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ON EMBEDDING ESSENTIAL ANNULI IN M^3

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1. Introduction. In [**7**; **8**; **11**] it is shown that an appropriate map of a planar surface into a 3-manifold can be replaced by an embedding. In [**1**; **4**; **6**; **7**; **9**; **10**] conditions are given so that a "non-trivial" map of a planar surface (2-sphere) can be replaced by a non-trivial embedding of a planar surface (2-sphere). In this paper we give conditions on an annular map which guarantee the existence of a non-trivial embedding of an annulus. It is reported that F. Waldhausen has proved a similar but stronger "annulus theorem".

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2. Notation. Throughout this paper all spaces are simplicial complexes and all maps are piecewise linear. We shall denote the boundary of a manifold N by $\mathrm{bd}(N)$. An embedding of a manifold X in a manifold Y is *proper* if $X \cap \mathrm{bd}(Y) = \mathrm{bd}(X)$.

We shall let A denote an annulus, c_1 and c_2 the components of $\mathrm{bd}(A)$, and M a compact 3-manifold throughout this paper. We shall let α be an arc properly embedded in A which meets both c_1 and c_2. Such an arc will be called a *spanning arc*. Let $f:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ be a map such that

(1) $f_*:\pi_1(A) \to \pi_1(M)$ is monic;

(2) the arc $f(\alpha)$ is not homotopic rel its boundary to a map into $\mathrm{bd}(M)$.

Then we shall say that f is an *essential* map. Note that condition 2 is independent of the choice of α. Let λ be a simple loop embedded in M. We shall say that λ is *orientable* if it has a neighborhood homeomorphic to a solid torus. Otherwise λ is *non-orientable*. A two-sided surface F embedded in M is *incompressible* if the natural map from $\pi_1(F)$ into $\pi_1(M)$ induced by inclusion is monic.

3. Principal results. We state below the principal results obtained in this paper.

THEOREM 3.1. *Let $f:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ be an essential map such that*

(1) *$f|\ \mathrm{bd}(A)$ is a homeomorphism;*

(2) *$[f(c_1)]$ generates a free factor of $H_1(M)$;*

(3) *$f(c_1)$ is an orientable loop.*

Then there exists an essential embedding $g:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ such that $f\ \mathrm{bd}(A) = g\ \mathrm{bd}(A)$.

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Corollary 3.2. *Let M be the closure of the complement of a tubular neighborhood of a knot k in S^3. Then k is a composite knot if and only if M admits an essential map $f:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ satisfying conditions* (1) *and* (2) *in Theorem* 3.1.

Theorem 3.3. *Let M be a compact, orientable* 3*-manifold and $f:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ an essential map such that*

(1) *$f \mid \mathrm{bd}(A)$ is a homeomorphism;*

(2) *there is an element ν in $H_1(M)$ which generates a free factor of $H_1(M)$ such that $2^n\nu = [f(c_1)]$ for some positive integer n. Then there exists an essential embedding $g:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ such that $f\,\mathrm{bd}(A) = g\,\mathrm{bd}(A)$.*

Remark. Theorem 3.3 is false if M is not required to be orientable [**2**].

Theorem 3.4. *Let M be a compact, orientable,* 3*-manifold and $f:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ an essential map such that*

(1) *$f(c_1)$ does not meet $f(c_2)$;*

(2) *there is an element ν in $H_1(M)$ such that $[f(c_1)] = 2^n\nu$ for some non-negative integer n and ν generates a free factor of $H_1(M)$. Then there is an essential embedding $g:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$.*

4. Supporting lemmas. We prove below a number of lemmas useful in the proof of the theorems above.

Lemma 4.1. *Let $(\tilde{M}, p)$ be a covering of M and $f:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ a map such that $f_*\pi_1(A) \subset p_*\pi_1(\tilde{M})$ and $[f(c_1)]$ generates a free factor of $H_1(M)$. Then if $f_1:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ is a map such that $pf_1 = f$, $[f_1(c_1)]$ generates a free factor of $H_1(M)$.*

Proof. This follows from the commutative diagram in Figure 1 since $[c_1]$ generates $H_1(A)$ and since the bottom row in this diagram is exact.

$$
\begin{array}{ccccc}
 & & H_1(\tilde{M}) & \xleftarrow{f_{1*}} & H_1(A) \\
 & & \Big\downarrow p_* & & \Big\downarrow f_* \\
0 \to \ker \Phi & \to & H_1(M) & \xrightarrow{\Phi} & Z \to 0
\end{array}
$$

Figure 1

It will be convenient to represent the closure of $\{x \in A : f^{-1}f(x) \neq \{x\}\}$ by $\bar{S}_f$.

Lemma 4.2. *Let $f:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ be an essential map such that $f \mid \mathrm{bd}(A)$ is an embedding. Let M^* be a compact* 3*-manifold, $q: M^* \to M$ a map, and $f_1:(A, \mathrm{bd}(A)) \to (M^*, \mathrm{bd}(M^*))$ a map such that $qf_1 = f$. Let $(\tilde{M}, p)$ be a two-sheeted cover of M^* and $f_2:(A, \mathrm{bd}(A)) \to (\tilde{M}, \mathrm{bd}(\tilde{M}))$ an embedding such that $pf_2 = f_1$. If $[f_1(c_1)]$ generates a free factor of $H_1(M^*)$, there is an embedding $g_1:(A, \mathrm{bd}(A)) \to (M^*, \mathrm{bd}(M^*))$ such that qg_1 is an essential map and $g_1\,\mathrm{bd}(A) = f_1\,\mathrm{bd}(A)$.*

Proof. We may assume that S_{f_1} is the disjoint union of a collection of simple loops and that the number of loops in S_{f_1} can not be reduced by a small motion of f_2. Suppose that λ_1 and λ_2 are distinct loops in S_{f_1} such that $f_1(\lambda_1) = f_1(\lambda_2)$. If λ_1, and thus λ_2, is nullhomotopic in A, we may suppose that the spanning arc α does not meet λ_1 or λ_2. It follows from the usual argument that S_{f_1} can be simplified by cuts. Thus we may suppose that neither λ_1 nor λ_2 is nullhomotopic on A, α meets λ_j in a single point x_j for $j = 1, 2$, and $f_1(x_1) = f_1(x_2)$. Denote the closures of the components of $\alpha - (\lambda_1 \cup \lambda_2)$ by α_1, α_2, and α_3 where α_2 lies between λ_1 and λ_2. Then either the arc $f(\alpha_1)$ followed by $f(\alpha_3)$ or the arc $f(\alpha_1)$ followed by the inverse of $f(\alpha_2)$ and then by $f(\alpha_3)$ is not homotopic rel its boundary to an arc in $\mathrm{bd}(M)$ since

$$f(\alpha) \backsim (f(\alpha_1)f(\alpha_3))(f(\alpha_1)f(\alpha_2)^{-1}f(\alpha_3))^{-1}(f(\alpha_1)f(\alpha_3)).$$

Thus one can simplify S_{f_1} by cuts.

Suppose that λ is a simple loop in S_{f_1} such that $\lambda = f_1^{-1}f_1(\lambda)$. If λ is nullhomotopic on A, λ bounds a disk $\mathscr{D}$ on A and we may assume that α does not meet λ. We can now choose a disk $\mathscr{D}_1$ containing $\mathscr{D}$ in its interior and not meeting α. One can now apply Dehn's lemma **[7]** to $f_1|\mathscr{D}_1$ and it can be seen that S_{f_1} could have been simplified. If λ is not nullhomotopic, λ and c_1 bound a subannulus of A. Let λ_1 be the simple loop $f_1(\lambda)$. Now $2[\lambda_1] = [f_1(c_1)]$ in $H_1(M^*)$ which contradicts our hypothesis that $[f_1(c_1)]$ generates a free factor of $H_1(M^*)$.

LEMMA 4.3. *Let* $f:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ *be an essential map such that* $f|\,\mathrm{bd}(A)$ *is an embedding. Let* M^* *be a compact, orientable* 3*-manifold,* $q: M^* \to M$ *a map, and* $f_1:(A, \mathrm{bd}(A)) \to (M^*, \mathrm{bd}(M^*))$ *a map such that* $qf_1 = f$. *Let* $(\tilde{M}, p)$ *be a two-sheeted cover of* M^* *and* $f_2:(A, \mathrm{bd}(A)) \to (\tilde{M}, \mathrm{bd}(\tilde{M}))$ *an embedding such that* $f_1 = pf_2$. *Then there exists an embedding* $g_1:(A, \mathrm{bd}(A)) \to (M^*, \mathrm{bd}(M^*))$ *such that* $g_1\,\mathrm{bd}(A) = f_1\,\mathrm{bd}(A)$ *and* qg_1 *is an essential map.*

Proof. The proof of Lemma 4.3 is quite similar to that of Lemma 4.2; the only difference is that one uses the orientability of M^* to show that no loop in S_{f_1}, which is not nullhomotopic on A, double covers its image in M^*. Suppose λ is a simple loop in S_{f_1} such that $\lambda = f_1^{-1}f_1\lambda$ and λ does not bound a disk on A. Since M^* is orientable we can find a neighborhood T of $f_1(\lambda)$ homeomorphic to a solid torus. Now $\tilde{T} = p^{-1}(T)$ can be seen to be a solid torus. Let $\rho: \tilde{T} \to \tilde{T}$ be the covering translation on $\tilde{T}$. We may assume that $T \cap S_{f_1} = \lambda$ and that $f_1^{-1}(T)$ is a subannulus A_1 of A. Denote the components of $\mathrm{bd}(A_1)$ by d_1 and ${}_2d$. We observe that $\mathrm{bd}(\tilde{T}) - (f_2d_1 \cup \rho f_2d_1)$ is the union of two open annuli and that, since $\mathrm{bd}(\tilde{T})$ is a cover of $\mathrm{bd}(T)$, f_2d_2 lies in one of these annuli and ρf_2d_2 lies in the other. However $f_2(A)$ crosses $\rho f_2(A)$ only in the loop $f_2(\lambda)$. Thus f_2d_1 and ρf_2d_1 bound an annulus in $\mathrm{bd}(\tilde{T})$ which does not meet $f_2d_2 \cup \rho f_2d_2$. It follows that no simple loop λ on A is a double cover of its image under f_1. This completes the proof of Lemma 4.3.

Remark. If one could prove Lemma 4.3 in case $(\tilde{M}, p)$ is a finite cyclic covering of M^*, one could remove condition 2 in Theorems 3.3 and 3.4.

LEMMA 4.4. *Let $f:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ be an essential map such that $f(c_1) \cap f(c_2)$ is empty. Let M^* be a compact, orientable 3-manifold. Let $f_1:(A, \mathrm{bd}(A)) \to (M^*, \mathrm{bd}(M^*))$ and $q: M^* \to M$ be maps such that*

(1) $qf_1 = f$;

(2) *q carries a neighborhood of $f_1(c_j)$ in $\mathrm{bd}(M^*)$ into a neighborhood R_j of $f(c_j)$ in $\mathrm{bd}(M)$ for $j = 1, 2$.*

Let $(\tilde{M}, p)$ be a two-sheeted cover of M^* and $f_2:(A, \mathrm{bd}(A)) \to (\tilde{M}, \mathrm{bd}(\tilde{M}))$ an embedding such that $pf_2 = f_1$. Then there is an embedding $g:(A, \mathrm{bd}(A)) \to (M^*, \mathrm{bd}(M^*))$ such that

(1) *qg is an essential map, and*

(2) *$qg(c_j)$ lies in R_j for $j = 1, 2$.*

Proof. The proof of Lemma 4.4 is essentially the same as that of 4.3. Thus one need only consider simple arcs α_1 and α_2 in S_{f_1} properly embedded in A such that $f_1(\alpha_1) = f_1(\alpha_2)$. If α_1 has both of its endpoints on c_1, α_2 will also have both its endpoints on c_1 since $f(c_1)$ does not meet $f(c_2)$. Thus we can simplify S_{f_1} by cuts. Note that we do not modify the image of the map f_1 near f_1c_2 so that $(qf_1)_*: \pi_1(A) \to \pi_1(M)$ will still be monic and that we may assume that we do not modify f_1 on α. Thus the composition of the resulting map with q will be essential. Similarly if α_1 has one endpoint on c_1 and one endpoint of c_2, α_2 will have the same property. Let a_1 and a_2 be the closures of the components of $c_1 - (\alpha_1 \cup \alpha_2)$. Now $f_1(a_j)$ is a loop for $j = 1, 2$ and since $f(c_1)$ is not nullhomotopic either the loop $f(a_1)$ or the loop $f(a_2)$ is not nullhomotopic. It follows that S_{f_1} can be simplified by cuts. Since S_{f_1} can have only finitely many components, Lemma 4.4 is proved.

The following lemma is well-known.

LEMMA 4.5. *Let $f:(A, \mathrm{bd}(A)) \to (M, \mathrm{bd}(M))$ be a map. If $f_*H_1(A)$ is of infinite index in $H_1(M)$, there is a two-sheeted cover $(\tilde{M}, p)$ of M and a map $f_1:(A, \mathrm{bd}(A)) \to (\tilde{M}, \mathrm{bd}(\tilde{M}))$ such that $pf_1 = f$.*

Let K be a fixed triangulation of M, F a surface embedded in M, and $f:(\mathscr{D}, \mathrm{bd}(\mathscr{D})) \to (M, \mathrm{bd}(M))$ a map of a disk $\mathscr{D}$ into M. Suppose that f and the embedding of F are simplicial maps with respect to one fixed triangulation K. Suppose further that $f^{-1}(F)$ is a simple arc in $\mathscr{D}$ which contains only finitely many points in the closure of S_f (and no triple or branch points). We shall then say that f is *transverse with respect to F*.

Let N be a regular neighborhood of $f(\mathscr{D}) \cap F$ in F. Let us suppose that we have applied the tower construction used in the proof of the loop theorem [**9**] to obtain an embedding $g:(\mathscr{D}, \mathrm{bd}(\mathscr{D})) \to (M, \mathrm{bd}(M))$. Of course $g(\mathscr{D}) \cap F \subset N$. In Figure 2 we show a decomposition of a neighborhood $N_0 \subset N$ of a point x in S_f into the union of five disks $\mathscr{D}_0$, $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$, $\mathscr{D}_4$. We assume that $g(\mathscr{D}) \cap F$

is in general position with respect to $\mathrm{bd}(\mathscr{D}_0)$.

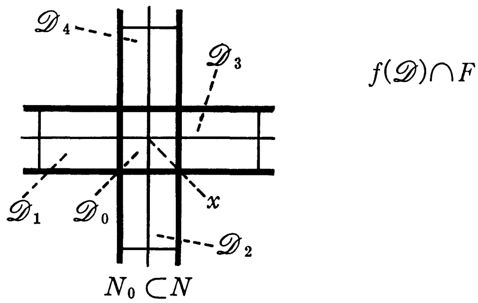

FIGURE 2

We recall that $g(\mathscr{D})$ is the image of pieces of spherical components of the boundary L of the manifold at the top of the tower. These pieces correspond to regions in $\mathscr{D} - S_f$ and there can be at most two pieces in L corresponding to any one region of $\mathscr{D} - S_f$. It follows that $\mathscr{D}_j \cap g(\mathscr{D})$ contains at most two arcs for $j = 1, 2, 3, 4$.

It can now be seen that $g(\mathscr{D}) \cap F$ can be decomposed into a collection $\mathscr{C}$ of arcs which correspond in a natural way to arcs in $(f(\mathscr{D}) \cap F) - S_f$. We observe that no more than two arcs in $\mathscr{C}$ correspond to a single arc in $(f(\mathscr{D}) \cap F) - S_f$.

Lemma 4.6 below can now be seen to be a consequence of the preceding paragraphs.

LEMMA 4.6. *Let F be a surface properly embedded in M. Let $\mathscr{D}$ be a disk and $f:(\mathscr{D}, \mathrm{bd}(\mathscr{D})) \to (M, \mathrm{bd}(M))$ a map which is transverse with respect to F such that $f^{-1}(F)$ is a simple arc. Then we may assume that any embedding $g:(\mathscr{D}, \mathrm{bd}(\mathscr{D})) \to (M, \mathrm{bd}(M))$, constructed via the tower argument from F, meets F in a collection (possibly empty) of simple arcs and simple loops and that there is a fixed finite number of possibilities for $g(\mathscr{D}) \cap F$ up to ambient isotopy in F. Furthermore the number of these possibilities is detei mined by the singular arc $ff^{-1}(F)$.*

5. Proof of major results.

Proof of Theorem 3.1. It is a consequence of a tower argument involving Lemma 4.1 and Lemma 4.5 that we may assume the existence of a 3-manifold M^* (the final stage in our tower of two-sheeted coverings), a map $q: M^* \to M$, and a map $f_1:(A, \mathrm{bd}(A)) \to (M^*, \mathrm{bd}(M^*))$ such that $f = qf_1$. Note that if f_1 is an embedding, Theorem 3.1 will follow after repeated applications of Lemma 4.2. It is a consequence of Lemma 4.5 that the image of $H_1(A)$ under $f_1{}^*$ must be of finite index in $H_1(M^*)$. It follows from standard arguments involving the first Betti number of M^* that there can be at most one non-spherical component of $\mathrm{bd}(M^*)$ and that component must be a torus or a klein bottle since M^* is the final stage of our tower of two-sheeted coverings. It follows from the arguments given in [7] or [9] that the tower has finite height.

Since $f(c_1)$ is orientable, f_1 could be lifted to an orientable two-sheeted covering of M^*. Thus we may assume that M^* and $\mathrm{bd}(M^*)$ are orientable manifolds and thus the non-spherical component of $\mathrm{bd}(M^*)$ is a torus.

Since $f_1(c_1)$ generates a free factor of $H_1(M^*)$, we can find a map of M^* onto S^1 which carries $f_1(c_j)$ homeomorphically onto S^1 for $j = 1, 2$. Using simplicial techniques, we may assume that the inverse image of some point in S^1 contains a connected, two-sided surface F properly embedded in M^* and meeting $f_1(c_j)$ in a single point for $j = 1, 2$. Of course $f_1(c_j)$ crosses F at $f_1(c_j) \cap F$ for $j = 1, 2$. We may apply standard techniques so that F may be assumed to be incompressible in M^*. After a homotopy of f_1 rel $\mathrm{bd}(A)$, we may assume that $f_1^{-1}(F)$ is the union of a spanning arc of A together with a collection of simple loops. We observe that each of these loops must be nullhomotopic on A and thus their image under f_1 is nullhomotopic on F. It follows that we may assume that $f_1^{-1}(F)$ is a spanning arc α_1 of A.

We will now finish the proof of Theorem 3.1 in two steps. In the first we will show that there is a positive integer n and an n-sheeted cyclic covering $(\tilde{M}, p)$ of M^* which admits an embedding $f_2:(A, \mathrm{bd}(A)) \to (\tilde{M}, \mathrm{bd}(\tilde{M}))$ such that

(1) qpf_2 is an essential map, and

(2) $pf_2(\mathrm{bd}(A)) = f_1(\mathrm{bd}(A))$.

In the second we show that n could have been chosen to be 1.

In the following paragraph we find an integer k to associate with $f_1(\alpha_1)$. Let M_1 be a 3-manifold and $h: F \to M_1$ a proper embedding such that $h_*: \pi_1(F) \to \pi_1(M_1)$ is 1-1. Let $\mathscr{D}$ be a disk and $\bar{f}: (\mathscr{D}, \mathrm{bd}(\mathscr{D})) \to (M_1, \mathrm{bd}(M_1))$ a map such that $\bar{f}(\mathscr{D}) \cap h(F) = hf_1(\alpha_1)$. (Recall that $\alpha_1 \subset A$ and $f_1(\alpha_1) \subset F$.) By Lemma 4.6, there is a positive integer k such that there are at most $k - 2$ ways for an embedded disk constructed via the tower argument from $\bar{f}$ to meet $h(F)$.

Let (N, p_1) be the k-sheeted cyclic covering of M^* and (A, q_1) a k-sheeted cover of A. Let $\bar{f}_1: A \to N$ be a map such that the diagram in Figure 3 commutes.

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}_1} & N \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ A & \xrightarrow{f_1} & M^* \end{array}$$

FIGURE 3

Let $F_2, \ldots, F_{k+1}$ be the k components of $p_1^{-1}(F)$. We cut N along F_{k+1} to obtain a 3-manifold N_1. We denote the two portions of $\mathrm{bd}(N_1)$ coming from F_{k+1} by F_0 and F_1. We denote the natural projection map from $N_1 \to N$ by p_2. We cut A along $f_1^{-1}(F_{k+1})$ to obtain a disk $\mathscr{D}$. Let $\bar{f}: (\mathscr{D}, \mathrm{bd}(\mathscr{D})) \to (N_1, \mathrm{bd}(N_1))$ be the map induced by $\bar{f}_1$.

We can now use a tower argument as in the proof of the loop theorem [**9**] to obtain an embedding $\bar{g}: (\mathscr{D}, \mathrm{bd}(\mathscr{D})) \to (N_1, \mathrm{bd}(N_1))$ such that

(1) $\bar{g}(\mathrm{bd}(\mathscr{D})) - (F_0 \cup F_1) = \bar{f}(\mathrm{bd}(\mathscr{D})) - (F_0 \cup F_1)$, and

(2) the image of the simple arc $\bar{g}(\mathscr{D}) \cap F_0$ under the map qp_1p_2 is not homotopic rel its boundary to an arc in $\mathrm{bd}(M)$. It is a consequence of our choice of k that there exist integers $1 < i < j \leqq k$ such that

$$p_1p_2(\bar{g}(\mathscr{D}) \cap F_i) = p_1p_2\bar{g}(\mathscr{D} \cap F_j).$$

Since F is incompressible, we may assume that

$$\bar{g}(\mathscr{D}) \cap \bigcap_{t=0}^{k} F_t$$

is a collection of simple arcs. Let $\beta_0 = \bar{g}(\mathscr{D}) \cap F_0$ and $\beta_1 = \bar{g}(\mathscr{D}) \cap F_t$ for $0 < t \leqq k$. Now $(qp_1p_2)\beta_1$ is homotopic rel its boundary to an arc in $\mathrm{bd}(M)$ followed by $(qp_1p_2)\beta_0$ followed by an arc in $\mathrm{bd}(M)$ across a portion of the singular disk $qp_1p_2\bar{g}(\mathscr{D})$. Thus $qp_1p_2\beta_1$ is not homotopic rel its boundary to an arc in $\mathrm{bd}(M)$. We observe that $F_t \cup F_j$ together with a portion of $\mathrm{bd}(N_1)$ bounds a connected 3-submanifold of N_1 which we will denote by X. We can now identify the two components of $p_2^{-1}p_1^{-1}(F)$ in $\mathrm{bd}(X)$ in the natural way to obtain an n-sheeted cyclic covering space $(\tilde{M}, p)$ of M^*. Note that $\bar{g}^{-1}(X)$ is a disk. Thus $\bar{g}$ induces an embedding $f_2: (A, \mathrm{bd}(A)) \to (\tilde{M}, \mathrm{bd}(\tilde{M}))$ in a natural way.

We claim that qpf_2 is an essential map. Since $qpf_2|\,\mathrm{bd}(A)$ covers $f(\mathrm{bd}(A))$, $(qpf_2)_*$ is monic. Since it has been shown above that $qp_1p_2\beta_1$ is not homotopic rel its boundary to an arc in $\mathrm{bd}(M)$, we may suppose that $qpf_2(\alpha)$ is not homotopic rel its boundary to an arc in $\mathrm{bd}(M)$. Our claim follows.

It remains to be shown that n could have been chosen to be 1. Assume that $n > 1$. By construction $(pf_2)^{-1}(F)$ is a collection of simple arcs $\alpha_1, \ldots, \alpha_n$ and we may assume that $pf_2|\alpha_t$ is a homeomorphism for $1 \leqq t \leqq n$. It is known that if $pf_2(\alpha_i)$ and $pf_2(\alpha_j)$ are homotopic in F rel their common boundary, they are ambient isotopic in F rel their common boundaries [**3**]. It is easily seen from the construction above that if $pf_2(\alpha_i)$ is ambient isotopic in F to $pf_2(\alpha_j)$ for $i \neq j$, one could have chosen n smaller.

Let $(N^*, \bar{p})$ be the covering space of M^* associated with the subgroup of $\pi_1(M^*)$ generated by $[f_1(c_1)]$. Since $(pf_2)_*\pi_1(A) \subset \bar{p}^*\pi_1(N^*)$, there is a map $\bar{f}: (A, \mathrm{bd}(A)) \to (N^*, \mathrm{bd}(N^*))$ such that the diagram in Figure 4 commutes.

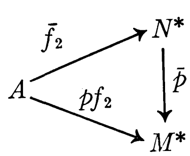

FIGURE 4

Since $pf_2(c_1) = f_1(c_1)$, we may assume that $\bar{f}_2|c_1: c_1 \to \bar{f}_2(c_1)$ is an n-fold covering of a simple loop which covers $f_1(c_1)$ once under the map $\bar{p}$. It is a consequence of [**11**, Theorem, p. 501] that there is an embedded annulus B in N^* such that $\text{bd}(B) = \bar{f}_2\,\text{bd}(A)$ since $\bar{f}_2(c_2)$ is a simple loop and $\bar{f}_2(c_j)$ has a annular neighborhood in $\text{bd}(N^*)$ for $j = 1, 2$. Let $\tilde{F}$ be the component of $\bar{p}^{-1}(F)$ which meets $\bar{f}_2(c_1)$.

We claim that each component of $\text{bd}(B)$ meets $\tilde{F}$ in a single point. Since $f_1(c_1)$ meets F in a single point and $\bar{p}|\bar{f}_2(c_1)$ is a homeomorphism onto $f_1(c_1)$, it is clear that one component of $\text{bd}(B)$ meets $\tilde{F}$ in a single point. We observe that the other component of $\text{bd}(B)$ is $\bar{f}_2(c_2)$ and it follows from an argument involving intersection numbers that $(c_2, \bar{f}_2|c_2)$ is an n-sheeted cover of $\bar{f}_2(c_2)$ since $(c_1, \bar{f}_2|c_1)$ is an n-sheeted cover of $\bar{f}_2(c_1)$. Since $c_2 \cap (\bar{p}\bar{f}_2)^{-1}F$ contains n points, $c_2 \cap \bar{f}_2^{-1}(\tilde{F})$ contains at most n points. Let k_1 be the number of points in $\bar{f}_2(c_2) \cap \tilde{F}$. Since each component of $\text{bd}(B)$ has intersection number one with $\tilde{F}$, $k_1 \geqq 1$. Now the number of points in $c_2 \cap \bar{f}_2^{-1}(\tilde{F})$ is $k_1 \cdot n$. It can now be seen that $k_1 = 1$. Thus our claim follows.

Recall that α_i, $i = 1, \ldots, n$ are the simple arcs in $\bar{f}_2^{-1}(\tilde{F})$. Now $\bar{f}_2(\alpha_1) \cup \bar{f}_2(\alpha_2)$ forms a loop λ lying on $\tilde{F}$ since $\bar{f}_2(\text{bd}(\alpha_1 \cup \alpha_2)) \subset \text{bd}(B) \cap \tilde{F}$. We observe that λ has intersection number zero with $\tilde{F}$. Since the generator of $\pi_1(N^*) \cong Z$ has intersection number one with $\tilde{F}$, λ is homotopic to a point in N^*. Thus the loop $\bar{p}\lambda$ is nullhomotopic in M^*. As has been observed above this completes the proof of Theorem 3.1.

The following lemma is useful in the proof of Theorem 3.3.

LEMMA 5.1. *Let σ be an element of $H_1(M)$ which generates a free factor of $H_1(M)$. Let $(\tilde{M}, p)$ be a two-sheeted cover of M and σ_1 an element of $H_1(\tilde{M})$ such that $p_*\sigma_1 = 2\sigma$. Suppose that $\sigma \notin p_*H_1(\tilde{M})$. Then σ_1 generates a free factor of $H_1(\tilde{M})$.*

Proof. The result above follows from the commutative diagram in Figure 5 where $\langle a \rangle$ represents the group generated by a and the bottom row in the diagram is exact.

$$\begin{array}{ccccccccc}
0 \rightarrow \ker(\phi p_*) & \rightarrow & H_1(\tilde{M}) & \xleftarrow{\text{inc}} & \langle \sigma_1 \rangle & & \\
& & \downarrow p_* & & \downarrow p_*\phi & & \\
0 \rightarrow \ker \phi & \rightarrow & H_1(M) & \xrightarrow{\phi} & \langle \sigma \rangle & \rightarrow & 0
\end{array}$$

FIGURE 5

Proof of Theorem 3.3. Let $\phi: H_1(M) \rightarrow Z$ be the projection of $H_1(M)$ onto the free factor of $H_1(M)$ generated by ν. Let $\mathscr{A}_k$ be the subgroup of Z generated by $2^k\nu$ for $0 \leqq k \leqq n$. Let $\phi_1: \pi_1(M) \rightarrow H_1(M)$ be the Hurewicz homomorphism and (M_k, p_k) the covering space of M associated with $(\phi\phi_1)^{-1}\mathscr{A}_k$. It is known that M_{k+1} is a two-sheeted covering of M_k for $0 \leqq k < n$. Note, as a consequence of 5.1, that $\phi p_{n*} H_1(M_n)$ is generated by $2^n \nu$ and that there is a map $f_n:(A, \mathrm{bd}(A)) \to (M_n, \mathrm{bd}(M_n))$ such that $p_n f_n = f$. It is easily seen that f_n is an essential map satisfying the conditions of Theorem 3.1. Thus there is an essential (in M_n) embedding $g_n:(A, \mathrm{bd}(A)) \to (M_n, \mathrm{bd}(M_n))$ such that $g_n \,\mathrm{bd}(A) = f_n \,\mathrm{bd}(A)$. Now $p_n g_n: A \to M$ can be seen to be an essential map. Thus we can apply Lemma 4.3 repeatedly to obtain essential (in M_i) embeddings $g_i: A \to M_i$ such that $p_i g_i: A \to M$ is essential and $p_i g_i \,\mathrm{bd}(A) = f\,\mathrm{bd}(A)$ for $i = 1, \ldots, n - 1$. This completes the proof of Theorem 3.3.

Proof of Theorem 3.4. As in the proof of Theorem 3.3 we can find covering spaces (M_n, p_n) of M and maps $f_k: A \to M_k$ such that

(1) $p_k f_k = f$,

(2) $[f_n(c_1)]$ generates a free factor of $H_1(M_n)$, and

(3) M_{k+1} is a 2-sheeted cover of M_k.

It can now be seen that if $f_n:(A, \mathrm{bd}(A)) \to (M_n, \mathrm{bd}(M_n))$ were an essential embedding, the theorem would follow by repeated applications of Lemma 4.4. Thus we may assume that $[\nu(c_1)]$ generates a free factor of $H_1(M)$.

We proceed using a tower argument. Suppose that M^* is the manifold at the top of the tower. As in the proof of Theorem 3.1 it can be seen that $\mathrm{bd}(M^*)$ has only one non-spherical component and that component is a torus. We let $q: M^* \to M$ be the composition of the tower's projection maps and $f_1: A \to M^*$ be a map such that $qf_1 = f$. We may assume that q maps a neighborhood of $f_1(c_j)$ in $\mathrm{bd}(M^*)$ into a neighborhood of $f(c_j)$ in $\mathrm{bd}(M)$ for $j = 1, 2$.

Let T be the component of $\mathrm{bd}(M^*)$ on which $f_1(c_1)$ lies and R a regular neighborhood of $f_1(c_1)$ in T. Now R is a 2-manifold with boundary and $f_1(c_2)$ lies in one component of the complement of R in T. We denote the closure of this component by R_2. We observe that R_2 is a compact 2-manifold. Let R_1 be the closure of the complement of R_2 in T. Note that R_1 is the union of R and the components of $T - R$ which do not meet $f_1(c_2)$. Now R_1 and R_2 are compact, connected 2-manifolds such that $R_1 \cup R_2 = T$ and $R_1 \cap R_2 = \mathrm{bd}(R_1)$. Since neither R_1 nor R_2 is a disk, R_1 and R_2 are annuli and $[f_1(c_j)]$ is a non-zero multiple of the generator $\bar{\nu}_j$ of $\pi_1(R_j)$ for $j = 1, 2$. Since $[qf_1 c_j]$ is a generator of a free factor of $H_1(M^*)$, we may suppose that $[f_1(c_j)] = \bar{\nu}_j$ for $j = 1, 2$. It follows that we can find a simple loop $\lambda_j \subset R_j$ such that $q\lambda_j$ lies in any chosen neighborhood of $f(c_j)$ for $j = 1, 2$ and which has at least one point in common with $f_1(c_j)$ for $j = 1, 2$. But now we can find a homotopy of $f_1(c_j)$ to λ_j leaving one point fixed for $j = 1, 2$. These homotopies together with f_1 give rise in a natural was to a map $f_2:(A, \mathrm{bd}(A)) \to (M^*, \lambda_1 \cup \lambda_2)$ such that $f_2(\alpha) = f_1(\alpha)$ and qf_2: $A \to M$ is an essential map.

We may assume that $qf_2(c_1)$ does not meet $qf_2(c_2)$. Using the arguments in the proof of Theorem 3.1 we may assume that f_2 is an embedding. We can then apply Lemma 4.4 to bring the embedding f_2 down the tower. This completes the proof of Theorem 3.4.

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