus V} e(p, q)(\Delta\psi)(p)\, d\lambda(p).$$

Then, $g \in \mathrm{Har}(K)$, since $\Delta\psi = 0$ on a neighbourhood of $K \setminus V$. We have in V

$$\Delta\psi = g_1 \cdot \Delta h + 2\,\mathrm{grad}\, h \cdot \mathrm{grad}\, g_1.$$

From [2, Lemma 6.1], there is an $a_1 > 0$, depending only on h and the Newtonian kernel e, such that, for $i = 1, 2$ and $j \in J$,

$$\|g - \psi\|_1(x) < a_1 \cdot M_j(g - \psi), \quad x \in \varphi_j(K_i \cap B_j).$$

Since $\|g_1 - g\|_1 \le \|g_1 - \psi\|_1 + \|g - \psi\|_1$, and ψ has already been seen to fuse g_1 and 0, and $M_j(g - \psi) \le a_2 \cdot M_j(g_1)$, for some constant a_2, the lemma follows.

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LEMMA 8 (LOCALIZATION). *Let $K \subset \Omega$ be a compact set with $\Omega^* \setminus K$ connected, f be a function which is C^1 in a neighbourhood of K, and $\{B_j, \varphi_j\}_{j \in J}$ be a finite cover of K by parametric balls. The following two conditions are equivalent.*

(i) For each set of ϵ_j, $j \in J$, there exists a family of $g_j \in \mathrm{Har}(\Omega)$ such that

$$\|f - g_j\|_1(x) < \epsilon_j, \quad x \in \varphi_j(K \cap \bar{B}_j).$$

(ii) For each set of ϵ_j, $j \in J$, there exists a $g \in \mathrm{Har}(\Omega)$ such that, for each $j \in J$,

$$\|f - g\|_1(x) < \epsilon_j, \quad x \in \varphi_j(K \cap \bar{B}_j).$$

PROOF. The implication (ii) $\longrightarrow$ (i) is trivial. The idea of proving localization via fusion was introduced by Alice Roth in the context of rational approximation (see [8] and [12]). In order to apply this technique, we shall refine the covering $\{B_j\}_{j \in J}$. We may suppose that Ω is embedded in some Euclidean space $\mathbf{R}^m$. Now we partition $\mathbf{R}^m$ into m-cubes in the usual way by hyperplanes parallel to the coordinate hyperplanes, with mesh size h. We choose h so small that, for each such hypercube Q, $K \cap Q$ is contained in some B_j. We may now prove localization via fusion following Roth's technique (see [8, p. 117].)

In general, the family of sets on which approximation is possible is not preserved under unions. However, we have the following special result.

LEMMA 9 (SATURN). *Suppose K is a compact set, H is a compact set situated on a smooth hypersurface and $\Omega^* \setminus (K \cup H)$ is connected. Then, for each function f which is C^1 in a neighbourhood of $K \cup H$ and harmonic on K and for each Carleman gauge $\{B_j, \varphi_j, \epsilon_j\}_{j \in J}$ for $K \cup H$, there exists a function $g \in \mathrm{Har}(\Omega)$ such that, for each $j \in J$,*

$$\|f - g\|_1(x) < \epsilon_j, \quad x \in \varphi_j\big((K \cup H) \cap \bar{B}_j\big)$$

PROOF. This follows immediately from the above localization lemma, the Hartogs-Rosenthal type theorem (see [5, Theorem 3.23], [13, Theorem 5.1], [23], [24, Theorem 3.4] and [21, Satz 1, p. 247]), and the Runge type lemma.

LEMMA 10 (CARLEMAN TYPE). *Suppose Γ is a smooth hypersurface in Ω with $\Omega^* \setminus \Gamma$ connected. Let $f \in \mathrm{Har}(\Gamma)$ and $\{B_j, \varphi_j, \epsilon_j\}_{j \in J}$ be a Carleman gauge for Γ. Then, there exists a function $g \in \mathrm{Har}(\Omega)$ such that*

$$\|f - g\|_1(x) < \epsilon_j, \quad x \in \varphi_j(\bar{B}_j \cap \Gamma). \tag{3}$$

We pause to explain in which sense we consider this lemma to be a result on Carleman type approximation. Let f be a C^1-function in a neighbourhood of a closed set $F \subset \Omega$ and let $\{B_j, \varphi_j\}$, $j \in J$ be a locally finite cover of F by parametric balls. We say that f admits C^1-Carleman approximation with respect to this cover $\{B_j, \varphi_j\}$, $j \in J$ by functions in $\mathrm{Har}(\Omega)$, if for each set $\{\epsilon_j\}$, $j \in J$, there is a $g \in \mathrm{Har}(\Omega)$, such that, for each $j \in J$,

$$\|f - g\|_1(x) < \epsilon_j, \quad x \in \varphi_j(F \cap \bar{B}_j).$$

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It is easy to see that, if f admits C^1-Carleman approximation with respect to one such cover, then it admits C^1-Carleman approximation with respect to any such cover. We are thus justified in saying that f admits C^1-Carleman approximation by functions in $\mathrm{Har}(\Omega)$, without reference to any particular cover of F. The above lemma then asserts that f restricted to Γ admits C^1-Carleman approximation by functions in $\mathrm{Har}(\Omega)$.

PROOF OF LEMMA. We may construct an exhaustion, $\Omega = \bigcup \Omega_i$, of Ω by bounded open sets Ω_i, $\overline{\Omega}_i \subset \Omega_{i+1}$, $i = 1, 2, \ldots$, which is compatible with Γ in the sense that, for $i = 1, 2, \ldots, \Omega^* \setminus (\overline{\Omega}_i \cup \Gamma)$ is connected.

Let $\{\delta_k\}$ be a sequence of positive numbers. Set $\Omega_0 = \emptyset$ and $\delta_0 = 0$. We shall construct inductively a sequence $g_k \in \mathrm{Har}(\Omega)$, such that, for $k = 1, 2, \ldots$ and $j \in J$,

1) $\|g_k\|_1 < \delta_k$ on $\varphi(\overline{B}_j \cap \overline{\Omega}_{k+1})$
2) $\|f - g_1 - \cdots - g_k\|_1 < \delta_k$ on $\varphi(\overline{B}_j \cap \Gamma \cap [\overline{\Omega}_{k+1} \setminus \Omega_k])$
3) $\|f - g_1 - \cdots - g_k\|_1 < 2\delta_{k-1} + \delta_k$ on $\varphi(\overline{B}_j \cap \Gamma \cap [\overline{\Omega}_k \setminus \Omega_{k-1}])$.

Indeed, set $X_k = \Gamma \cap \overline{\Omega}_k$. By the harmonic Runge lemma, there is a $g_1 \in \mathrm{Har}(\Omega)$ such that $\|f - g_1\|_1 < \delta_1$ on X_2. Let h_2 be a function C^1 on $\overline{\Omega}_1 \cup X_3$, with: $\|h_2\|_1 < \delta_1$ on each $\varphi_j(\overline{B}_j \cap [\overline{\Omega}_1 \cup X_3])$; $h_2 = 0$ on a neighbourhood of $\overline{\Omega}_1$, and $h_2 = f - g_1$ on $X_3 \setminus \Omega_2$. By the saturn lemma, there exists a $g_2 \in \mathrm{Har}(\Omega)$ such that $\|h_2 - g_2\|_1 < \delta_2$ on each $\varphi_j(\overline{B}_j \cap [\overline{\Omega}_1 \cup X_3])$. Then, 1), 2) and 3) are satisfied for $k = 1$.

Suppose now that 1), 2) and 3) have been established for $k = 1, \ldots, m-1$. We establish 1), 2) and 3) for $k = m$. Let h_m be: C^1 on $\overline{\Omega}_{m-1} \cup X_{m+1}$; $\|h_m\|_1 < \delta_{m-1}$ on each $\varphi(\overline{B}_j \cap [\Omega_{m-1} \cup X_{m+1}])$; and equal to $f - g_1 - \cdots - g_{m-1}$ on $X_{m+1} \setminus \Omega_m$. By the Saturn lemma, there exists a $g_m \in \mathrm{Har}(\Omega)$ such that $\|h_m - g_m\|_1 < \delta_m$ on each $\varphi_j(\overline{B}_j \cap [\overline{\Omega}_{m-1} \cup X_{m+1}])$. Then, g_m satisfies 1), 2) and 3) for $k = m$. By induction, then, we may construct a sequence satisfying 1), 2) and 3).

We may choose $\{\delta_k\}$ such that $\sum \delta_k < \infty$. Thus, by 1), $g = \sum g_k$ is in $\mathrm{Har}(\Omega)$. Now, fix f and a sequence $\{\epsilon_k\}_{j \in J}$. If the $\{\delta_k\}$ are chosen to decrease sufficiently rapidly, it follows from 2) and 3) that, for all j, we have (3). This completes the proof of the lemma.

PROOF OF THEOREM 1 ON RIEMANNIAN MANIFOLDS. The proof now proceeds in complete analogy to the proof of Theorem 1, which we presented on Riemann surfaces. We have only to replace the Cauchy integral by the Poisson-Green integral and Carleman approximation by C^1-Carleman approximation.

As in the proof of Theorem 1 on Riemann surfaces, we may also prove a harmonic analog of Theorem 3. However, this is of less interest than Theorem 3, for, in the harmonic case, a much better Runge theorem is known [2] (see [9] also).

4. **Compact Riemann surfaces.** Let $C(e_-, e_+)$ and $C(E_-, E_+)$ be two condensers (*cf.* Introduction) on a manifold Ω. We say that the condenser $C(e_-, e_+)$ *contains* the condenser $C(E_-, E_+)$ if $e_- \subset E_-$ and $e_+ \subset E_+$.

Now suppose that Ω is a Riemann surface or a Riemannian manifold and suppose $C(e_-, e_+)$ and $C(E_-, E_+)$ are two condensers on Ω such that $C(e_-, e_+)$ contains $C(E_-, E_+)$. If $C(e_-, e_+)$ admits linking, then $C(E_-, E_+)$ also admits linking. Indeed, let $f \in H(E_-, E_+)$. Then by the weak linking lemma applied to the noncompact manifold $\Omega \setminus e_-$, we may

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write $f = f_- + f_+$, where $f_- \in H\big((\Omega \setminus e_-) \setminus E_-\big) = H(\Omega \setminus E_-)$ and $f_+ \in H\big((\Omega \setminus e_-) \setminus E_+\big)$. Again applying the weak linking lemma to the manifold $\Omega \setminus e_+$, we may write $f_+ = g_- + g_+$, where $g_- \in H\big((\Omega \setminus e_+) \setminus e_-\big) = H(e_-, e_+)$ and $g_+ \in H\big((\Omega \setminus e_+) \setminus E_+\big) = H(\Omega \setminus E_+)$. Now, since, by hypothesis, $C(e_-, e_+)$ admits linking, we have $g_- = h_- + h_+$, with $h_\pm \in H(\Omega \setminus e_\pm)$. Thus,

$$f = (f_- + h_-) + (h_+ + g_+),$$

with $f_- + h_- \in H(\Omega \setminus E_-)$ and $h_+ + g_+ \in H(\Omega \setminus E_+)$. Hence, $C(E_-, E_+)$ admits linking as claimed.

Because of the preceding remarks, it is particularly interesting to consider the largest possible condensers, namely the two-point condensers. These are condensers $C(e_-, e_+)$ for which $e_\pm$ are singletons $\{p_\pm\}$. For simplicity, we denote such a two-point condenser by $C(p_-, p_+)$.

On the Riemann sphere $\Omega = \overline{\mathbf{C}}$, any two-point condenser admits holomorphic linking. This follows immediately from the Laurent expansion (after a Möbius transformation taking one of the two points to the point at infinity). Since any condenser is contained in a two-point condenser, it follows that the weak linking lemma holds, not only for *open* Riemann surfaces, but also on the Riemann sphere, which is *compact*.

We shall show that the Riemann sphere is the *only* compact Riemann surface for which the weak holomorphic linking lemma holds. In fact, we shall show that if Ω is any compact Riemann surface other than the Riemann sphere, then there is a two-point condenser on Ω which does not admit linking.

Suppose, then, that Ω is a compact Riemann surface other than $\overline{\mathbf{C}}$. Then, the genus g of Ω is at least 1. We may choose two distinct points $p_1, p_2 \in \Omega$ which are *not* Weierstrass points [20, p. 274]. Consider the divisor $D = -gp_1 - gp_2$. By Riemann's inequality (see [22, p. 266] and [16, p. 197], $\dim D \geq 2g - g + 1 \geq 2$. Hence, there are at least two linearly independent meromorphic functions whose only singularities are poles at p_1 and p_2 of order at most g. At most one of these can be a constant. Let ψ be such a nonconstant function. Since p_1 and p_2 are not Weierstrass points, ψ has a pole at both p_1 and p_2. Suppose we have a linking of ψ:

$$\psi = f_1 + f_2, \quad f_j \in \mathrm{Hol}(\Omega \setminus \{p_j\}).$$

Note that f_j extends to be meromorphic on Ω. Then, the only singularity of f_j is a pole at p_j of the same order as the pole of ψ at p_j. Since this order is at most g, this contradicts the choice of p_j as points which are not Weierstrass points. Thus, the above linking is impossible. We have shown that every compact Riemann surface other than the Riemann sphere has a two-point condenser which does not admit holomorphic linking, whereas on the Riemann sphere, every condenser admits holomorphic linking.

Let us consider, for a moment, the analogous question of linking *meromorphic* functions. We shall say that the weak holomorphic (meromorphic) linking lemma holds for meromorphic functions on a Riemann surface Ω if, for every condenser $C(e_-, e_+)$ on Ω and every function f *meromorphic* on $C(e_-, e_+)$, we can write $f = f_- + f_+$, with $f_\pm$

holomorphic (meromorphic) on $\Omega \setminus e_\pm$. If we form a new condenser by adding the poles of f to one of the sides $e_\pm$ of the condenser $C(e_-, e_+)$, it is clear that the weak *holomorphic* linking lemma holds for meromorphic functions on Ω if and only if it holds for holomorphic functions on Ω and the latter problem is precisely the one we have been investigating. Thus, this linking holds on open Riemann surfaces and the Riemann sphere, but fails on all other compact surfaces. On the other hand, the weak *meromorphic* linking lemma for meromorphic functions is a different issue and, since compact Riemann surfaces are more hospitable to meromorphic functions than to holomorphic ones, *meromorphic* linking might be more successful on compact Riemann surfaces than *holomorphic* linking. We shall not, however, pursue this question.

The situation for harmonic linking on the Riemann sphere is different from that for holomorphic linking. In fact, the two-point condenser $C(0,\infty)$ does *not* admit harmonic linking. Indeed, suppose it were possible to write $\log|z| = u_0(z) + u_\infty(z)$, with $u_0 \in \mathrm{Har}(\overline{\mathbf{C}} \setminus \{0\})$ and $u_\infty \in \mathrm{Har}(\overline{\mathbf{C}} \setminus \{\infty\})$. Then, since both u_0 and u_∞ have single-valued harmonic conjugates, so would $\log|z|$, which is absurd.

5**. Isolated singularities.** In this section, we look more closely at the classical case of isolated singularities. In the (punctured) neighbourhood of an isolated (possibly artificial) singularity z, a holomorphic function f of a single complex variable ζ has a Laurent expansion

$$f(\zeta) = \sum_{-\infty}^{+\infty} a_j(\zeta - z)^j.$$

For any integer J, we shall call a series of the form

$$\sum_{-\infty}^{J} a_j(\zeta - z)^j$$

a left tail at the point z. If a left tail is convergent in some deleted neighbourhood of z then we say that it is an *admissible* left tale. If the coefficients of a left tail at z are the Laurent coefficients of a function f holomorphic in a deleted neighbourhood of z, then we say that the left tail is the left tail of f at the point z. Clearly such a left tail is admissible. The following is a more precise version of the Mittag-Leffler Theorem.

THEOREM 4. *Let Z be a discrete set in an open subset Ω of the complex plane. For each $z \in Z$ let t_z be any admissible left tail at z. Then there exists a function f holomorphic on Ω except for isolated (possible artificial) singularities at the points of Z such that for each $z \in Z$, t_z is a left tail of f at z.*

If each tail t_z is of the form

$$\sum_{-n_z}^{-1} a_j(\zeta - z)^j,$$

where n_z is some positive integer, then we have the usual Mittag-Leffler theorem. If each tail t_z is of the form

$$\sum_{0}^{j_z} a_j(\zeta - z)^j,$$

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where j_z is a non-negative integer, then we obtain a holomorphic function whose Taylor polynomial of degree j_z is prescribed at each $z \in Z$. Thus we interpolate not only the values of the function f at the points $z \in Z$ but also the values of finitely many (depending on z) derivatives. Such interpolations are well known (see [19]), however, the possibility of also specifying *essential* singularities for holomorphic functions as above is not so generally known. Arakelian stated such theorems in private discussions with the author during the late 70's. An analogous result (Theorem 6 below) was proved for harmonic functions by Goldstein, Gauthier and Ow [11]. The reader can reconstruct the proof of Theorem 4 by looking at that of Theorem 7.

By a singularity function at a point $x \in \mathbf{R}^n$ we mean a function s_x which is harmonic in a deleted neighbourhood of x. If a singularity function at x can be extended harmonically to the point x, then we say that the point x is an artificial singularity of s_x. In this case, we also say by anticipation that s_x is harmonic at x.

THEOREM 5 ([11]). *Let X be a discrete set in an open subset Ω of $\mathbf{R}^n$ and for each $x \in X$ let s_x be any singularity function at x. Then, there exists a function $u \in \mathrm{Har}(\Omega \setminus X)$ such that at each $x \in X$ the function $u - s_x$ is harmonic.*

Of course, Theorem 5 is a particular case of Theorem 1. In the remaining pages, we shall prove a harmonic analog of Theorem 4, which does not follow directly from Theorem 1 and which improves Theorem 5.

Recall that if u is a C^∞-function in a neighbourhood of a point $x \in \mathbf{R}^n$, then, for each $j = 0, 1, \ldots,$ the Taylor polynomial of order j at x is the polynomial

$$p(y) = \sum_{\alpha|\leq j} \frac{1}{\alpha!}(\partial^\alpha u)(x) \cdot (y - x)^\alpha.$$

where ∂^α denotes as usual the partial differentiation operator with multi-index α.

THEOREM 6. *Let X be a discrete set in an open subset Ω of $\mathbf{R}^n$ and for each $x \in X$ let p_x be any harmonic polynomial whose degree we denote by j_x. Then there exists a function $u \in \mathrm{Har}(\Omega)$ such that at each $x \in X$ its Taylor polynomial of order j_x is p_x.*

The following Runge type lemma for harmonic functions is due to Walsh (see [9] and [12]).

LEMMA 11 (RUNGE TYPE). *Let $W \subset \Omega$ be open subsets of $\mathbf{R}^n$ with $\Omega^* \setminus W$ connected. Then, $\mathrm{Har}(\Omega)$ is dense in $\mathrm{Har}(W)$.*

The following assures us that if we can approximate, then we may simultaneously approximate *and* interpolate.

LEMMA 12 (WALSH TYPE [7]). *Suppose S is a dense subspace of a locally convex linear topological space H. Then, for every $h \in H$, neighbourhood U of zero in H, and continuous linear functionals $T_1, \ldots, T_m$ on H, there is an $s \in S$ such that $s \in h + U$ and $T_j(s) = T_j(h)$, $j = 1, \ldots, m$.*

PROOF OF THEOREM 6. Let $\{\Omega_k\}_{k=0}^{\infty}$ be a regular exhaustion of Ω by relatively compact open sets. We may assume that $X \cap \Omega_0 = \emptyset$. We may also assume that

$$X \cap (\overline{\Omega}_{2k+1} \setminus \Omega_{2k}) = \emptyset$$

for $k = 0, 1, \ldots$. Thus, for each $x \in X$, there is some $k = k_x$ such that

$$x \in \Omega_{2k} \setminus \overline{\Omega}_{2k-1}.$$

Hence, for each $x \in X$ we may choose a radius r_x such that the closed balls $\overline{B}_x$ of center x and radius r_x are mutually disjoint and $\overline{B}_x \subset \Omega_{2k_x} \setminus \overline{\Omega}_{2k_{x-1}}$. Set

$$W_k = \Omega_{2k-1} \cup \bigcup \{B_x : k_x = k\}$$

and

$$K_k = \overline{\Omega}_{2k-2} \cup (X \cap \Omega_{2k}).$$

Let ϵ_k, $k = 1, 2, \ldots$ be any sequence of positive numbers whose sum converges.

Define a function h_1 on W_1 by setting h_1 equal to any harmonic function on Ω_1 and by setting $h_1 = p_x$ on B_x for each $x \in \Omega_2 \setminus \Omega_1$. By the harmonic Runge lemma, the harmonic functions on Ω_3 are dense in the harmonic functions on W_1. By the Walsh lemma, then, there is a function u_1 harmonic on Ω_3 such that

$$|u_1 - h_1| < \epsilon_1 \text{ on } K_1$$

and for each $x \in X \cap \Omega_2$,

$$\partial^\alpha u_1(x) = \partial^\alpha h_1(x), \quad \text{for } |\alpha| \le j_x,$$

It follows that at each $x \in X \cap \Omega_2$ the Taylor polynomial of order j_x of the function u_1 is p_x.

Next, define a function h_2 on W_2 by setting $h_2 = h_1$ on Ω_3 and by setting $h_2 = p_x$ on B_x for each $x \in \Omega_4 \setminus \Omega_3$. By the harmonic Runge lemma, the harmonic functions on Ω_5 are dense in the harmonic functions on W_2. By the Walsh lemma, then there is a function u_2 harmonic on Ω_5 such that

$$|u_2 - h_2| < \epsilon_2 \quad \text{on } K_2$$

and for each $x \in X \cap \Omega_4$,

$$\partial^\alpha u_1(x) = \partial^\alpha h_1(x), \quad \text{for } |\alpha| \le j_x,$$

It follows that at each $x \in X \cap \Omega_4$ the Taylor polynomial of order j_x of the function u_2 is p_x.

Proceeding by induction, we construct a sequence u_k of functions with the following properties:

1) u_k is harmonic on Ω_{2k+1};

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2) $|u_k - u_{k-1}| < \epsilon_k$ on K_k, for $k = 2, 3, \ldots$;

3) at each $x \in X \cap \Omega_{2k}$, the Taylor polynomial of order j_x of the function u_2 is p_x.

Since the series $\sum \epsilon_k$ converges, the sequence $\{u_k\}$ is uniformly Cauchy on compact subsets of Ω and hence converges to a function u which has the required properties. This completes the proof.

We note here one of the differences between the holomorphic situation and the harmonic one. Any complex polynomial is holomorphic, whereas not every real polynomial is harmonic. Thus, we can specify any (complex) polynomials as the Taylor polynomials of a holomorphic function at a discrete set of points. However, it can be shown that the Taylor polynomial of a harmonic function must also be harmonic. Thus, the restriction to harmonic polynomials in the preceding theorem is a necessary one.

COROLLARY 1. *Let X be a discrete set in an open subset Ω of $\mathbf{R}^n$ and, for each $x \in X$, let y_x be any real number. Then there exists a function $u \in \mathrm{Har}(\Omega)$ such that $u(x) = y_x$ for each $x \in X$.*

THEOREM 7. *Let X be a discrete set in an open subset Ω of $\mathbf{R}^n$. For each $x \in X$, let s_x be any singularity function at x and let p_x be any harmonic polynomial whose degree we denote by j_x. Then there exists a function $u \in \mathrm{Har}(\Omega \setminus X)$ such that at each $x \in X$ the function $u - s_x$ is harmonic and its Taylor polynomial of order j_x is p_x.*

PROOF. By Theorem 5, there is a function $u_1 \in \mathrm{Har}(\Omega \setminus X)$ such that at each $x \in X$ the function $u_1 - s_x$ is harmonic. By Theorem 6, there is a function $u_2 \in \mathrm{Har}(\Omega)$ such that at each $x \in X$ its Taylor polynomial of order j_x is the same as that of $p_x + s_x - u_1$. The function $u = u_1 + u_2$ has the required properties.

In (deleted) neighbourhoods of isolated singularities, harmonic functions have Laurent-type expansions resembling those of holomorphic functions and Theorem 7 can be formulated in the same way as Theorem 4 was for holomorphic functions. One can prove Theorem 4 in the same way as Theorem 7, replacing the harmonic Runge theorem in the proof by the classical complex Runge theorem. In the special case where Ω is a simply connected domain in the plane, Theorem 6 can also be deduced from Theorem 4. However, for multiply connected plane domains, it is not clear that one could deduce the harmonic case from the holomorphic one since a harmonic function in a multiply connected domain need not be the real part of a (global) holomorphic function.

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