Appendix D

Useful operator identities

Identity 1

We wish to prove[1]

$$e^{A+B} = e^{A} e^{B} e^{C/2} \tag{D.1}$$

where $C = [B, A]$ is assumed to commute with A and B.

Let

$$S(x) = e^{(A+B)x} \tag{D.2}$$

where x is a parameter. Write

$$S(x) = e^{Ax} U(x) \tag{D.3}$$

where U is an unknown matrix-valued function. Then

$$(A + B)S(x) = \frac{dS}{dx} = Ae^{Ax} U(x) + e^{Ax} \frac{dU}{dx} \tag{D.4}$$

which leads to

$$\frac{dU}{dx} = e^{-Ax} B e^{+Ax} U(x) \tag{D.5}$$

Now

$$Be^{Ax} = B \sum_n \frac{(Ax)^n}{n!} = \sum_n \frac{(Ax)^n}{n!} B + \sum_n \frac{[B, A^n]x^n}{n!} \tag{D.6}$$

Also,

$$[B, A^n] = A^{n-1}[B, A] + [B, A^{n-1}]A = \cdots = [B, A]nA^{n-1} \tag{D.7}$$

provided $C = [B, A]$ commutes with A. Therefore

$$Be^{Ax} = e^{Ax} B + [B, A]xe^{Ax} \tag{D.8}$$

and

$$\frac{dU}{dx} = (B + [B, A]x)U(x) \tag{D.9}$$

[1] Several of the proofs in this Appendix were provided by Harsh Mathur, private communication (2005).

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This equation can be solved to get

$$U(x) = \exp(Bx + [B, A]x^2/2) \tag{D.10}$$

This solution satisfies the boundary condition $U(0) = 1$.

Hence

$$S(x) = \mathrm{e}^{(A+B)x} = \mathrm{e}^{Ax}\mathrm{e}^{Bx}\mathrm{e}^{[B,A]x^2/2} \tag{D.11}$$

With $x = 1$, we get the desired result.

Identity 2

Here we outline a proof of the identity

$$:\mathrm{e}^{A+B}:=:\mathrm{e}^{A}::\mathrm{e}^{B}:\mathrm{e}^{D} \tag{D.12}$$

where $D = [A^+, B^-]$ is assumed to be a c-number. Also, a linear decomposition of A and B is assumed $A = A^+ + A^-$, $B = B^+ + B^-$ and the superscripts $\pm$ refer to terms proportional to creation $(+)$ and annihilation $(-)$ operators. Expressions sandwiched between : : are normal ordered, and so annihilation operators are placed to the right of creation operators.

The first step is the identity

$$:\mathrm{e}^{A}:=\mathrm{e}^{A^-}\mathrm{e}^{A^+} \tag{D.13}$$

This can be proved by explicit expansion of the exponentials.

Then

$$:\mathrm{e}^{A}::\mathrm{e}^{B}:=\mathrm{e}^{A^-}\mathrm{e}^{A^+}\mathrm{e}^{B^-}\mathrm{e}^{B^+} \tag{D.14}$$

and

$$:\mathrm{e}^{A+B}:=\mathrm{e}^{A^-}\mathrm{e}^{B^-}\mathrm{e}^{A^+}\mathrm{e}^{B^+} \tag{D.15}$$

since $[A^-, B^-] = 0 = [A^+, B^+]$.

Now we use the identity, Eq. (D.1) proved in the previous section to exchange the order of the middle two factors in Eq. (D.14) and together with Eq. (D.15) gives the identity in Eq. (D.12).

Identity 3

Here we wish to show

$$A:\mathrm{e}^{B}:=:\{A + [A^+, B^-]\}\mathrm{e}^{B}: \tag{D.16}$$

$$A:\mathrm{e}^{B}:=\sum_{n=0}^{\infty}\frac{(A^+ + A^-)}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(B^-)^k(B^+)^{n-k} \tag{D.17}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\{(B^-)^k A^+ + [A^+, (B^-)^k] + A^-(B^-)^k\}\,(B^+)^{n-k} \tag{D.18}$$

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Now use

$$[A^+, (B^-)^k] = k(B^-)^{k-1}[A^+, B^-] \tag{D.19}$$

provided that $[A^+, B^-]$ is a c-number. Therefore

$$A : e^B : = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \{(B^-)^k A^+ + k(B^-)^{k-1}[A^+, B^-] + A^-(B^-)^k\} (B^+)^{n-k} \tag{D.20}$$

$$= : (A + [A^+, B^-])e^B : \tag{D.21}$$

Similarly we can show

$$: e^B : A =: e^B(A + [B^+, A^-]) : \tag{D.22}$$

Putting this together with Eq. (D.16) we get

$$[A, : e^B :] =: [A, e^B] : +([A^+, B^-] - [B^+, A^-]) : e^B : \tag{D.23}$$

In the case of interest for deriving Eq. (4.80) we have

$$A^+ = (A^-)^\dagger, \quad B^+ = -(B^-)^\dagger \tag{D.24}$$

then

$$[A^+, B^-] - [B^+, A^-] = [A^+, B^-] + [A^+, B^-]^\dagger \tag{D.25}$$

With $A = \phi(y)$ and $: e^B := \psi(x)$, $[A^+, B^-]$ is purely imaginary and the right-hand side vanishes. Then Eq. (D.23) gives the identity

$$[A, : e^B :] =: [A, e^B] : \tag{D.26}$$

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