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DIAGONALS OF NILPOTENT OPERATORS

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The purpose of the present note is to answer the following question of T. A. Gillespie, learned from G. J. Murphy [**4**]: for which sequences $\{a_n\}$ of complex numbers does there exist a quasinilpotent operator Q on a (separable, infinite-dimensional, complex) Hilbert space H, which has $\{a_n\}$ as a diagonal, that is $(Qe_n, e_n)=a_n$ for some orthonormal basis $\{e_n\}$ in H?

It was pointed out in [**4**] that $(a_1,\dots,a_n)\in\mathbb{C}^n$ is the diagonal of a nilpotent operator on a n-dimensional space if and only if $a_1+\cdots+a_n=0$. In fact, if $(a_1,\dots,a_n)$ is a diagonal of a nilpotent operator N, then $a_1+\cdots+a_n$ is $\mathrm{tr}(N)$, the trace of N, and hence must be zero. Conversely, if we have $a_1+\cdots+a_n=0$, then $(a_1,\dots,a_n)$ is the diagonal of the matrix

$$N=\begin{pmatrix} a_1 & a_1 & \dots & a_1 \\ a_2 & a_2 & \dots & a_2 \\ \dots & \dots & \dots & \dots \\ a_n & a_n & \dots & a_n \end{pmatrix}$$

and a direct computation gives $N^2=0$.

It turns out that the question has a simple-minded answer:

Theorem. *For each bounded sequence $\{a_n\}$ in $\mathbb{C}$, there is a nilpotent operator N on H such that $N^4=0$ and $(Ne_n, e_n)=a_n$ for some orthonormal basis $\{e_n\}$.*

For the proof of this theorem, we need a result taken from [**2**; Corollary 4]. For the reader's convenience, we provide a sketch of the proof based on an idea in [**1**].

Lemma 1. *If T is an operator on H and if $0\in W_e(T)^0$, the interior of the essential numerical range of T, then there is an orthonormal basis $\{e_n\}$ such that $(Te_n, e_n)=0$ for all n.*

Proof. Notice that, for any sequence $\{c_n\}$ in $W_e(T)^0$, there is an orthonormal sequence $\{f_n\}$ in H such that $(Tf_n, f_n)=c_n$. Since $0\in W_e(T)^0$, there is an orthonormal basis $\{e_n\}$ such that the sequence $\{(Te_n, e_n)\}$ contains $1/k$, $-1/k$, i/k, $-i/k$ for sufficiently large k, say, for $k\geqq k_0$. By making a rearrangement if necessary, we may assume that the partial sums

$$s_n=a_1+a_2+\cdots+a_n, \qquad n=1,2,\dots$$

where we write a_n for (Te_n, e_n) for brevity, have a subsequence converging to zero.

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Choose n_1 such that $a_1, a_2, \ldots, a_{n_1}$ contains $1/k_0$, $-1/k_0$, i/k_0, $-i/k_0$ whose convex hull contains s_{n_1}. Let T_1 be the compression of T to the subspace M_1 spanned by $e_1, \ldots, e_{n_1}$. Then $\mathrm{tr}(T_1) = s_{n_1} \in W(T_1)$, the numerical range of T_1. There is a unit vector f_1 such that $(T_1 f_1, f_1) = s_{n_1}$. Let $N_1 = M_1 \ominus \mathbb{C} f_1$ and A_1 be the compression of T to N_1. Since $\mathrm{tr}(A_1) = 0$, by Fillmore [3], A_1 has a zero diagonal. We have shown that T_1 has a diagonal consisting of

$$\underbrace{0, 0, \ldots, 0,}_{n_1 - 1 \text{ times}}\ s_{n_1}$$

Next, we replace $\{a_n\}$ by s_{n_1}, $a_{n_1+1}, a_{n_1+2}, \ldots$ and argue in the same way as before to obtain n_2 such that the compression of T to the subspace spanned by f_1, $e_{n_1+1}, \ldots, e_{n_2}$ has a diagonal consisting of

$$\underbrace{0, 0, \ldots, 0,}_{n_2 - n_1 \text{ times}}\ s_{n_2}$$

and the unit vector f_2 satisfies $(Tf_2, f_2) = s_{n_2}$ is a linear combination of $e_{n_1+1}, \ldots, e_{n_2}$. Continuing in this manner, we obtain an orthonormal basis $\{g_n\}$ such that $(Tg_n, g_n) = 0$ for all n. The detailed argument is left to the reader.

Lemma 2. *Let $\{c_n\}$ be a bounded sequence in $\mathbb{C}$ and A be the diagonal operator with $c_1, c_2, \ldots$ as its diagonal elements. If $0 \in W_e(A)^0$, then there is a nilpotent operator N and an orthonormal basis $\{e_n: -\infty < n < \infty\}$ such that $N^2 = 0$ and $(Ne_n, e_n) = c_n$ for $n > 0$ while $(Ne_n, e_n) = 0$ for $n \leqq 0$.*

Proof. Let N be the block matrix

$$\begin{pmatrix} A & A \\ -A & -A \end{pmatrix}.$$

By Lemma 1, $-A$ has a zero diagonal and hence the lemma follows.

Lemma 3. *If a_1, a_2, b, c are given complex numbers satisfying $a_1 + a_2 = b + c$, then there are numbers r, s such that*

$$|r|, |s| \leqq 2 \max(|a_1|, |a_2|, |b|, |c|) \quad \textit{and}$$

$$\begin{pmatrix} a_1 & r \\ s & a_2 \end{pmatrix} \cong \begin{pmatrix} b & * \\ 0 & c \end{pmatrix}$$

where the symbol "$\cong$" stands for "unitarily equivalent".

Proof. It suffices to show that, for suitable r and s, b is an eigenvalue of the left-hand-side matrix. The characteristic polynomial of the left-hand-side matrix is given by

$$p(X) = (X - a_1)(X - a_2) - rs.$$

Thus we may choose r, s in such a way that $p(b)=0$ and $|r|=|s|=|b-a_1|^{1/2}|c-a_2|^{1/2}$ which is not greater than $2\max(|b|,|c|,|a_1|,|a_2|)$.

Now we are ready to prove the theorem. Assume that $|a_n| \leqq M$ for all n. Take two bounded sequences $\{b_n\}$, $\{c_n\}$ in such a way that

(i) $b_n+c_n=a_{2n-1}+a_{2n}$ for all n, and

(ii) each of the numbers 0, 1, -1, i, $-i$ occurs infinitely many times in both $\{b_n\}$ and $\{c_n\}$.

By Lemma 3, there exist sequences $\{r_n\}$ and $\{s_n\}$ such that

$$\begin{pmatrix} a_{2n-1} & r_n \\ s_n & a_{2n} \end{pmatrix} \cong \begin{pmatrix} b_n & d_n \\ 0 & c_n \end{pmatrix}$$

for some bounded sequence $\{d_n\}$. In view of (ii), it follows from Lemma 2 that there exist nilpotent operators N_b and N_c such that $N_b^2=N_c^2=0$, $\{b_n\}$ is a diagonal of N_b and $\{c_n\}$ is a diagonal of N_c. Let D be the diagonal operator with $d_1, d_2, \ldots$ as its diagonal elements and let

$$N=\begin{pmatrix} N_b & D \\ 0 & N_c \end{pmatrix}.$$

Then $N^4=0$ and

$$N=\left(\begin{array}{cc|cc} b_1 & * & d_1 & 0 \\ * & b_2 & 0 & d_2 \\ & & \ddots & & \ddots \\ \hline & & c_1 & * \\ & \bigcirc & * & c_2 \\ & & & & \ddots \end{array}\right) \cong \left(\begin{array}{cc|cc} b_1 & d_1 & & * \\ 0 & c_1 & & \\ \hline & & b_2 & d_2 \\ & * & 0 & c_2 \\ & & & & \ddots \end{array}\right)$$

$$\cong \left(\begin{array}{cc|cc} a_1 & r_1 & & * \\ s_1 & a_2 & & \\ \hline & & a_3 & r_2 \\ & * & s_2 & a_4 \\ & & & & \ddots \end{array}\right)$$

The proof is complete.

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