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BOUNDARIES FOR REAL BANACH ALGEBRAS

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Introduction. Let A be a commutative real Banach algebra with unit, and M_A its maximal ideal space. The existence of the Silov boundary S_A for A was established in [**5**] by resorting to the complexification of A. We give here an intrinsic proof of this result which exhibits the close connection between the absolute values and the real parts of 'functions' in A (Theorem 1.3).

For a subset B of A, we define the Silov boundary for B relative to A, and use it, together with the method of complexification, to extend to the real case some recent results in [**6**] for complex function algebras. These determine M_B and S_B in terms of M_A and S_A if B is a closed subalgebra of A and contains an ideal J of A such that $\text{hull}_A\, J$ contains no non-empty perfect subset (Theorems 3.1 and 3.4). They also extend a result in [**5**] where B is a particular type of real subalgebra of a complex function algebra A (Corollary 3.5 and Example 3.6).

1. Choquet sets and Silov boundaries. Let A be a commutative real Banach algebra with 1, and M_A the set of all maximal ideals of A. For each f in A, $|\hat{f}|$ and $\text{Re}\,\hat{f}$ are well defined functions on M_A. (See, e.g., [**1**].) Let $|\hat{A}| = \{|\hat{f}| : f \text{ in } A\}$ and $\text{Re}\,\hat{A} = \{\text{Re}\,\hat{f} : f \text{ in } A\}$.

PROPOSITION 1.1. *The weak $|\hat{A}|$ topology on M_A is the same as the weak* Re $\hat{A}$ *topology on M_A, and it makes M_A a compact Hausdorff space.*

Proof. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be the weak $|\hat{A}|$ and the weak $\text{Re}\,\hat{A}$ topologies on M_A. Let L_A denote the set of all real linear maps of A into the complex numbers, and $\mathscr{T}$ the weak A topology on it. If $T : \phi_A \rightarrow M_A$, by $T(k) = k^{-1}(0)$, where ϕ_A is the closed set $\{k \text{ in } L_A : k \text{ multiplicative}, k(1) = 1\}$, then ϕ_A is onto M_A, and T is continuous if L_A and M_A are given the topologies $\mathscr{T}$ and $\mathscr{T}_1$ respectively. Since the closed unit ball in L_A is compact by the Banach-Alaoglu theorem, $(M_A, \mathscr{T}_1)$ is compact. We next show that $(M_A, \mathscr{T}_2)$ is Hausdorff. This follows since Re $\hat{A}$ separates points of M_A: Let $y_1 \neq y_2$ be in M_A, and f in A which belongs to y_1 but not to y_2. If k_1 and k_2 are in L_A such that $T(k_1) = y_1$ and $T(k_2) = y_2$, then $k_1(f) = 0$, and $k_2(f) = a + ib$, for some real numbers a and b, which are not both zero. Then $\text{Re}\,\hat{f}(y_1) = 0$, $\text{Re}\,\hat{f}(y_2) = a$, $\text{Re}(f^2)\hat{}(y_1) = 0$, and $\text{Re}(f^2)\hat{}(y_2) = a^2 - b^2$. Hence either $\text{Re}\,\hat{f}$ or $\text{Re}(f^2)\hat{}$ separates y_1 and y_2.

Now, the identity map from $(M, \mathscr{T}_1)$ to $(M, \mathscr{T}_2)$ is continuous, since for each f in A, $\text{Re}\,\hat{f} = \log|(\exp f)\hat{}|$. All is proven.

Definition 1.2. A subset S of M_A is called a *Choquet set for A* if each element of

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Re $\hat{A}$ assumes its maximum on S. S is called a *boundary for A* if each element of $|\hat{A}|$ assumes its maximum on S.

THEOREM 1.3.

(i) *Every boundary for A is a Choquet set for A.*

(ii) *Every closed Choquet set for A is a boundary for A.*

(iii) *There exists a (unique) smallest closed Choquet set for A, and it is the smallest closed boundary for A.*

Proof. (i) follows since, for each f in A, $\operatorname{Re}\hat{f} = \log|(\exp f)\hat{}\,|$.

Let S be a closed Choquet set for A. If it were not a boundary for A, there exists f in A, $\epsilon < 1$, and y in M_A such that $|\hat{f}| \leqq \epsilon$ on S, but $|\hat{f}|(y) = 1$. Since, for each positive integer n, $|\operatorname{Re}(f^n)\hat{}\,| \leqq |(f^n)\hat{}\,| \leqq \epsilon^n$ on S, and S is a Choquet set for A, $|\operatorname{Re}(f^n)\hat{}\,| \leqq \epsilon^n$ on M_A, and in particular at y. If k is a real homomorphism of A with null space y, then, since $|\hat{f}|(y) = 1$, $k(f) = \exp(ia)$, for some real number a. Thus, $|R(f^n)\hat{}(y)| = |\cos na| \leqq \epsilon^n$, for each positive integer n. But, as n tends to infinity, ϵ^n tends zero while $\cos na$ does not, a contradiction. It follows that S is a boundary for A.

Finally, it follows from (i) and (ii) that if a smallest closed Choquet set for A exists, it must also be the smallest closed boundary for A. That a smallest closed Choquet set for A exists follows from these results of Choquet: Let Y be a compact Hausdorff topological space, and H a linear subspace of the space of all real valued continuous functions on Y. Let $\operatorname{Ch}(H)$ be the set of all points of Y which admit unique representing measures with respect to H. Then, if H separates points of Y, each h in H attains its maximum on $\operatorname{Ch}(H)$ and the closure in Y of $\operatorname{Ch}(H)$ is the smallest closed subset of Y on which each h in H attains its maximum [**2**, Corollary 29.6 and Proposition 29.8].

Definition 1.4. The smallest closed boundary for A is called the *Silov boundary for A*. We shall denote it by S_A.

If B is a subset of A, and if there exists a (unique) closed subset of S_A such that each element of $|\hat{B}| = \{|\hat{f}| : f \text{ in } B\}$ attains its maximum on it, then such a set will be called the *Silov boundary for B relative to A*, and denoted by ${}_AS_B$.

PROPOSITION 1.5. *Let B be a closed subalgebra of A containing* 1, M_B *its maximal ideal space, and* $r : M_A \to M_B$, *the restriction map. Then,*

(i) S_B *is contained in* $r(S_A)$.

(ii) *If r is one to one on* S_A, *then* ${}_AS_B$ *exists and equals* $S_A \cap r^{-1}(S_B)$.

Proof. (i). That S_B is contained in $r(M_A)$; i.e., every maximal ideal of B which is in S_B can be extended to a maximal ideal of A follows as in the case where A and B are complex algebras [**4**, p. 78–80]. Hence $r(M_A)$ is a boundary for B. This together with the fact that S_A is a boundary for A shows that $r(S_A)$ is a boundary for B.

(ii). Since S_B is contained in $r(S_A)$, each $|\hat{f}|$, f in B, attains its maximum on $S_A \cap r^{-1}(S_B)$. On the other hand, if F is a closed subset of S on which each $|\hat{f}|$, f

in B, attains its maximum, then $r(F)$ is a boundary for B, so that S_B is contained in $r(F)$. But r is one to one on S_A, hence $S_A \cap r^{-1}(S_B)$ is contained in F. This shows that ${}_AS_B$ exists and equals $S_A \cap r^{-1}(S_B)$.

Remark 1.6. It is well known that if A is a complex subspace of $C(Y)$, the set of all complex-valued continuous functions on a compact Hausdorff space Y, such that A contains constants and separates points, then there exists a (unique) smallest closed subset of Y on which every f in A attains its maximum modulus. If A is only a real subspace of $C(Y)$, then the proof of Theorem 1.3 shows that if A is a ring such that f in A implies $\exp f$ also in A, and if Re A separates points of Y then the same conclusion holds.

2. Complexifications. Let A be a commutative real Banach algebra with unit 1. Under the natural operations

$$\text{cx } A \equiv \{1 \otimes f + i \otimes g : f, g \text{ in } A\}$$

becomes a commutative complex algebra with unit $1 \otimes 1$, and there exists a norm on cx A for which cx A becomes a Banach algebra, and the natural injection of A into cx A is an isometry. If $\text{cx}^* : M_{\text{cx } A} \to M_A$ is the restriction map, then cx* is surjective, and if $M_{\text{cx } A}$ is given the Gelfand topology, it is continuous as well as open [**1**, 3.3 and 3.9]. Let $\sigma : \text{cx } A \to \text{cx } A$, by

$$\sigma(1 \otimes f + i \otimes g) = 1 \otimes f - i \otimes g, \text{ for every } 1 \otimes f + i \otimes g \text{ in cx } A,$$

and $\tau : M_{\text{cx } A} \to M_{\text{cx } A}$, by

$$\tau(x) = \{h : \sigma(h) \text{ in } x\}, \text{ for every } x \text{ in } M_{\text{cx } A}.$$

Then $\text{cx}^* \circ \tau = \text{cx}^*$, and if $S_{\text{cx } A}$ is the Silov boundary for cx A, $\tau(S_{\text{cx } A}) = S_{\text{cx } A}$.

PROPOSITION 2.1. $\text{cx}^*(S_{\text{cx } A}) = S_A$, *and* $(\text{cx}^*)^{-1}(S_A) = S_{\text{cx } A}$.

Proof. Since $\text{cx}^*(S_{\text{cx } A})$ is compact, it is closed and is clearly a boundary for A, so it contains S_A. Conversely, we show that $S_{\text{cx } A}$ is contained in $(\text{cx}^*)^{-1}(S_A) = F$, say. For this it is enough to prove that F is a boundary for cx A. If it were not a boundary, there exists h in cx A, $\epsilon < 1$, and x in $M_{\text{cx } A}$ such that $|\hat{h}| \leqq \epsilon < 1$ on F, but $\hat{h}(x) = 1$. Let $c = \sigma(h)\hat{}(x)$. For each positive integer n,

$$|(\hat{h})^n + (\sigma(h)\hat{})^n| \leqq |(\hat{h})^n| + |(\sigma(h)\hat{})^n| \leqq 2\epsilon^n$$

on F. Since $h + \sigma(h)$ 'belongs' to A, this inequality is valid on all of $M_{\text{cx } A}$, in particular at x. This gives $|1 + c^n| \leqq 2\epsilon^n$, for each positive integer n. But, as n tends to infinity, $2\epsilon^n$ tends to zero, while $1 + c^n$ does not, a contradiction. Thus, $S_{\text{cx } A}$ is contained in $(\text{cx}^*)^{-1}(S_A)$.

Note. Compare the above result with Proposition 1.0 of [**5**]. The proof given there uses the trace map taking h to $h + \sigma(h)$, and the norm map taking h to $h \cdot \sigma(h)$, while the above proof uses only the trace map. The proof in [**1**, 3.16] is incorrect, for it uses the inequality $|u(x)| \leqq |u(x) + iv(x)|$, for complex functions u and v.

As an application of the above proposition we prove the following result which will be used in § 3.

PROPOSITION 2.2. *If S_A contains no non-empty perfect subset, then $S_A = M_A$.*

Proof. Since $\text{cx}^* : S_{\text{cx } A} \to S_A$ is at most two to one, it is clear that if K is a perfect subset of $S_{\text{cx } A}$ then $\text{cx}^*(K)$ is a perfect subset of S_A. Since S_A contains no non-empty perfect subset, neither does $S_{\text{cx } A}$. Now $\text{cx } A$ is a complex commutative Banach algebra with a unit, and if $S_{\text{cx } A} = M_{\text{cx } A}$, then

$$S_A = \text{cx}^*(S_{\text{cx } A}) = \text{cx}^*(M_{\text{cx } A}) = M_A.$$

Thus, it is enough to prove the proposition when A is a complex commutative Banach algebra with 1. But this is given in [**8**, p. 107].

Let now B be a real subalgebra A containing 1, and

$$\text{cx } B = \{1 \otimes f + i \otimes g : f \text{ and } g \text{ in } B\}.$$

Then $\text{cx } B$ is a complex subalgebra of $\text{cx } A$ containing $1 \otimes 1$.

PROPOSITION 2.3. *Let $r : M_A \to M_B$ and $r_{\text{cx}} : M_{\text{cx } A} \to M_{\text{cx } B}$ be the restriction maps. Then*

(i) $\text{cx}^* \circ r_{\text{cx}} = r \circ \text{cx}^*$, *and* $\tau_B \circ r_{\text{cx}} = r_{\text{cx}} \circ \tau_A$, *where τ_A and τ_B are the involutions on $M_{\text{cx } A}$ and $M_{\text{cx } B}$ respectively.*

(ii) *r_{cx} is surjective if and only if r is surjective.*

(iii) *If r_{cx} is injective, then r is injective. If r is injective, then $r_{\text{cx}}(x_1) = r_{\text{cx}}(x_2)$ implies $x_2 = x_1$ or $x_2 = \tau_A(x_1)$.*

(iv) $\text{cx } B$ *is closed in* $\text{cx } A$ *if and only if B is closed in A. In that case, $S_{\text{cx } B} = r(S_{\text{cx } A})$ if and only if $S_B = r(S_A)$.*

Proof. (i) For M in $M_{\text{cx } A}$,

$$\text{cx}^* \circ r_{\text{cx}}(M) = (M \cap \text{cx } B) \cap B = (M \cap A) \cap B = r \circ \text{cx}^*(M),$$
$$\tau_B \circ r_{\text{cx}}(M) = \{1 \otimes f + i \otimes g \text{ in } M \text{ with } f \text{ and } g \text{ in } B\} = r_{\text{cx}} \circ \tau_A(M).$$

(ii) and the first part of (iii) are clear. If r is injective, and $r_{\text{cx}}(x_1) = r_{\text{cx}}(x_2)$, then $\tau_B \circ r_{\text{cx}}(x_j) = r \circ \tau_A(x_j)$, for $j = 1, 2$. Hence, $\tau_A(x_1) = \tau_A(x_2)$, so that $x_2 = x_1$ or $x_2 = \tau_A(x_1)$.

If $\text{cx } B$ is closed in $\text{cx } A$, then since the injection of A into $\text{cx } A$ is an isometry, B is closed in A. Conversely, let B be closed in A, and $1 \otimes f_n + i \otimes g_n$ tend to $1 \otimes f + i \otimes g$, where f_n and g_n, for each n, are in B, and f and g are in A, then since σ is continuous $1 \otimes f_n - i \otimes g_n$ tends to $1 \otimes f - i \otimes g$. This shows that f_n tends to f and g_n tends to g, so that f and g are in B. Thus, $\text{cx } B$ is closed in $\text{cx } A$. The last statement follows from Proposition 2.1 and the definition of a Silov boundary.

3. Ideals and subalgebras. Throughout this section, unless otherwise stated, B will be a closed subalgebra of a commutative real Banach algebra A with unit 1 in B, and $r : M_A \to M_B$ the restriction map. We find conditions under which $r(M_A) = M_B$, and $r(S_A) = S_B$. If A and B are complex function algebras, this was done by Lund [**6**, 2.1 and 2.3]. We shall use many of his arguments in conjunction with the results in § 1 and § 2 to treat the real case.

Since every y in S_B can be extended to an x in M_A (Proposition 1.5), $M_B = S_B$ implies $r(M_A) = M_B$. More generally if J is a closed ideal of A contained in B, $M_{B/J} = S_{B/J}$ implies $r(M_A) = M_B$. The proof of the following theorem is modelled after this observation. If J is an ideal of A, we let

$$\text{hull}_A\, J = \{y \text{ in } M_A : y \text{ contains } J\}.$$

THEOREM 3.1. *Let J be an ideal of A contained in B such that either*

(a) $\text{hull}_A\, J$ *contains no non-empty perfect subset, and r restricted to* $\text{hull}_A\, J$ *is one to one, or*

(b) $\text{hull}_A\, J$ *is at most countable.*

Then $r(M_A) = M_B$.

Proof. Since $\text{hull}_A\, J = \text{hull}_A\, \bar{J}$, and B is closed, we can assume without loss of generality that J itself is closed. The cannonical map $c_A : A \rightarrow A/J$ induces a homeomorphism $c_A{}^* : M_{A/J} \rightarrow \text{hull}_A\, J$, and similarly for $M_{B/J}$ and $\text{hull}_B\, J$, by considering $c_B{}^*$. The injection map from B/J to A/J induces the restriction map $r' : M_{A/J} \rightarrow M_{B/J}$. Moreover, $r = c_B{}^* \circ r' \circ (c_A{}^*)^{-1}$ on $\text{hull}_A\, J$, and $r' = (c_B{}^*)^{-1} \circ r \circ c_A{}^*$. Our assumption implies that $M_{A/J}$ and hence $r'(M_{A/J})$ contains no non-empty perfect subset. But then $S_{B/J}$, which is contained in $r'(A/J)$ by (i) of Proposition 1.5, cannot contain a non-empty perfect subset. Now, Proposition 2.2 gives $M_{B/J} = S_{B/J}$.

Now, again by (i) of Proposition 1.5, r' is surjective, so that $r(\text{hull}_A\, J) = \text{hull}_B\, J$. On the other hand, if z belongs to M_B but not to $\text{hull}_B\, J$, then the ideal generated by z in A, say I, is proper: Let f belong to J, but not to z. If

$$1 = a_1 f_1 + \ldots + a_n f_n$$

with a_j in A and f_j in z, for $1 \leqq j \leqq n$, then

$$f = (fa_1)f_1 + \ldots + (fa_n)f_n.$$

Since J is an ideal of A, fa_j belongs to J, and hence to B. Since z is an ideal of B, $(fa_j)f_j$ belongs to z, for $1 \leqq j \leqq n$. This implies that f is in z, a contradiction. Thus, I is a proper ideal of A. Then I is contained in some y in M_A, and $r(y) = z$. We thus have $r(M_A) = M_B$.

We now turn our attention to the Silov boundaries for A and B. First we state a result involving S_A and $\text{hull}_A\, J$. Although its proof is the same as in the complex case [**8**, p. 44], we present it here for the sake of completeness.

LEMMA 3.2. *Let A be a commutative real Banach algebra with* 1, *and J an ideal of A. If B is a subset of A which contains J and such that ${}_AS_B$ exists, then* $S_A - \text{hull}_A\, J$ *is contained in* ${}_AS_B$.

Proof. Let y be in $S_A - \mathrm{hull}_A J$, and U a neighbourhood of y in M_A. Since $\mathrm{hull}_A J$ is closed in M_A, we can assume without loss of generality that U does not intersect $\mathrm{hull}_A J$. Let f be in A such that $|\hat{f}|(y') = 1$ for some y' in U, $|\hat{f}| \leqq 1$ on M_A and $|\hat{f}| \leqq 1/2$ on $M_A - U$. Since y' does not belong to $\mathrm{hull}_A J$, there exists g in J such that $|\hat{g}|(y') = 1$. Then $g_n = gf^n$ belongs to J and hence to B, for each n, and for large enough n, $|\hat{g}_n|$ assumes its maximum only on U. Thus, y belongs to ${}_AS_B$.

Before we state our final theorem which gives sufficient conditions for $r(S_A) = S_B$, we prove another lemma which seems interesting in itself.

LEMMA 3.3. *Let the map r be one to one and onto. If y in S_A is isolated in S_A, then $r(y)$ belongs to S_B.*

Proof. Let y in S_A be isolated in S_A, and let $\mathrm{cx}^*(x) = y$, for some x in $S_{\mathrm{cx}\,A}$. Then it is clear that x is isolated in $S_{\mathrm{cx}\,A}$. We show that there exists an open as well as closed subset E of $M_{\mathrm{cx}\,A}$ such that $E \cap S_{\mathrm{cx}\,A} = \{x\}$.

Since $F = S_{\mathrm{cx}\,A} - \{x\}$ is closed and is strictly contained in $S_{\mathrm{cx}\,A}$, F is not a boundary for cx A. Hence there exists an h in cx A such that $\hat{h}(x) = 1$, but $|\hat{h}| < 1$ on F. Since the topological boundary of $\hat{h}(M_{\mathrm{cx}\,A})$ is contained in $\hat{h}(S_{\mathrm{cx}\,A})$ [**3**, p. 10], it follows that $\{1\}$ is open in $\hat{h}(M_{\mathrm{cx}\,A})$. Let $E = \{x' \text{ in } M_{\mathrm{cx}\,A} : \hat{h}(x') = 1\}$, which is as required.

Let $G = E \cup \tau(E)$. Then G is also open and closed in $M_{\mathrm{cx}\,A}$, and $G \cap S_{\mathrm{cx}\,A} = \{x, \tau(x)\}$. If $r_{\mathrm{cx}} : M_{\mathrm{cx}\,A} \to M_{\mathrm{cx}\,B}$ is the restriction map, then clearly $r_{\mathrm{cx}}(G)$ is closed in $M_{\mathrm{cx}\,B}$. Since r is one to one and onto, by (ii) and (iii) of Proposition 2.3 we obtain

$$M_{\mathrm{cx}\,B} - r_{\mathrm{cx}}(G) = r_{\mathrm{cx}}(M_{\mathrm{cx}\,A} - G).$$

Hence $r_{\mathrm{cx}}(G)$ is also open. By Silov's idempotent theorem [**3**, p. 88], there exists h in cx B such that $\hat{h} = 1$ on $r_{\mathrm{cx}}(G)$, and $\hat{h} = 0$ on $M_{\mathrm{cx}\,B} - r_{\mathrm{cx}}(G)$. Thus, $r_{\mathrm{cx}}(G)$ is a peak set for cx B, and as such has non-empty intersection with $S_{\mathrm{cx}\,B}$. Since $\tau(G) = G$, it follows that either $r_{\mathrm{cx}}(x)$ or $r_{\mathrm{cx}}(\tau(x))$ belongs to $S_{\mathrm{cx}\,B}$. By (i) of Proposition 2.3, then, $\mathrm{cx}^*(x) = y$ belongs to S_B.

THEOREM 3.4. *Let the map r be one to one. Let J be an ideal of A contained in B such that* $\mathrm{hull}_A J$ *contains no non-empty perfect subset. Then $r(S_A) = S_B$.*

Proof. First, since B is closed, by (ii) of Proposition 1.5, ${}_AS_B$ exists and equals $S_A \cap r^{-1}(S_B)$. Also, by Lemma 3.2, $S_A - \mathrm{hull}_A J$ is contained in it. Thus, $r(S_A - \mathrm{hull}_A J)$ is contained in S_B. If we let $E = S_A - r^{-1}(S_B)$, this implies that $\mathrm{hull}_A J$ contains E.

Next, by Theorem 3.1, $r(M_A) = M_B$, so that Lemma 3.3 applies, and if y is isolated in S_A, then $r(y)$ belongs to S_B. This shows that no y in E is isolated in E. Since $\mathrm{hull}_A J$ contains no non-empty perfect subset, we conclude that E must be empty, so that $r(S_A) = S_B$.

COROLLARY 3.5. *Let A be a complex function algebra on a compact Hausdorff*

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space Y. *Let* E *be a subset of* M_A, *and for each* y *in* E, *let* D_y *be a continuous point derivation of* A *at* y. *Let*

$$B = \{f \text{ in } A : \hat{f}(y) \text{ and } D_y(f) \text{ real for each } y \text{ in } E\}.$$

Assume that for $y_1 \neq y_2$ *in* M_A, *there exists* f *in* B *such that* $\hat{f}(y_1) = 1$, *and* $\hat{f}(y_2) = 0$, *and that the set*

$$\{y \text{ in } M_A : \hat{f} = 0 \text{ on } E \text{ implies } \hat{f}(y) = 0\}$$

is at most countable. Then M_B *is homeomorphic to* M_A, *and* S_B *to* S_A.

Proof. First, $r : M_A \to M_B$ is one to one. Let

$$J = \{f \text{ in } A : \hat{f}(y) = D_y(f) = 0 \text{ for each } y \text{ in } E\}.$$

Then by Theorem 3.1 $r(M_A) = M_B$, and by Theorem 3.4, $r(S_A) = S_B$.

Example 3.6. The above corollary generalizes Proposition 2.2 of [**5**] where the set E was finite. We give here an example to show that it is a strict generalization. Let A be the standard algebra on the unit circle, and let (y_n) be a sequence in the open unit disk such that $\sum_{n=1}^{\infty} (1 - |y_n|)$ converges and (y_n) has only one limit point y on the circle.

Let $D_{y_n}(f) = (\hat{f})'(y_n)$, and let B and J be as in the above corollary with $E = \{y_n\}$. Then, by the factorization theorem for functions in A, $\text{hull}_A J$ consists of $\{y_n\}$ together with the limit point y, and for $y' \neq y''$ in the closed unit disk, there exists f in B such that $\hat{f}(y') = 1$ and $\hat{f}(y'') = 0$. Hence M_B is the closed unit disk and S_B is the unit circle.

Added in proof. We have stated in the beginning of § 2 that $\text{cx}^* : M_{\text{cx } A} \to M_A$ is an open map, and referred to Lemma 3.9 of [**1**] for a proof. We now notice that this proof is incorrect since it assumes that if $u(x_0) + iv(x_0) = 0$, then $u(x_0) = v(x_0) = 0$, where u and v are complex-valued functions. We supply here a valid proof for the openness of cx*. Let V be an open subset of $M_{\text{cx } A}$. To prove $\text{cx}^*(V)$ is open in M_A. Since $\text{cx}^*(V) = \text{cx}^*(\tau(V))$, we assume without loss of generality that $V = \tau(V)$. Let $y_0 = \text{cx}^*(x_0)$, with x_0 in V. There exist $h_1, \ldots, h_k$ in cx A such that $\hat{h}_1(x_0) = \ldots = \hat{h}_k(x_0) = 0$, and an ϵ, $0 < \epsilon \leqq 1/3$, such that if $U_m \equiv \{x \text{ in } M_{\text{cx } A} : |\hat{h}(x)| < \epsilon, m = 1, \ldots, k\}$ then $\bigcap_{m=1}^{k} U_m$ is contained in V.

Let $f_{m,n} \equiv h_m\sigma(h_n) + h_n\sigma(h_m)$, $m, n = 1, \ldots, k$. Then $f_{m,n}$ is in A, and $|\hat{f}_{m,n}|(y_0) = |\hat{f}_{m,n}(x_0)| = 0$. If $W \equiv \{y \text{ in } M_A : |\hat{f}_{m,n}|(y) < 2\epsilon^4, m, n = 1, \ldots, k\}$, then W is an open set in M_A containing y_0. We show that W is contained in $\text{cx}^*(V)$. Let $y = \text{cx}^*(x)$, with y in W. We have

$$|\hat{h}(x)||\hat{\sigma}(h_m)(x)| = 1/2|\hat{f}_{m,m}|(y) < \epsilon^4, \quad 1 \leqq m \leqq k.$$

Fix m and n, $1 \leqq m, n \leqq k$. We prove that either x belongs to $U_m \cap U_n$, or to $\tau(U_m) \cap \tau(U_n)$. Now, either $|\hat{h}_m(m)| < \epsilon^2$, or $|\hat{\sigma}(h_m)(x)| < \epsilon^2$. Assume first that $|\hat{h}_m(x)| < \epsilon^2$. If $|\hat{h}_n(x)| < \epsilon^2$, then since $\epsilon < 1$, x belongs to $U_m \cap U_n$, while if

$|\hat{h}_n(x)| \geqq \epsilon^2$, then $|\hat{\sigma}(h_n)(x)| < \epsilon^2$. In this case we claim that $|\hat{\sigma}(h_m)(x)| < \epsilon$, so that x belongs to $\tau(U_m) \cap \tau(U_n)$. For, if $|\hat{\sigma}(h_m)(x)| \geqq \epsilon$, then

$$\begin{aligned}\epsilon^3 - \epsilon^4 &< |\hat{h}_n(x)\hat{\sigma}(h_m)(x)| - |\hat{h}_m(x)\hat{\sigma}(h_n)(x)| \\ &\leqq |\hat{f}_{m,n}(x)| = |\hat{f}_{m,n}|(y) < 2\epsilon^4.\end{aligned}$$

But this is impossible since $\epsilon \leqq 1/3$. Next, assume that $|\hat{\sigma}(h_m)| < \epsilon^2$. Then the above argument goes through if we interchange h_m and $\sigma(h_m)$, and h_n and $\sigma(h_n)$. Thus we see that x belongs to $U_m \cap U_n$, or to $\tau(U_m) \cap \tau(U_n)$. Since this is true for every $m, n = 1, \ldots, k$, either x belongs to $\bigcap_{m=1}^{k} U_m$, or to $\bigcap_{m=1}^{k} \tau(U_m)$. In any case x belongs to V, since $V = \tau(V)$. Hence cx*(V) contains W.

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