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Philipp Gross

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The resolution property of algebraic surfaces

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Abstract

We prove that on separated algebraic surfaces every coherent sheaf is a quotient of a locally free sheaf. This class contains many schemes that are neither normal, reduced, quasiprojective nor embeddable into toric varieties. Our methods extend to arbitrary two-dimensional schemes that are proper over an excellent ring.

Introduction

A noetherian scheme (or complex analytic space, or, more generally, a locally ringed site) has the *resolution property* or *enough locally free sheaves* if every coherent sheaf $\mathcal{M}$ admits a surjection $\mathcal{E} \twoheadrightarrow \mathcal{M}$ by some coherent locally free sheaf $\mathcal{E}$ (also called a vector bundle). For an introduction to this property and applications to K-theory we refer to the seminal paper of Totaro [Tot04].

The aim of this article is to prove the resolution property for all separated algebraic surfaces (see Theorem 5.2 and Corollary 5.3). This generalizes a result of Schröer and Vezzosi who verified the resolution property for *normal* separated algebraic surfaces [SV04, Theorem 2.1]. Our methods extend to all two-dimensional schemes that are proper over an excellent base ring, throughout denoted by A.

Unless stated otherwise, all schemes are assumed to be noetherian. The word *surface* refers to a two-dimensional separated A-scheme of finite type.

It is known that a scheme satisfies the resolution property if it has an ample line bundle, or, more generally, if it has an ample family of line bundles; that is, a family of invertible sheaves where the whole collection behaves like an ample line bundle (see [BGI71, Exposé II, 2.2] and [BS03]). This includes all schemes that are quasiprojective over a noetherian ring (hence separated) and all $\mathbb{Q}$-factorial schemes with affine diagonal [BS03]. The latter means that the intersection of two affine open subsets is affine. The special case of regular, separated schemes is also known as Kleiman's theorem [Bor63, Theorem 4.2] (independently proven by Illusie [BGI71, Corollaire II.2.2.7.1]).

Schröer and Vezzosi showed that the resolution property can fail for regular schemes that do not have affine diagonal [SV04, 4.2]. Totaro observed that the latter is a *necessary* condition for the resolution property to hold [Tot04, 1.3]. In particular, a $\mathbb{Q}$-factorial scheme has the resolution property if and only if it has affine diagonal.

Concerning the category of analytic spaces, the resolution property does not hold automatically for regular and separated spaces. Schuster showed that every smooth, compact, complex surface satisfies the resolution property [Sch82]. However, it fails for generic complex tori of dimension greater than or equal to 3 (see [Voi02, A.5]).

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Starting in dimension two, it can happen that a scheme has no effective Cartier divisor or even no non-trivial line bundle at all [Sch99]. Therefore, it is not possible to construct locally free resolutions by invertible sheaves, in general. The lack of control for vector bundles of higher rank makes it difficult to tackle the resolution property for singular and non-projective schemes. To the authors knowledge, it is not known if there exists a proper algebraic scheme of dimension greater than or equal to 3 which has an affine diagonal, but does not satisfy the resolution property, or does not admit non-trivial vector bundles. Even the case of normal toric threefolds is completely open [Pay09].

In our work we give a clarifying picture of the resolution property for separated schemes of dimension less than 3. Our strategy is not a simple reduction to the normal case treated by Schröer and Vezzosi, but we generalize their arguments by using pinching techniques and constructing families of vector bundles that satisfy a partial cohomologically vanishing condition. We do not have a direct application in mind, but our result underlines the fact that separated surfaces have many non-trivial vector bundles.

The paper is organized as follows. In the first section we shall investigate quasiprojectivity of large open subsets. Given a separated A-scheme of finite type, we prove that every point has a quasiprojective neighborhood whose complement has codimension greater than 1 (Theorem 1.5). This fact is well known if X is normal [Jel05]. Indeed, since X admits only trivial finite birational covers in that case, this is a consequence of Chow's lemma and Zariski's main theorem. We will deduce the general case by using Ferrand's pinching techniques [Fer03] and deformation theory of vector bundles [Ill05, 5.A]. Consequently, every coherent sheaf is a quotient of a coherent sheaf which is locally free outside a closed subset of codimension greater than 1. Restricting ourselves to surfaces, we generalize the gluing methods of Schuster, Schröer and Vezzosi in § 2 to construct locally free resolutions of coherent sheaves that are already locally free outside finitely many points of codimension two. The cohomological obstructions appearing here lie in second cohomology groups of coherent sheaves. In order to control these, we study in § 3 a partial cohomological vanishing condition for families of coherent sheaves $(\mathcal{E}_n)_{n\in\mathbb{N}}$ on a proper scheme X of any dimension. Given an integer $1 \leqslant d < \dim(X)$, we call $(\mathcal{E}_n)$ *cohomologically d-ample* if for every coherent sheaf $\mathcal{M}$ the groups $\mathrm{H}^i(X, \mathcal{M} \otimes \mathcal{E}_n)$ vanish for all $i > d$. Here our main result states that $(\dim(X) - 1)$-ampleness is preserved and reflected under pullback by alterations. This is used in § 4 to construct a 1-ample family of vector bundles of rank two on an arbitrary proper surface. In the last section we collect the preceding results to prove the resolution property for a large class of two-dimensional separated schemes.

1. Thick quasiprojective open subsets

The aim of this section is to provide an existence result of thick quasiprojective open subschemes of a scheme X that is separated and of finite type over A (see Theorem 1.5). Here, we call a subset $V \subset X$ *thick* if it is open in X and $\mathrm{codim}(X - V, X) \geqslant 2$. Clearly, a finite intersection of thick subsets is thick. If $W \subset V$, $V \subset X$ are thick then W is thick in X.

We will frequently use birational auxiliary schemes Y that are quasiprojective over A and are endowed with a birational map $Y \to X$. Let us call a morphism of schemes $f : Y \to X$ (*U-admissible*) *birational* if there exists a dense open subscheme $U \subset X$ such that $f^{-1}(U) \subset Y$ is dense and the restriction f_U is an isomorphism.

We start with a sequence of preparatory lemmas.

Lemma 1.1. *Let $f : Y \to X$ be a birational morphism of finite type of locally noetherian schemes. Then f is quasifinite over all points of codimension less than* 2.

Proof. Let $x \in X$ be a point of codimension one. By applying the base change with $\operatorname{Spec} \mathcal{O}_{X,x} \to X$, we may assume that X is local of Krull dimension one with closed point x. Moreover, we may assume that X and Y are irreducible by a second base change. Let $y \in f^{-1}(x)$ be a maximal point. Then $\dim \mathcal{O}_{Y,y} \geqslant 1$ because y cannot be a generic point of Y, whereas $0 \leqslant \dim \mathcal{O}_{Y,y} + \operatorname{deg.tr}_{k(x)} k(y) \leqslant \dim \mathcal{O}_{X,x} = 1$ by [Gro65, 5.6.5.1]. Consequently, $\operatorname{deg.tr}_{k(x)} k(y) = 0$, which shows that $f^{-1}(x)$ is finite and discrete. □

Lemma 1.2. *Let $f : Y \to X$ be a proper birational morphism of locally noetherian schemes which is U-admissible for a dense open subset $U \subset X$. Then f is finite over a thick subset $V \subset X$ containing U.*

Proof. Let $V \subset X$ be the locus over which f has discrete fibers. Then V is open and the restriction $f^{-1}(V) \to V$ is finite [Gro61b, 4.4.11]. Lemma 1.1 implies that V is thick. □

Lemma 1.3. *Let $f : Y \to X$ be a proper birational morphism of locally noetherian schemes which is U-admissible for a dense open subset $U \subset X$. Suppose that $\mathcal{O}_{X,x}$ is normal (equivalently regular) for all $x \in X - U$ with $\dim \mathcal{O}_{X,x} = 1$. Then f is V-admissible for a thick subset $V \subset X$ containing U.*

Proof. Using Lemma 1.2, we may assume that f is finite by replacing X with a suitable thick subset. Then for all $x \in X$ with $\dim \mathcal{O}_{X,x} \leqslant 1$ the map $\mathcal{O}_{X,x} \to (f_*\mathcal{O}_Y)_x$ is an isomorphism. This is obvious if $x \in U$, but if $x \in X - U$, then $\mathcal{O}_{X,x}$ is normal by assumption and the assertion follows from [Gro61b, 4.4.9]. Thus, $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is an isomorphism outside a closed subset $Z \subset X$ of codimension greater than 1 and it follows that f is an isomorphism over $X - Z$. □

Recall that a scheme is called an *AF-scheme* if every finite set of points is contained in an affine open neighborhood. This is also known as the Kleiman–Chevalley condition [Kle66] and beyond that related to embeddings into toric varieties [Wło93] and étale cohomology [Art71, § 4]. It is always satisfied if there exists an ample line bundle [Gro61a, 4.5.4]; for example, if X is quasiprojective over a noetherian ring. The converse holds for proper, smooth algebraic schemes (see [Kle66] and its generalization by Wlodarczyk [Wło99]), but not for non-normal schemes.

Remark 1.4. Suppose we have a finite, birational map with schematically dense image of noetherian schemes $f : X' \to X$. The latter means $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ is injective, which is automatic if X is reduced. Let $Y \subset X$ be the *conductor subscheme*, defined by the conductor ideal $\operatorname{Ann} \operatorname{coker}(\mathcal{O}_X \to f_*\mathcal{O}_{X'}) \subset \mathcal{O}_X$ and define $g := f_Y : f^{-1}(Y) \to Y$. Then X is isomorphic to the pinching $Y \coprod_g X'$ of Y with X' along g by Ferrand [Fer03, 4.3]. By Ferrand's theorem [Fer03, 5.4], it follows that X is an AF-scheme if and only if X' and Y are AF-schemes. The latter is satisfied if X' and Y are quasiprojective over a noetherian ring but then it does not follow that X is quasiprojective (see Example 1.7 below).

Theorem 1.5. *Let X be a separated scheme of finite type over A. Then every point $x \in X$ has a thick neighborhood $x \in V \subset X$ which is quasiprojective over A.*

Proof. First, assume that X is reduced. Let $x \in U \subset X$ be an affine open set. By enlarging U with disjoint affine open sets, we may assume that U is dense. Then by Nagata there exists a U-admissible blow up $f : X' \to X$ with center contained in $X - U$ such that X' is quasiprojective over A (see [Con07, 2.6]).

By Lemma 1.2, we may assume that f is finite by replacing X with a thick subset containing U. Moreover, f is birational and has schematically dense image. Let $Y \subset X$ be the conductor subscheme of f. Choose a closed subset $Z \subset Y$ such that $Y - Z \subset Y$ is a dense, affine and hence quasiprojective open subscheme. Then $\operatorname{codim}(Z, X) \geqslant \operatorname{codim}(Z, Y) + \operatorname{codim}(Y, X) \geqslant 2$. So, by replacing X with $X - Z$, we may assume that Y and X' are quasiprojective over A.

In light of Remark 1.4, we may therefore assume that X is an AF-scheme. Using the fact that A is excellent, the dense subset of all regular points of X is open. In particular, the set of non-regular points $z \in X$ of codimension one is finite and thus contained in a dense affine open neighborhood $U_1 \subset X$ of x. So, by repeating the previous arguments with U replaced by U_1, we may assume that for all $z \in X - U$ with $\dim \mathcal{O}_{X,z} = 1$ the stalk $\mathcal{O}_{X,z}$ is regular. However, f is then an isomorphism over a thick subset by Lemma 1.3.

Let X now be arbitrary. Replacing X with a thick neighborhood of x, we may assume that X_{red} is quasiprojective over A by the special case. By removing closed subsets $Z \subset X - U$ with $\operatorname{codim}(Z, X - U) \geqslant 1$, we may assume that $X - U$ is affine so that $X = U \cup (X - U)$ has cohomological dimension less than 2 (see [RV04, 2.8]). Therefore, every invertible $\mathcal{O}_{X_{\mathrm{red}}}$-module lifts to an invertible $\mathcal{O}_X$-module because the obstructions lie in second cohomology groups of coherent sheaves [Ill05, Theorem 5.3]. So, by choosing an ample line bundle on X_{red}, we find an invertible $\mathcal{O}_X$-module $\mathcal{L}$ whose restriction $\mathcal{L}|_{X_{\mathrm{red}}}$ is ample. Then $\mathcal{L}$ is ample by [Gro61a, 4.5.13] and we conclude that X is quasiprojective over A. □

We derive an existence result for affine open neighborhoods, generalizing a result of Raynaud [Ray70, VIII 1].

Corollary 1.6. *Let X be a separated scheme of finite type over A. Then every point $x_0 \in X$ and every finite set of points $x_1, \ldots, x_n \in X$ of codimension less than 2 are contained in a common dense affine open subset.*

Proof. By Theorem 1.5, there is a thick quasiprojective neighborhood $x_0 \in V \subset X$. Then $x_1, \ldots, x_n \in V$. Using that V is an AF-scheme, we can find an affine open neighborhood $U \subset V$ which contains $x_0, \ldots, x_n$ and all generic points of X. □

The bound on the codimension in Theorem 1.5 and Corollary 1.6 is sharp because there exist normal separated algebraic surfaces having two closed points which do not admit a common affine open neighborhood [Sch99].

In the proof of Theorem 1.5 we removed closed subsets from a scheme X, from outside the locus over which a birational quasiprojective modification $f : Y \to X$ is an isomorphism, in order to obtain a quasiprojective thick subset. In general, it does not suffice to remove subsets from outside the locus where f is finite, i.e. where X is the pinching of Y. The following example illustrates that pinching may destroy many Cartier divisors, so that the thick subset is *a priori* not even *divisorial*; that is, the complements of effective Cartier divisors define a base for the Zariski topology ([BGI71, Exposé II, 2.2], [Bor63]).

Example 1.7 (A non-divisorial proper algebraic surface whose normalization is projective). We work over an algebraically closed field k, say $k := \mathbb{C}$ for simplicity. Let E be an elliptic curve and consider the surface $X := E \times \mathbb{P}^1$ with projections $p : X \to E$ and $q : X \to \mathbb{P}^1$. Choose distinct fibers E_0 and E_∞ over $\mathbb{P}^1$. Let $t_x : E \to E$ be the translation with respect to a rational point $x \in E$ of infinite order. Define the map $g : E_0 \coprod E_\infty \to E$ as the identity on E_0 and as t_x on E_∞.

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By Ferrand [Fer03, 5.4], the pushout S of the closed immersion $i: E_0 \coprod E_\infty \hookrightarrow X$ along the finite map g exists in the category of schemes and fits in the cartesian and cocartesian square.

$$\begin{tikzcd} E_0 \coprod E_\infty \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, "f"] \\ E \arrow[r, "j"] & S \end{tikzcd}$$

Here j is a closed immersion and f is finite with schematically dense image. It follows that S is an integral, proper surface with normalization f and having E as a singularity of codimension one.

Let us show that S is not divisorial by contradiction. Assume that for a given point $y \in E \subset S$ there exists an effective Cartier divisor $C \subset S$ with $y \notin C$ and $S - C$ affine. Then $C \cap E$ is non-empty and zero-dimensional. It follows that the line bundle $\mathcal{L} := \mathcal{O}_S(C)|_E$ has positive degree and hence is ample. Now, the natural isomorphism $g^*\mathcal{L} \simeq i^* f^* \mathcal{O}_S(C)$ induces the isomorphisms $\mathcal{L} \simeq g^*\mathcal{L}|_{E_0} \simeq f^*\mathcal{O}_S(C)|_{E_0}$ and $t_x{}^*\mathcal{L} \simeq g^*\mathcal{L}|_{E_\infty} \simeq f^*\mathcal{O}_S(C)|_{E_\infty}$.

Since X is a ruled surface, we can write $f^*\mathcal{O}_S(C) \simeq p^*\mathcal{M} \otimes q^*\mathcal{O}_{\mathbb{P}^1}(n)$ for some $\mathcal{M} \in \mathrm{Pic}(E)$ and $n \in \mathbb{N}$. Thus, $f^*\mathcal{O}_S(C)|_{E_\infty} \simeq \mathcal{M} \simeq f^*\mathcal{O}_S(C)|_{E_0}$ and consequently $\mathcal{L} \simeq t_x{}^*\mathcal{L}$. However, x must then have finite order by the theory of abelian varieties [Mum70, p. 60, Application 1], contradicting the choice of x.

As a consequence of Theorem 1.5 we obtain, in the following proposition, resolutions by coherent sheaves that are invertible outside a closed subset of codimension greater than 1, by extending a sufficiently anti-ample invertible sheaf beyond a thick quasiprojective subset.

Proposition 1.8. *Let X be a separated scheme that is of finite type over A. Then for every $x \in X$ there exists a thick quasiprojective neighborhood $x \in V \subset X$ and a coherent sheaf $\mathcal{F}$ with the following properties.*

(i) *Every coherent sheaf $\mathcal{M}$ admits for every $m \gg 0$ a map*

$$(\mathcal{F}^{\otimes m})^{\oplus n} \to \mathcal{M},$$

which is surjective over V and $n \in \mathbb{N}$.

(ii) *The restriction $\mathcal{F}|_V$ is invertible and $\mathcal{F}|_V^\vee$ is an ample $\mathcal{O}_V$-module.*

(iii) *There exists a V-admissible blow-up $f: X' \to X$ such that $f_V{}^*\mathcal{F}|_V^\vee$ extends to an ample $\mathcal{O}_{X'}$-module.*

Proof. By Theorem 1.5, there exists a thick neighborhood $x \in V \subset X$ which is quasiprojective over A. Let $f: X' \to X$ be a V-admissible blow-up such that X' is quasiprojective over A. Choose an ample $\mathcal{O}_V$-module $\mathcal{L}$ such that $f_V{}^*\mathcal{L}$ extends to an ample $\mathcal{O}_{X'}$-module. We shall construct $\mathcal{F}$ as a certain coherent extension of $\mathcal{L}^{-m}$ for $m \gg 0$, independently from $\mathcal{M}$. Then (ii) and (iii) hold immediately.

Using the fact that $\mathcal{L}$ is ample, there exist $s_i \in \Gamma(V, \mathcal{L}^n)$, $1 \leqslant i \leqslant p$, for some $n \in \mathbb{N}$, such that the non-vanishing sets $V_i := V_{s_i} \subset V$ define an affine open covering of V. We shall extend the isomorphisms $s_i^\vee|_{V_i}: \mathcal{L}^{-n}|_{V_i} \xrightarrow{\simeq} \mathcal{O}_{V_i}$ simultaneously to X. Denote by $\mathcal{I}_i \subset \mathcal{O}_X$ the coherent ideal defining on $X - V_i$ some closed subscheme structure. Then each $(s_i^{\otimes k})^\vee|_{V_i}$ extends to a map $\varphi_i: \mathcal{L}^{-nk} \to \mathcal{I}_i|_V$ for $k \gg 0$ (see [GD71, 6.8.1]). We take one k that works for all i and put $m = kn$. Now choose an arbitrary extension $\mathcal{F}_0$ of $\mathcal{L}^{-m}$ and let $\mathcal{J} \subset \mathcal{O}_X$ be the coherent ideal defining on $X - V$ some closed subscheme structure. Then the sum $\bigoplus_i \varphi_i : \mathcal{F}_0^{\oplus p}|_V \to \bigoplus_i \mathcal{I}_i|_V$ extends to

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a morphism of $\mathcal{O}_X$-modules $\psi : \mathcal{J}^\ell \cdot \mathcal{F}_0^{\oplus p} = (\mathcal{J}^\ell \cdot \mathcal{F}_0)^{\oplus p} \to \bigoplus_i \mathcal{I}_i$ for $\ell \gg 0$ (see [GD71, I.6.9.17]). Define $\mathcal{F} := \mathcal{J}^\ell \mathcal{F}_0$ and $\psi_i := \pi_i \circ \psi \circ \iota_i$, where $\iota_i : \mathcal{F} \to \mathcal{F}^{\oplus p}$ denotes the canonical inclusion and $\pi_i : \bigoplus_j \mathcal{I}_j \to \mathcal{I}_i$ the projection. Then $\mathcal{F}$ satisfies $\mathcal{F}|_V \simeq \mathcal{L}^{-m}$ and each φ_i extends to a map $\psi_i : \mathcal{F} \to \mathcal{I}_i$.

In order to prove (i), let $\mathcal{M}$ be an arbitrary coherent sheaf on X. It suffices to find for each $t \in \Gamma(V_i, \mathcal{M})$, $1 \leqslant i \leqslant p$, and every $m \gg 0$ a map $\mathcal{F}^{\otimes m} \to \mathcal{M}$ whose image contains t over V_i. Then $t : \mathcal{O}_{V_i} \to \mathcal{M}|_{V_i}$ extends for sufficiently large m to a map $\mathcal{I}_i^m \cdot \mathcal{O}_X \to \mathcal{M}$ (see [GD71, I.6.9.17]). Using the fact that $\psi_i : \mathcal{F} \to \mathcal{I}_i$ is an isomorphism over V_i, it follows that the induced map $\mathcal{F}^{\otimes m} \to \mathcal{I}_i^{\otimes m} \twoheadrightarrow \mathcal{I}_i^m$ is surjective over V_i. Thus, the composition $\mathcal{F}^{\otimes m} \to \mathcal{I}_i^m \to \mathcal{M}$ satisfies the desired properties. □

Definition 1.9. A coherent sheaf $\mathcal{F}$ on a scheme X is called *almost anti-ample (with respect to V)* if it satisfies properties 1.8(i)–(iii).

2. Gluing resolutions

In this section we formulate conditions that are sufficient for the existence of locally free resolutions of coherent sheaves which are locally free away from finitely many closed points of codimension two.

We pursue the strategy of constructing surjections $\varphi : \mathcal{E} \twoheadrightarrow \mathcal{M}$ with predefined kernel $\mathcal{S}$, the *first syzygy* of φ, generalizing the methods of Schröer and Vezzosi [SV04]. Such a map φ is determined by an extension class $\gamma \in \operatorname{Ext}^1(\mathcal{M}, \mathcal{S})$ up to isomorphism of $\mathcal{E}$ over $\mathcal{M}$. Instead of gluing morphisms one glues extension classes. This is controlled by the exact sequence

$$\operatorname{Ext}^1(\mathcal{M}, \mathcal{S}) \xrightarrow{\ell} \mathrm{H}^0(X, \mathcal{E}xt^1(\mathcal{M}, \mathcal{S})) \longrightarrow \mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{S}))$$

which can be read off the local to global spectral sequence for Ext. Here, the function ℓ maps a class $[e]$, represented by an extension $e : 0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{M} \to 0$, to a global section $\ell(e)$ whose stalk $\ell(e)_x$ at $x \in X$ is given by the class of the localized sequence $[e_x] \in \operatorname{Ext}^1_{\mathcal{O}_{X,x}}(\mathcal{M}_x, \mathcal{S}_x) \simeq \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{M}, \mathcal{S})_x$.

If $\mathcal{M}$ is locally free outside finitely many closed points $Z \subset X$, the local extensions of $\mathcal{M}$ by $\mathcal{S}$ appear in a simple form. In fact, there is a canonical isomorphism $\bigoplus_{z \in Z} \operatorname{Ext}^1_{\mathcal{O}_{X,z}}(\mathcal{M}_z, \mathcal{S}_z) \simeq \mathrm{H}^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{M}, \mathcal{S}))$. Therefore, an arbitrary stalk of extension classes near Z extends to a global section.

We will use a more convenient formulation of the obstruction space for gluing local resolutions of $\mathcal{M}$ by $\mathcal{S}$. For that, observe that the canonical map $\mathcal{M}^\vee \otimes \mathcal{S} \to \mathcal{H}om(\mathcal{M}, \mathcal{S})$ is an isomorphism over the open subset where $\mathcal{M}$ is locally free. In general, this is not an isomorphism everywhere because the tensor product may have torsion sections. Nevertheless, if $\mathcal{M}$ is locally free up to finitely many closed points, we deduce from the succeeding Lemma 2.1 an isomorphism between the top cohomology groups

$$\mathrm{H}^2(X, \mathcal{M}^\vee \otimes \mathcal{S}) \xrightarrow{\simeq} \mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{S})).$$

Analogously, the obstruction space does not change up to isomorphism if $\mathcal{S}$ is modified over finitely many closed points.

Lemma 2.1. *Let X be a noetherian scheme and $\mathcal{F}$, $\mathcal{F}'$ be coherent sheaves that are isomorphic outside a closed subscheme $Z \subset X$. Then for all $i \in \mathbb{N}$ with $i \geqslant \dim(Z)$ we have $\mathrm{H}^{i+2}(X, \mathcal{F}) \simeq \mathrm{H}^{i+2}(X, \mathcal{F}')$.*

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Proof. If there exists a map $u : \mathcal{F} \to \mathcal{F}'$ which is an isomorphism on $X - Z$, then the support of $\ker u$ and $\operatorname{coker} u$ has dimension less than or equal to $\dim(Z)$. Hence, the assertion follows by taking cohomology of the two exact sequences

$$0 \to \ker u \to \mathcal{F} \to \operatorname{im} u \to 0 \quad \text{and} \quad 0 \to \operatorname{im} u \to \mathcal{F}' \to \operatorname{coker} u \to 0.$$

In general, there exists just an isomorphism $\mathcal{F}|_{X-Z} \xrightarrow{\simeq} \mathcal{F}'|_{X-Z}$. However, this extends to a morphism $\varphi : \mathcal{I}\mathcal{F} \to \mathcal{F}'$ for some coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ that defines a closed subscheme structure on Z (see [GD71, I.6.9.17]). Then φ as well as the inclusion $\mathcal{I}\mathcal{F} \hookrightarrow \mathcal{F}$ are isomorphisms over $X - Z$. So, the assertion follows by the previous case. □

Let us begin with gluing local resolutions of coherent sheaves which are almost locally free in the following sense.

DEFINITION 2.2. We say that a coherent sheaf $\mathcal{N}$ on a scheme X has property F_k if the condition $\operatorname{pd}(\mathcal{N}_x) \leqslant \max\{0, \dim \mathcal{O}_{X,x} - k\}$ holds. Then $\mathcal{N}$ is free in codimension less than or equal to k and has everywhere locally bounded projective dimension.

Remark 2.3. If X is a Cohen–Macaulay scheme, then the Auslander–Buchsbaum formula implies that a coherent sheaf of positive rank satisfies F_k if and only if it has locally finite projective dimension and property S_k is fulfilled.

PROPOSITION 2.4. *Let X be a d-dimensional scheme and let $\mathcal{M}$ be a coherent sheaf satisfying F_{d-1}. Then for every vector bundle $\mathcal{S}$ and every integer $m \gg 0$, there exists an obstruction $o \in \mathrm{H}^2(X, \mathcal{M}^\vee \otimes \mathcal{S}^{\oplus m})$ whose vanishing is sufficient for the existence of a locally free resolution*

$$0 \to \mathcal{S}^{\oplus m} \to \mathcal{E} \to \mathcal{M} \to 0.$$

Proof. Since $\mathcal{M}$ satisfies F_{d-1}, there exists a finite subset of closed points $Z \subset X$ such that $\mathcal{M}|_{X-Z}$ is locally free and $\operatorname{pd}(\mathcal{M}_z) = 1$ for each $z \in Z$. Using the fact that $\mathcal{S}$ is locally free, we can choose for every $m \gg 0$ and every $z \in Z$ a locally free resolution

$$\gamma_z : 0 \to \mathcal{S}_z^{\oplus m} \to \mathcal{F}_z \to \mathcal{M}_z \to 0$$

for some finite free $\mathcal{O}_{X,z}$-module $\mathcal{F}_z$. Then the extension classes $[\gamma_z]$ glue to the desired extension if the obstruction $o \in \mathrm{H}^2(X, \mathcal{M}^\vee \otimes \mathcal{S}^{\oplus m}) \simeq \mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{S}^{\oplus m}))$ vanishes. □

Next, we focus on two-dimensional schemes. In light of Proposition 2.4, it suffices to resolve coherent sheaves by coherent sheaves satisfying F_1. For that, we construct local resolutions first, and then apply the gluing procedure to get a global resolution.

As a preparation, we provide a generalization of the Bourbaki Lemma to non-normal rings ([Bou65, p. 76], see [BV75] for torsion-free modules). It says that for a normal noetherian ring, every torsion-free module of rank r has a free submodule of rank $r - 1$ such that its quotient is isomorphic to an ideal, hence has rank one.

LEMMA 2.5 (Modified Bourbaki lemma). *Let R be a noetherian ring and $k \geqslant 0$. Then the following properties are equivalent.*

(i) *The ring R has no associated prime of height greater than k.*

(ii) *For every finitely generated R-module M, which is free of rank $r \geqslant k$ at all primes of height less than or equal to k, there is a free submodule F of M of rank $r - k$ such that M/F is free of rank k at all primes of height less than or equal to k.*

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Proof. (ii) ⇒ (i). Suppose that there is an associated prime ideal $\mathfrak{q} = \mathrm{Ann}(x) \subset R$, $x \in R$, with $\mathrm{ht}(\mathfrak{q}) = k+1 > 0$. Then the R-module $M := \mathfrak{q}^{\oplus k+1}$ is free of rank $k+1$ at all primes of height less than or equal to k. However, every R-linear map $R \to M$ sends x to zero, so M cannot have a free submodule of rank one.

(i) ⇒ (ii). The case $r = k$ is trivial, so let us assume $r \geqslant k+1$. We will apply basic element theory (cf. [EG85]). Denote by $\mu(\,\cdot\,)$ the minimal number of generators of a module. A submodule $N \subset M$ is called *w-fold basic* at a prime ideal $\mathfrak{p} \subset R$ if $\mu((M/N)_{\mathfrak{p}}) \leqslant \mu(M_{\mathfrak{p}}) - w$. A set of generators $x_1, \ldots, x_s$ of N is called *basic up to height k* if N is $\min(s, k - \mathrm{ht}\,\mathfrak{p} + 1)$-fold basic in M at each prime ideal $\mathfrak{p} \subset R$ with $\mathrm{ht}\,\mathfrak{p} \leqslant k$.

Given a set $x_1, \ldots, x_s$ of generators for M and a prime ideal $\mathfrak{p} \subset R$ with $\mathrm{ht}\,\mathfrak{p} \leqslant k$ the number $w := \min(s, k - \mathrm{ht}\,\mathfrak{p} + 1) \leqslant k+1$ satisfies $\mu(M_{\mathfrak{p}}) - w \geqslant r - (k+1) \geqslant 0$. This shows that M itself is basic up to height k. Then, by [EG85, Theorem 2.3], there is an element $y \in M$ such that the spanned submodule $Ry \subset M$ is basic up to height k. Consider, the induced short exact sequence

$$0 \to Ry \to M \to M/Ry \to 0. \tag{2.5.1}$$

Then for every prime ideal $\mathfrak{p} \subset R$ the exact sequence

$$(Ry)_{\mathfrak{p}} \otimes_R k(\mathfrak{p}) \to M_{\mathfrak{p}} \otimes_R k(\mathfrak{p}) \to (M/Ry)_{\mathfrak{p}} \otimes_R k(\mathfrak{p}) \to 0$$

gives $\mu((M/Ry)_{\mathfrak{p}}) \geqslant \mu(M_{\mathfrak{p}}) - \mu((Ry)_{\mathfrak{p}}) \geqslant r-1$. Let us now suppose that $\mathrm{ht}\,\mathfrak{p} \leqslant k$. Then from the choice of y it follows that $\mu(M/Ry)_{\mathfrak{p}} \leqslant \mu(M)_{\mathfrak{p}} - \min\{1, k - \mathrm{ht}\,\mathfrak{p} + 1\} = r-1$, and we conclude that $\mu(M/Ry)_{\mathfrak{p}} = r-1$ and $\mu((Ry)_{\mathfrak{p}}) = 1$. Consequently, $(M/Ry)_{\mathfrak{p}}$ is free of rank $r-1$ and Ry is free of rank one, using the fact that R has no associated prime of height greater than k.

By induction, there is a free submodule $F' \subset M/Ry$ of rank $(r-1)-k$ such that $(M/Ry)/F'$ is free of rank k at all primes of height less than or equal to k. Pulling back the short exact sequence (2.5.1) along the inclusion $F' \hookrightarrow M/Ry$ gives a submodule $F \subset M$ that is an extension of F' by Ry, thus free of rank $r-k$, and that satisfies $M/F \simeq (M/Ry)/F'$. □

Using the modified Bourbaki lemma, we obtain the following decomposition result for finitely generated modules over two-dimensional rings.

LEMMA 2.6. *Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension two whose closed point is not associated. Then for every finitely generated R-module M that is locally free of rank $r \geqslant 1$ on $U := \mathrm{Spec}\,R - \{\mathfrak{m}\}$, there exists a short exact sequence of finitely generated R-modules*

$$0 \to L \to N \to M \to 0, \tag{2.6.1}$$

satisfying the following properties.

(i) *The restriction $L|_U$ is invertible and $L|_U \simeq \det M^{\vee}|_U$.*

(ii) *The module N has property F_1.*

Proof. Every choice of a generating set for M gives rise to a short exact sequence

$$0 \to S \to R^{\oplus n} \to M \to 0. \tag{2.6.2}$$

Then $S|_U$ is locally free of rank $n-r$. So, by applying Lemma 2.5 with $k=1$, there exists a free submodule $F \subset S$ of rank $n-r-1$ such that $L := S/F$ is invertible in codimension less than 2. It follows that $L|_U = \det L|_U \otimes \det F|_U \simeq \det S|_U \simeq \det M^{\vee}|_U$. Then the pushout of (2.6.2) along the quotient map $S \twoheadrightarrow L$ gives the desired sequence (2.6.1). For the module N, $N \simeq R^{\oplus n}/F$ holds, showing that $\mathrm{pd}\,N \leqslant 1$. As (2.6.1) is locally split over U, we infer that $N|_U$ is locally free. It follows that N satisfies F_1. □

PROPOSITION 2.7. *Let X be a two-dimensional scheme, and let $\mathcal{M}$ be a coherent sheaf which is locally free of constant rank outside a finite subset $Z \subset X$ of non-associated closed points of codimension two. Denote by $\mathcal{F}$ a coherent extension of* $\det \mathcal{M}^\vee|_{X-Z}$.

Then there exists an obstruction $o \in \mathrm{H}^2(X, \mathcal{M}^\vee \otimes \mathcal{F})$, whose vanishing is sufficient for the existence of a short exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{M} \to 0, \tag{2.7.1}$$

satisfying the following properties.

(i) *The $\mathcal{O}_X$-module $\mathcal{L}$ is a coherent extension of* $\det \mathcal{M}^\vee|_{X-Z}$, *possibly different from $\mathcal{F}$.*

(ii) *The $\mathcal{O}_X$-module $\mathcal{N}$ satisfies F_1.*

Proof. Denote the rank of $\mathcal{M}$ by r. By Lemma 2.6, there exists for each $z \in Z$ an extension

$$\gamma_z : 0 \to L_z \to N_z \to M_z \to 0,$$

such that $L_z|_{\mathrm{Spec}\,\mathcal{O}_{X,z}-\{z\}} \simeq \det \mathcal{M}^\vee|_{\mathrm{Spec}\,\mathcal{O}_{X,z}-\{z\}}$ and N_z satisfies F_1. Then the family L_z, $z \in Z$, and $\det \mathcal{M}^\vee|_{X-Z}$ glue to a coherent $\mathcal{O}_X$-module $\mathcal{L}$, i.e. $\mathcal{L}_z \simeq L_z$ and $\mathcal{L}|_{X-Z} \simeq \det \mathcal{M}^\vee|_{X-Z}$, and the extension classes glue to a global section γ of $\mathcal{E}xt^1(\mathcal{M}, \mathcal{L})$.

Since $\mathcal{M}^\vee \otimes \mathcal{F}|_{X-Z} \simeq \mathcal{H}om(\mathcal{M}, \mathcal{L})|_{X-Z}$, we deduce, by Lemma 2.1, that $\mathrm{H}^2(X, \mathcal{M}^\vee \otimes \mathcal{F}) \simeq \mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{L}))$. Thus, we can identify the obstruction $o(\gamma) \in \mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{L}))$ for gluing the extension classes $[\gamma_z]$ with an element $o \in \mathrm{H}^2(X, \mathcal{M}^\vee \otimes \mathcal{F})$. If the obstruction vanishes, we can choose an extension (2.7) whose localization is isomorphic to γ_z for each $z \in Z$. It follows that $\mathcal{N}$ satisfies F_1. □

So, up to cohomological obstructions, Propositions 2.4, 2.7 and 1.8 enable us to construct locally free resolutions of coherent sheaves on surfaces.

3. Cohomologically ample families of coherent sheaves

In this section we shall investigate families of coherent sheaves with a partial cohomological vanishing condition. This enables us to control the cohomological constructions for gluing resolutions later on in the case of surfaces.

DEFINITION 3.1. Let X be a scheme which is proper over A and let $d > 0$ be an integer with $d \leqslant \dim(X) - 1$. A family of coherent sheaves $(\mathcal{E}_n) = (\mathcal{E}_n)_{n \in \mathbb{N}}$ is called *(cohomologically) d-ample* if for every coherent sheaf $\mathcal{M}$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and $i \geqslant d+1$ it is true that $\mathrm{H}^i(X, \mathcal{E}_n \otimes \mathcal{M}) = 0$.

A coherent sheaf $\mathcal{E}$ is called *(cohomologically) d-ample* if $(\mathcal{E}^{\otimes n})$ is a cohomologically d-ample family.

For the case where $\mathcal{E}$ is a vector bundle, this cohomological vanishing condition was studied in [Som78, Ste98]. For a recent treatment of line bundles we refer to [Tot10].

We intend to show that the dual of an almost anti-ample sheaf is $(\dim(X) - 1)$-ample (Corollary 3.5) and that $(\dim(X) - 1)$-ampleness is preserved and reflected by alterations (Proposition 3.7). Recall that an *alteration* is a morphism of schemes $f : Y \to X$ which is proper and finite over a dense open subset $U \subset X$ such that $f^{-1}(U) \subset Y$ is dense.

Example 3.2. An invertible sheaf on X is ample if and only if it is 0-ample.

We will frequently exploit the fact that d-ampleness does not change if the sheaves of the family in question or if the scheme itself is modified at closed subsets of dimension less than or equal to $d-1$. The following is a direct consequence of Lemma 2.1.

Lemma 3.3. *Let X be a scheme that is proper over A and let $(\mathcal{E}_n)$, $(\mathcal{E}'_n)$ be families of coherent sheaves such that for every $n \in \mathbb{N}$, $\mathcal{E}_n$ and $\mathcal{E}'_n$ are isomorphic outside a closed subscheme $Z \subset X$ with $\dim(Z) \leqslant d-1$. Then $(\mathcal{E}_n)$ is d-ample if and only if $(\mathcal{E}'_n)$ is d-ample.*

Serre's vanishing theorem on projective birational models provides plenty of d-ample coherent sheaves.

Proposition 3.4. *Let $f: Y \to X$ be a morphism of proper A-schemes such that Y has an ample line bundle $\mathcal{L}$. Suppose that f is an isomorphism away from a closed subset $Z \subset X$ with $\dim(Z) < d$. Then $f_*\mathcal{L}$ is d-ample.*

Proof. Let $\mathcal{M}$ be a coherent sheaf on X and $i \geqslant d+1$. Then by Lemma 2.1, $\mathrm{H}^i(X, (f_*\mathcal{L})^{\otimes n} \otimes \mathcal{M}) \simeq \mathrm{H}^i(X, f_*(\mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}))$ holds, because $\dim(Z) \leqslant d-1 \leqslant i-2$. The latter group vanishes for $n \gg 0$ using Grothendieck's spectral sequence and Serre's Vanishing theorem, because $\mathcal{L}$ is ample, *a fortiori*, f-ample. □

It follows that almost anti-ample coherent sheaves (cf. Proposition 1.8) satisfy a cohomological vanishing condition for the top cohomology.

Corollary 3.5. *Let X be a scheme of dimension $d \geqslant 1$ that is proper over A. Then for every almost anti-ample coherent sheaf $\mathcal{F}$ the dual $\mathcal{F}^\vee$ is $(d-1)$-ample.*

Proof. By definition, there exists a proper morphism $f: X' \to X$, which is an isomorphism over a thick open subset $V \subset X$ and whose domain carries an ample line bundle $\mathcal{L}'$ such that $\mathcal{F}^\vee|_V \simeq f_*\mathcal{L}'|_V$. Since $\dim(X - V) \leqslant d-2$, the assertion follows from Proposition 3.4 and Lemma 3.3. □

Concerning pullbacks along finite maps, we will see next that d-ampleness behaves as usual ampleness.

Proposition 3.6. *Let $f: Y \to X$ be a* finite *map of schemes that are proper over A, let $(\mathcal{E}_n)$ be a family of coherent $\mathcal{O}_X$-modules and d an integer with $0 \leqslant d \leqslant \dim(Y) - 1$.*

(i) *If $(\mathcal{E}_n)$ is d-ample, then $(f^*\mathcal{E}_n)$ is d-ample.*

(ii) *Conversely, suppose that f is a surjective nilimmersion. Assume that each $\mathcal{E}_n$ is locally free, or that $d = \dim(Y) - 1$. If $(f^*\mathcal{E}_n)$ is d-ample, then $(\mathcal{E}_n)$ is d-ample.*

Proof. Let us prove (i) first. Given a coherent $\mathcal{O}_Y$-module $\mathcal{N}$ the projection formula $f_*\mathcal{N} \otimes \mathcal{E}_n = f_*(\mathcal{N} \otimes f^*\mathcal{E}_n)$ holds by exactness of f_*. Therefore, it induces for all $i, n \geqslant 0$, an isomorphism of abelian groups $\mathrm{H}^i(Y, \mathcal{N} \otimes f^*\mathcal{E}_n) \simeq \mathrm{H}^i(X, f_*\mathcal{N} \otimes \mathcal{E}_n)$ which proves the assertion.

So let us prove (ii) next. The closed immersion $f: Y \hookrightarrow X$ is given by a nilpotent coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ since X is noetherian. We may assume that $\mathcal{I}^2 = 0$ by factoring f. Let $\mathcal{M}$ be a given coherent $\mathcal{O}_X$-module. Then by applying $\cdot \otimes_{\mathcal{O}_X} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}_n$ to the short exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$, we obtain an exact sequence

$$\mathcal{I}\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}_n \xrightarrow{\varphi} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}_n \to \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}_n \to 0.$$

Taking cohomology gives, for each $i \geqslant d+1$, the exact sequences

$$\mathrm{H}^i(X, \operatorname{im}\varphi) \to \mathrm{H}^i(X, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}_n) \to \mathrm{H}^i(Y, \mathcal{M}|_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_n|_Y),$$
$$\mathrm{H}^i(X, \ker\varphi) \longrightarrow \mathrm{H}^i(X, \mathcal{I}\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}_n) \xrightarrow{\varphi_*} \mathrm{H}^i(X, \operatorname{im}\varphi).$$

If $i = \dim(X)$, then φ_* is surjective. If $\mathcal{E}_n$ is locally free, then φ_* is an isomorphism because $\ker\varphi = 0$.

Since $\mathcal{I}^2 = 0$, the $\mathcal{O}_X$-module $\mathcal{I}\mathcal{M}$ carries the structure of an $\mathcal{O}_Y$-module so that we may identify the middle group with $\mathrm{H}^i(Y, \mathcal{I}\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{E}_n|_Y)$. Hence, if $(\mathcal{E}_n|_Y)$ is d-ample, then $(\mathcal{E}_n)$ is d-ample, too. $\square$

Finally, we show that $(\dim(X)-1)$-ampleness is preserved and reflected by pullback along alterations.

PROPOSITION 3.7. *Let $f : Y \to X$ be an alteration of schemes of dimension $d \geqslant 1$ that are proper over A. Then a family of coherent $\mathcal{O}_X$-modules $(\mathcal{E}_n)$ is $(d-1)$-ample if and only if $(f^*\mathcal{E}_n)$ is a $(d-1)$-ample family.*

Proof. First, assume that $(\mathcal{E}_n)$ is $(d-1)$-ample. If f is finite, the assertion follows from Proposition 3.6(i). So, by applying Stein factorization to f, we may assume that f is Stein. Using the fact that f is proper and birational, we infer from Lemma 3.9 below that $\mathrm{H}^d(Y, \mathcal{N} \otimes f^*\mathcal{E}_n) \simeq \mathrm{H}^d(X, f_*(\mathcal{N} \otimes f^*\mathcal{E}_n))$ as abelian groups for every coherent $\mathcal{O}_Y$-module $\mathcal{N}$. The latter group is isomorphic to $\mathrm{H}^d(X, f_*\mathcal{N} \otimes \mathcal{E}_n)$ by Lemma 2.1 because f is an isomorphism over all points of codimension less than 2. Thus, $(f^*\mathcal{E}_n)$ is $(d-1)$-ample.

Conversely, assume that $(f^*\mathcal{E}_n)$ is $(d-1)$-ample. By replacing f with its base change along the reduction $X_{\mathrm{red}} \to X$, we may assume that X is reduced, using Proposition 3.6(ii). In particular, f has schematically dense image. Denote by $\mathcal{M}$ a given coherent $\mathcal{O}_X$-module.

First, suppose that f is finite. Let $\varphi : \mathcal{O}_X \hookrightarrow f_*\mathcal{O}_Y$ be the natural map and put $\mathcal{C} := \operatorname{coker}\varphi$. Since f is generically flat and finitely presented, $\mathcal{C}$ is generically locally free. Hence, φ is generically split injective. Using the identification $\mathcal{H}om(f_*\mathcal{O}_Y, \mathcal{M}) = f_*f^!\mathcal{M}$, we conclude that the transpose $\varphi^t : f_*f^!\mathcal{M} \to \mathcal{M}$ is generically split surjective. It follows that $\varphi^t \otimes 1 : f_*f^!\mathcal{M} \otimes \mathcal{E}_n \to \mathcal{M} \otimes \mathcal{E}_n$ is generically surjective so that $\dim(\operatorname{coker}(\varphi^t \otimes 1)) \leqslant d-1$. Therefore, taking cohomology gives a surjection $\mathrm{H}^d(X, f_*f^!\mathcal{M} \otimes \mathcal{E}_n) \twoheadrightarrow \mathrm{H}^d(X, \mathcal{M} \otimes \mathcal{E}_n)$. The left-hand side is isomorphic to $\mathrm{H}^d(X, f_*(f^!\mathcal{M} \otimes f^*\mathcal{E}_n)) \simeq \mathrm{H}^d(Y, f^!\mathcal{M} \otimes f^*\mathcal{E}_n)$ using the projection formula which holds since f_* is exact. This proves the case where f is finite.

Let us now turn to the general case. By invoking the finite case, we may replace X with its integral components, and hence assume that X is integral. Then $\Gamma(X, \mathcal{O}_X)$ is an integral domain and the structure map $p : X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X) =: S$ is proper. We have to show that the abelian group $\mathrm{H}^d(X, \mathcal{M} \otimes \mathcal{E}_n) = \mathrm{H}^0(S, R^dp_*(\mathcal{M} \otimes \mathcal{E}_n))$ vanishes for $n \gg 0$. The coherent sheaf $R^dp_*(\mathcal{M} \otimes \mathcal{E}_n)$ is zero if its stalks at all closed points vanishes. So we may assume that $\dim p^{-1}(s) \geqslant d$ for some closed point $s \in S$ by the theorem on formal functions. Then, however, $\dim p^{-1}(s) = d$, and since X is irreducible it follows that for the generic point $\eta \in X$ we have $p(\eta) = s$. Using the fact that s is closed, we infer $p(X) = \{s\}$, so that S consists of a single point. Consequently, S is the spectrum of a field.

Thus, X and Y are algebraic schemes. By the finite case, we may replace X as well as Y by the normalization. Then the assertion follows from the succeeding Lemma 3.8. $\square$

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Lemma 3.8. *Let $f : Y \to X$ be a birational map of d-dimensional, normal schemes that are proper over a field k. Then for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists a natural surjection of k-vector spaces*

$$\mathrm{H}^d(Y, \mathcal{H}om(f^*\omega_X, \omega_Y) \otimes f^*\mathcal{F}) \twoheadrightarrow \mathrm{H}^d(X, \mathcal{F}).$$

Proof. Since X and Y are proper over a field, they admit dualizing sheaves ω_X and ω_Y. Consider the $\mathcal{O}_X$-linear map, which is natural in $\mathcal{F}$:

$$\sigma : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \longrightarrow f_*\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{H}om_{\mathcal{O}_Y}(f^*\omega_X, \omega_Y) \otimes f^*\mathcal{F}, \omega_Y), \sigma(\varphi) : \lambda \otimes s \mapsto \lambda(f^*(\varphi)[s]).$$

Now, σ is generically bijective because f is birational and ω_X, ω_Y are generically invertible. However, $\mathcal{H}om(\mathcal{F}, \omega_X)$ is torsion-free, since ω_X is by normality of X, and we conclude that σ is injective everywhere. By taking global sections, this gives an injective k-linear map:

$$\Gamma(\sigma) : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{H}om_{\mathcal{O}_Y}(f^*\omega_X, \omega_Y) \otimes f^*\mathcal{F}, \omega_Y).$$

Finally, we obtain the desired surjection by applying Serre duality. □

Lemma 3.9. *Let $f : Y \to X$ be a proper and birational Stein morphism of d-dimensional schemes. Then for every coherent $\mathcal{O}_Y$-module $\mathcal{F}$*

$$\mathrm{H}^d(Y, \mathcal{F}) \simeq \mathrm{H}^d(X, f_*\mathcal{F}).$$

Proof. Let $x \in X$ with $\dim \mathcal{O}_x \geqslant 1$. Then $\dim f^{-1}(x) \leqslant \dim \mathcal{O}_x - 1$ since f is birational and of finite type. Hence, $(R^q f_*\mathcal{F})_x = 0$ for all $q \geqslant \dim \mathcal{O}_x$ by the theorem on formal functions. It follows that $\mathrm{codim}(\mathrm{Supp}\, R^q f_*\mathcal{F}, X) \geqslant q + 1$, hence $\dim \mathrm{Supp}\, R^q f_*\mathcal{F} < d - q$, so that $\mathrm{H}^p(X, R^q f_*\mathcal{F}) = 0$ for all $p \geqslant 1$, $q \geqslant 0$ with $p + q \geqslant d$. Applying the Grothendieck spectral sequence settles the result. □

4. Existence of positive vector bundles on non-projective surfaces

In this section we construct a 1-ample family of vector bundles $(\mathcal{E}_n)_{n\in\mathbb{N}}$ of rank two on an arbitrary surface X that is proper over a A (see Theorem 4.5). The idea is to work on a projective surface Y which admits a birational map $Y \to X$.

Let us first discuss descent conditions of vector bundles for a proper birational map of surfaces. For that, we have to introduce some terminology. Let $f : Y \to X$ be a proper birational morphism. The closed subscheme $B \subset X$, given by the conductor ideal $\mathrm{Ann}_{\mathcal{O}_X} \mathrm{coker}(\mathcal{O}_X \to f_*\mathcal{O}_Y)$, is called the *branching subscheme*. The union of all integral one-dimensional closed subschemes contracted by f is called the *exceptional curve* $E \subset Y$.

Lemma 4.1. *Let $f : Y \to X$ be a proper birational morphism of two-dimensional schemes which satisfies the following conditions.*

(i) *The branching subscheme $B \subset X$ is empty or zero-dimesional.*

(ii) *There exists an effective Cartier divisor $D \subset Y$ that contains the exceptional curve $E \subset Y$ and $\mathcal{O}_E(-D)$ is ample.*

Then for every $r \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that for every vector bundle $\mathcal{F}$ on Y of rank r whose restriction $\mathcal{F}|_{mD}$ is trivial, there exists a vector bundle $\mathcal{E}$ on X of rank r with $f^*\mathcal{E} \simeq \mathcal{F}$.

Proof. Let $Y \xrightarrow{f_0} X_0 \xhookrightarrow{\iota} X$ be the factorization of f over its schematic image. Then every vector bundle on X_0 lifts to a vector bundle on X. To see this, we may assume that the

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nilimmersion ι is given by a coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ with $\mathcal{I}^2 = 0$, so that $\mathcal{I}$ carries the structure of an $\mathcal{O}_{X_0}$-module. Then the obstruction of lifting a locally free $\mathcal{O}_{X_0}$-module $\mathcal{E}_0$ is an element of $\mathrm{H}^2(X_0, \mathcal{I} \otimes_{\mathcal{O}_{X_0}} \mathcal{E}nd(\mathcal{E}_0))$ (see [Ill05, Theorem 5.3]). However, this group vanishes as the support of $\mathcal{I}$ has dimension less than or equal to 1. The branching subscheme $B_0 \subset X_0$ of f_0 equals $\iota^{-1}(B)$ and the f- and f_0-exceptional curve coincides, so the conditions (i), (ii) on f carry over to f_0. So, we may assume that f has schematically dense image. Then the same proof as for [SV04, Proposition 1.2] applies. □

We begin with the preparations for the proof of Theorem 4.5.

LEMMA 4.2. *Let $\mathcal{L}$ be an ample line bundle on a one-dimensional scheme X and let $P \subset X$ be a set of finitely many points. Then there exists a short exact sequence*

$$0 \to \mathcal{O}_X^{\oplus 2} \to (\mathcal{L}^n)^{\oplus 2} \to \mathcal{O}_D \to 0$$

for some effective Cartier divisor $D \subset X$ with $\mathcal{O}_X(D) \simeq \mathcal{L}^{2n}$, $n > 0$, and $D \cap P = \emptyset$.

Proof. Let $s \in \mathrm{H}^0(X, \mathcal{L}^m)$, $m \gg 0$, be a global section that is non-zero over $\mathrm{Ass}(X) \cup P$ (see [Gro61a, 4.5.4]). Then $s : \mathcal{O}_X \to \mathcal{L}^m$ is injective and $\operatorname{coker} s \simeq \mathcal{O}_{V(s)}$ where we consider $V(s) \subset X$ as a Cartier divisor that satisfies $\mathcal{O}_X(V(s)) \simeq \mathcal{L}^m$. Let $t \in \mathrm{H}^0(X, \mathcal{L}^n)$, $n = mk$, $k \gg 0$, be a second global section that is nonzero over $V(s) \cup \mathrm{Ass}(X) \cup P$. Then $t : \mathcal{O}_X \to \mathcal{L}^n$ is injective too, and $V(t) \subset X$ is a Cartier divisor which is disjoint from $V(s)$, and satisfies $\mathcal{O}_X(V(t)) \simeq \mathcal{L}^n$ and $\operatorname{coker} t \simeq \mathcal{O}_{V(t)}$. It follows that $t \oplus s^{\otimes k} : \mathcal{O}_X^{\oplus 2} \to (\mathcal{L}^n)^{\oplus 2}$ is also injective and its cokernel is isomorphic to $\mathcal{O}_{V(t)+kV(s)}$. Then $D := V(t) + kV(s)$ is the desired Cartier divisor. □

LEMMA 4.3. *Let X be a one-dimensional scheme that is proper over A and has an ample line bundle $\mathcal{L}$. Then for every discrete closed subscheme $Z \subset X$ and $a_i \in \mathrm{H}^0(Z, \mathcal{O}_Z)$, $i = 1, 2$, there exist $t_i \in \mathrm{H}^0(X, \mathcal{L}^n)$, $n > 0$, and a regular section $t' \in \mathrm{H}^0(Z, \mathcal{L}^n|_Z)$ such that:*

(i) *$t_i|_Z = a_i t'$ for $i = 1, 2$; and*
(ii) *$X_{t_1} \cup X_{t_2} \supseteq X - Z$.*

Proof. By enlarging Z and extending each a_i by 1, we may assume that every one-dimensional irreducible component of X meets Z in a point where each $a_i = 1$.

Choose $f \in \mathrm{H}^0(X, \mathcal{L}^p)$, $p > 0$, such that $X_f \subset X$ is a dense affine open neighborhood of Z (see [Gro61a, 4.5.4]). Then $V(f)$ is discrete since $\dim(X) \leqslant 1$. Next, pick $g \in \mathrm{H}^0(X, \mathcal{L}^q)$, $q > 0$, such that $X_g \subset X$ is a dense affine open neighborhood of $Z \cup V(f)$. By replacing f with f^q and g with g^p, we found $f, g \in \mathrm{H}^0(X, \mathcal{L}^{pq})$ satisfying $X = X_f \cup X_g$ and $Z \subseteq X_f \cap X_g$. We may assume that $p = q = 1$ by replacing $\mathcal{L}$ with $\mathcal{L}^{pq}$.

For every reduced closed subscheme $P \subset X$ with $\dim(P) \leqslant 0$ and $P \cap Z = \emptyset$ the restriction map $\mathrm{H}^0(X, \mathcal{L}^n) \to \mathrm{H}^0(Z, \mathcal{L}^n|_Z) \oplus \bigoplus_{x \in P} \mathrm{H}^0(\{x\}, \mathcal{L}^n \otimes k(x))$ is surjective for $n \gg 0$ because its cokernel is contained in $\mathrm{H}^1(X, \mathcal{I}_{Z \cup P} \otimes \mathcal{L}^n)$, which is eventually zero. Thus, every $s_0 \in \mathrm{H}^0(Z, \mathcal{L}^n|_Z)$ lifts to global section $s \in \mathrm{H}^0(X, \mathcal{L}^n)$ with $s(x) \neq 0$ for all $x \in P$. Applying this to $P = V(f)$ gives $s_1 \in \mathrm{H}^0(X, \mathcal{L}^{n_1})$, $n_1 \gg 0$, such that $s_1|_Z = a_1 f^{n_1}|_Z$ and $V(s_1) \cap V(f) = \emptyset$. Then $V(s_1)$ is discrete by the initial assumption on Z. By a second application with $P = V(g) \cup (V(s_1) - Z)$, we obtain $s_2 \in \mathrm{H}^0(X, \mathcal{L}^{n_2})$, $n_2 \gg 0$, such that $s_2|_Z = a_2 g^{n_2}|_Z$, $V(s_2) \cap V(g) = \emptyset$ and $V(s_2) \cap V(s_1) \subset Z$.

Define $t' := f^{n_1} g^{n_2}|_Z$ and $n := n_1 n_2$. Then t' is a regular section of $\mathcal{L}^n|_Z$ because $Z \subseteq X_f \cap X_g$. For the restriction of $t_1 := s_1 g^{n_2}$ and $t_2 := s_2 f^{n_1}$ to Z it is true that $t_1 = a_1 f^{n_1} g^{n_2} = a_1 t'$

and $t_2 = a_2 g^{n_2} f^{n_1} = a_2 t'$, proving (i). Also, (ii) follows because the subset $V(s_1 g^{n_2}) \cap V(s_2 f^{n_1})$ is equal to

$$(V(s_1) \cap V(s_2)) \cup (V(s_1) \cap V(f)) \cup (V(s_2) \cap V(g)) \cup (V(f) \cap V(g))$$

and this is a subset of Z by choice of s_1, s_2. □

Proposition 4.4. *Let X be a surface that is proper over A and has an ample line bundle $\mathcal{L}$. Then for every one-dimensional closed subscheme $Y \subset X$, and $N \gg 0$, there exists a vector bundle $\mathcal{E}$ on X whose restriction $\mathcal{E}|_Y$ is trivial and that fits in a short exact sequence*

$$0 \to \mathcal{E} \to (\mathcal{L}^a)^{\oplus 2} \to \mathcal{L}^b|_C \to 0, \tag{4.4.1}$$

for some effective Cartier divisor $C \subset X$ with $\mathcal{O}_X(C) \simeq \mathcal{L}^{2a}$, and $a \geqslant N$, $b \geqslant 3a$.

Proof. By enlarging Y, we may assume that every component of X contains a component of Y and that every embedded point of X lies on Y. Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the coherent ideal defining $Y \subset X$ and suppose that $\mathrm{H}^1(X, \mathcal{I}_Y \otimes \mathcal{L}^k) = 0$ for all $k \geqslant 1$ by replacing $\mathcal{L}$ with a suitable multiple. By replacing $\mathcal{L}$ with $\mathcal{L}^N$, we may assume that $N = 1$.

By Lemma 4.2, there exists an $n \in \mathbb{N}$ and a short exact sequence

$$0 \to \mathcal{O}_Y^{\oplus 2} \to \mathcal{L}^n|_Y^{\oplus 2} \to \mathcal{O}_Z \to 0$$

for some Cartier divisor $Z \subset Y$ satisfying $\mathcal{O}_Y(Z) \simeq \mathcal{L}^{2n}|_Y$ and $Z \cap \mathrm{Ass}(X) = \emptyset$. Then applying the functor $\cdot \otimes \mathcal{L}^{-n}|_Y$ induces a short exact sequence

$$0 \to (\mathcal{L}^{-n})|_Y^{\oplus 2} \longrightarrow \mathcal{O}_Y^{\oplus 2} \xrightarrow{\varphi} \mathcal{O}_Z \to 0. \tag{4.4.2}$$

Claim. There exists an effective Cartier divisor $C \subset X$ with $C \cap Y = Z$ and $\mathcal{O}_X(C) = \mathcal{L}^{2n}$.

By assumption on $\mathcal{L}$, the right-hand side of the exact sequence

$$\mathrm{H}^0(X, \mathcal{L}^{2n}) \to \mathrm{H}^0(Y, \mathcal{L}^{2n}|_Y) \to \mathrm{H}^1(X, \mathcal{I}_Y \otimes \mathcal{L}^{2n})$$

vanishes, so that the regular section $z' \in \mathrm{H}^0(Y, \mathcal{L}^{2n}|_Y)$ defining $Z \subset Y$ lifts to a global section $z \in \mathrm{H}^0(X, \mathcal{L}^{2n})$. Then $X_z \cap Y = Y_{z'} = Y - Z$ meets every irreducible component of X and contains all embedded points of X by assumption on Y and Z. Thus, z is a *regular* section, so that $C = V(z)$ is a Cartier divisor satisfying the asserted properties.

Claim. There exists a surjection $\Phi : \mathcal{O}_X^{\oplus 2} \twoheadrightarrow \mathcal{L}^m|_C$ extending φ, for some $m \geqslant 2n$.

The map φ is given as $(y_1, y_2) \mapsto y_1|_Z a_1 + y_2|_Z a_2$ with $a_i \in \mathrm{H}^0(Z, \mathcal{O}_Z)$. Using Lemma 4.3 for the ample sheaf $\mathcal{L}^{2n}$, we can choose a trivialization $\sigma : \mathcal{O}_Z \xrightarrow{\sim} \mathcal{L}^m|_Z, 1 \mapsto t'$, and $t_i \in \mathrm{H}^0(C, \mathcal{L}^m|_C)$, $i = 1, 2$, with the properties $t_i|_Z = a_i t'$ and $C_{t_1} \cup C_{t_2} \supseteq C - Z$, where $2n|m$. Define the map $\Phi : \mathcal{O}_X^{\oplus 2} \to \mathcal{L}^m|_C$ by $(b_1, b_2) \mapsto b_1|_C t_1 + b_2|_C t_2$. Then $\Phi|_Y = \sigma \circ \varphi$ and the cokernel satisfies $\mathrm{Supp\,coker}\,\Phi \subseteq C - (C_{t_1} \cup C_{t_2}) \subseteq Z$. Using the fact that φ is surjective, we conclude that Φ is surjective and extends φ up to isomorphism of the codomain, as claimed.

Consider now the induced short exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{F} \xrightarrow{\psi} \mathcal{O}_X^{\oplus 2} \xrightarrow{\Phi} \mathcal{L}^m|_C \to 0. \tag{4.4.3}$$

Then $\mathcal{F}$ is a locally free of rank two since $\mathrm{pd}\,\mathcal{L}^m|_C \leqslant 1$.

Claim. For the restriction, $\mathcal{F}|_Y \simeq \mathcal{L}^{-n}|_Y^{\oplus 2}$.

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Using the fact that $\mathcal{F}|_Y$ has no section supported on Z, the restriction of (4.4.3) to Y gives an exact sequence

$$0 \to \mathcal{F}|_Y \xrightarrow{\psi|_Y} \mathcal{O}_Y^{\oplus 2} \xrightarrow{\sigma\circ\varphi} \mathcal{L}^m|_Z \to 0.$$

Then the claim is a consequence of (4.4.2).

Finally, by applying $\cdot \otimes \mathcal{L}^n$ to (4.4.3), we obtain (4.4.1), where $a := n$, $b := m + n$, and $\mathcal{E} := \mathcal{F} \otimes \mathcal{L}^n$. □

THEOREM 4.5. *Let X be a surface that is proper over A. Then there exists a cohomologically 1-ample family of vector bundles of rank two.*

Proof. By Theorem 1.5, there exists an open subset $V \subset X$ which is quasiprojective over A, and $X - V$ consists of finitely many points of codimension two. Then by Nagata there exists a V-admissible blow up $f : Y \to X$ with center Z supported on $X - V$ such that Y is projective over A (see [Con07, 2.6]).

Let us verify for f the descent conditions of Lemma 4.1. Clearly, the branching subscheme of f has dimension less than or equal to 0 since V is thick. Also, the inverse image $D := f^{-1}(Z)$ contains the f-exceptional curve $E \subset Y$. By construction of the blow-up, the closed subscheme $D \subset Y$ is an effective Cartier divisor given by the f-ample invertible inverse image ideal $\mathcal{I}_Z \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$. Thus, $\mathcal{O}_E(-D)$ is ample. Then by Lemma 4.1 there exists an $m \in \mathbb{N}$ such that every vector bundle $\mathcal{F}$ of rank two on Y descends to a vector bundle $\mathcal{E}$ on X if each $\mathcal{F}|_{mD}$ is trivial.

Choose an ample $\mathcal{O}_Y$-module $\mathcal{L}$. By applying Proposition 4.4 to the closed subscheme $mD \subset Y$, there exists for every $n \in \mathbb{N}$ a vector bundle $\mathcal{F}_n$ on Y whose restriction $\mathcal{F}_n|_{mD}$ is trivial, and that fits in a short exact sequence

$$0 \to \mathcal{F}_n \to (\mathcal{L}^{a_n})^{\oplus 2} \to \mathcal{L}^{b_n}|_{C_n} \to 0$$

for some Cartier divisor $C_n \subset Y$ with $\mathcal{O}_Y(C_n) \simeq \mathcal{L}^{2a_n}$ and $a_n \geqslant n$, $b_n \geqslant 3a_n$.

We claim that $(\mathcal{F}_n)$ is cohomologically 1-ample. For that, let $\mathcal{N}$ be a given coherent $\mathcal{O}_Y$-module. Then applying $\cdot \otimes \mathcal{N}$ results in an exact sequence

$$0 \to \mathcal{T}_n \to \mathcal{F}_n \otimes \mathcal{N} \xrightarrow{\varphi_n} (\mathcal{L}^{a_n})^{\oplus 2} \otimes \mathcal{N} \to \mathcal{L}^{b_n}|_{C_n} \otimes \mathcal{N} \to 0$$

for some coherent $\mathcal{O}_Y$-module $\mathcal{T}_n$ supported on C_n. Taking the associated long exact cohomology sequence gives two exact sequences:

$$\mathrm{H}^2(Y, \mathcal{T}_n) \longrightarrow \mathrm{H}^2(Y, \mathcal{F}_n \otimes \mathcal{N}) \longrightarrow \mathrm{H}^2(Y, \operatorname{im} \varphi_n),$$

$$\mathrm{H}^1(Y, \mathcal{L}^{b_n}|_{C_n} \otimes \mathcal{N}) \xrightarrow{\partial} \mathrm{H}^2(Y, \operatorname{im} \varphi_n) \longrightarrow \mathrm{H}^2(Y, (\mathcal{L}^{a_n})^{\oplus 2} \otimes \mathcal{N}).$$

Note that $\mathrm{H}^2(Y, \mathcal{T}_n) = 0$ as $\dim \operatorname{Supp} \mathcal{T}_n \leqslant 1$. Moreover, $\mathrm{H}^2(Y, (\mathcal{L}^{a_n})^{\oplus 2} \otimes \mathcal{N})$ vanishes for $n \gg 0$ using the fact that $\mathcal{L}$ is ample and $a_n \to \infty$. Therefore it suffices to show that $\mathrm{H}^1(Y, \mathcal{L}^{b_n}|_{C_n} \otimes \mathcal{N})$ is eventually zero.

By construction of C_n, there is a short exact sequence

$$0 \to \mathcal{L}^{-2a_n} \to \mathcal{O}_Y \to \mathcal{O}_{C_n} \to 0,$$

and applying $\cdot \otimes \mathcal{L}^{b_n} \otimes \mathcal{N}$ gives the exact sequence

$$0 \to \ker \varphi_n \to \mathcal{L}^{-2a_n+b_n} \otimes \mathcal{N} \xrightarrow{\varphi_n} \mathcal{L}^{b_n} \otimes \mathcal{N} \to \mathcal{O}_{C_n} \otimes \mathcal{L}^{b_n} \otimes \mathcal{N} \to 0.$$

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Taking cohomology and using $\dim \operatorname{Supp} \ker \varphi_n \leqslant 1$ leads to an exact sequence:

$$\mathrm{H}^1(Y, \mathcal{L}^{b_n} \otimes \mathcal{N}) \to \mathrm{H}^1(Y, \mathcal{L}^{b_n}|_{C_n} \otimes \mathcal{N}) \to \mathrm{H}^2(Y, \mathcal{L}^{b_n - 2a_n} \otimes \mathcal{N}).$$

Using the fact that $b_n \geqslant 3a_n$ and $a_n \to \infty$ it follows that $\mathrm{H}^1(Y, \mathcal{L}^{b_n}|_{C_n} \otimes \mathcal{N})$ vanishes for sufficiently large n. This shows that $(\mathcal{F}_n)$ is a 1-ample family of vector bundles of rank two.

Finally, by Lemma 4.1 there exists a family of vector bundles $(\mathcal{E}_n)$ of rank two on X such that $f^*\mathcal{E}_n \simeq \mathcal{F}_n$ for each $n \in \mathbb{N}$, and Proposition 3.7 implies that $(\mathcal{E}_n)$ is 1-ample. □

5. Proof of the resolution property for surfaces

Let us finally collect the preceding results to prove the resolution property for proper surfaces.

Lemma 5.1. *Let $Y \hookrightarrow X$ be a surjective nilimmersion of noetherian schemes given by a coherent ideal $\mathcal{I}$. Suppose that $\dim \operatorname{Supp} \mathcal{I} = 0$. If the resolution property holds for Y, then it also holds for X.*

Proof. Let $\mathcal{M}$ be a given coherent $\mathcal{O}_X$-module. By assumption on Y, there exists a vector bundle $\mathcal{E}_Y$ on Y and a surjection $\mathcal{E}_Y \twoheadrightarrow \mathcal{M}|_Y$. Then $\mathcal{E}_Y$ extends to a vector bundle $\mathcal{E}$ on X because $\dim \operatorname{Supp} \mathcal{I} < 2$. The pullback of the exact sequence $0 \to \mathcal{I}\mathcal{M} \to \mathcal{M} \to \mathcal{M}|_Y \to 0$ along $\mathcal{E} \twoheadrightarrow \mathcal{M}|_Y$ gives a short exact sequence $0 \to \mathcal{I}\mathcal{M} \to \mathcal{N} \to \mathcal{E} \to 0$ for some coherent $\mathcal{O}_X$-module $\mathcal{N}$ that surjects on $\mathcal{M}$. This sequence is split, because $\operatorname{Ext}^1(\mathcal{E}, \mathcal{I}\mathcal{M}) = \mathrm{H}^1(X, \mathcal{E}^\vee \otimes \mathcal{I}\mathcal{M}) = 0$, using the fact that $\dim \operatorname{Supp} \mathcal{I} < 1$. Since $\mathcal{I}\mathcal{M}$ is globally generated it follows that $\mathcal{N} \simeq \mathcal{I}\mathcal{M} \oplus \mathcal{E}$ is a quotient of $\mathcal{O}_X^n \oplus \mathcal{E}$ for some n. This gives a surjection $\mathcal{O}_X^{\oplus n} \oplus \mathcal{E} \twoheadrightarrow \mathcal{N} \twoheadrightarrow \mathcal{M}$. □

Theorem 5.2. *Every surface, which is proper over A, has the resolution property.*

Proof. In light of Lemma 5.1, we may assume that X has no associated point of codimension two, by replacing X with the closed subscheme whose ideal is generated by all local sections with zero-dimensional support.

Let $\mathcal{M}$ be a coherent sheaf and $x \in X$ be an arbitrary point. By Proposition 1.8, there exists a coherent sheaf $\mathcal{F}$ which is almost anti-ample near x. In particular, for every $m \gg 0$ there exists a map $(\mathcal{F}^{\otimes m})^{\oplus n} \to \mathcal{M}$, for some $n \in \mathbb{N}$, which is surjective near x. Therefore, it suffices to find a locally free resolution for every $\mathcal{F}^{\otimes m}$ and every $m \gg 0$ using the fact that X is quasicompact.

By definition, $\mathcal{F}$ is invertible outside finitely many points of codimension two. So, by Proposition 2.7, for every $m \gg 0$ there exists a surjection $\mathcal{G} \to \mathcal{F}^{\otimes m}$ for some coherent sheaf $\mathcal{G}$ satisfying F_1 because $\mathrm{H}^2(X, (\mathcal{F}^{\otimes m})^\vee \otimes (\mathcal{F}^{\otimes m})^\vee) \simeq \mathrm{H}^2(X, (\mathcal{F}^\vee)^{\otimes 2m})$ (Lemma 2.1) vanishes using the fact that $\mathcal{F}^\vee$ is 1-ample by Corollary 3.5.

By Theorem 4.5, there exists a 1-ample family $\mathcal{E}_n$, $n \in \mathbb{N}$, of vector bundles of rank two. So, for all $n \gg 0$ and all $m \geqslant 0$, it is true that $\mathrm{H}^2(X, \mathcal{G}^\vee \otimes \mathcal{E}_n^{\oplus m}) \simeq \mathrm{H}^2(X, \mathcal{G}^\vee \otimes \mathcal{E}_n)^{\oplus m} = 0$. Consequently, $\mathcal{G}$ admits a surjection $\mathcal{H} \twoheadrightarrow \mathcal{G}$ by a vector bundle $\mathcal{H}$, as a result of Proposition 2.4. □

As the resolution property descends along immersions, it holds for all two-dimensional schemes X which are embeddable into two-dimensional schemes that are proper over a noetherian base ring.

Corollary 5.3. *Suppose that A is an excellent Jacobson ring such that each irreducible component of $\operatorname{Spec} A$ is equicodimensional (for example, if $A = \mathbb{Z}$, or if A is a field). Then every two-dimensional scheme, which is separated and of finite type over A, satisfies the resolution property.*

Proof. Since X is separated and of finite type over a noetherian ring, there exists a proper A-scheme $\overline{X}$ together with an open immersion $X \hookrightarrow \overline{X}$, which identifies X as a dense open

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subscheme of $\overline{X}$, by Nagata's embedding theorem [Con07, 4.1]. The assumptions on A guarantee that $\dim(\overline{X}) = \dim(X) = 2$ since X is of finite type over A (see [Gro66, 10.6.2]). Thus, Theorem 5.2 implies that $\overline{X}$, and hence X has the resolution property. $\square$

Remark 5.4. In the first section we extended the resolution property from a dense affine open neighborhood $U_0 \subset X$ of a given point $x \in X$ to a thick open quasiprojective subset $U_0 \subset U_1 \subset X$, i.e. we added all points of codimension one. The results of § 2 can be adapted to formulate conditions for extending the resolution property to an open subset $U_1 \subset U_2 \subset X$ which contains all points of codimension two. The cohomological obstructions lie then in second cohomology groups of coherent $\mathcal{O}_{U_2}$-modules. By shrinking U_2, one can arrange that U_2 has cohomological dimension less than or equal to 2 and we believe that U_2 will satisfy the resolution property. For technical reasons we assumed here that X is two-dimensional and proper over a noetherian ring, so that $U_2 = X$ is proper, allowing us to control the cohomological obstructions.

Even the case when the cohomological dimension, the affine covering number or the affine stratification number of X is equal to one seems to be difficult with our approach, because removing closed subschemes may increase such an invariant.

Finally, our methods rely on the fact that the resolution property holds Zariski locally. We do not know how to extend the techniques to algebraic spaces (or algebraic stacks).

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Philipp Gross gross@math.uni-duesseldorf.de
Mathematisches Institut, Heinrich-Heine-Universität,
D-40225 Düsseldorf, Deutschland, Germany