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A FREE BOUNDARY PROBLEM IN AN ANNULUS

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Abstract

If Ω is a ring region with starlike boundary components α and β, then we show for each $\lambda > 0$ there exists a ring region $\omega \subset \Omega$ with $\partial\omega = \alpha \cup \gamma$, $\alpha \cap \gamma = \varnothing$ such that there is a harmonic function V in ω satisfying (a) $V(z) = 0$ for $z \in \alpha$, (b) $V(z) = 1$ for $z \in \gamma$, (c) $|\operatorname{grad} V(z)| = \lambda$ for $z \in \gamma \cap \Omega$. Furthermore, we show when ω is not equal to Ω; that is, there is a non-trivial solution.

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1. Introduction

Let $\mathcal{D}$ be an unbounded doubly connected region which has for a boundary a connected and compact set α that is not equal to single point. Suppose $\mathcal{C}$ is the collection of all doubly connected regions $\omega \subset \mathcal{D}$ which have α as one boundary component. This paper concerns a type of free boundary problem. Let us fix $\Omega \in \mathcal{C}$. Given $\lambda > 0$, do there exist bounded $\omega \in \mathcal{C}$ and a harmonic function V_ω such that $\omega \subset \Omega$ and V_ω satisfies

(a) $V_\omega(z) = 0$ for $z \in \alpha$,

(b) $V_\omega(z) = 1$ for $z \in \partial\omega - \alpha$,

(c) $|\operatorname{grad} V_\omega(z)| = \lambda$ for $z \in (\partial\omega - \alpha) \cap \Omega$.

For each $\omega \in \mathcal{C}$, the harmonic function V_ω satisfying (a) and (b) which exists by the Riemann-Dirichlet principle will be called the stream function of ω. Also, for $\omega \in \mathcal{C}$, $\partial\omega - \alpha$ will be called the free boundary of ω. In [4], Beurling studied the more general problem where (c) is replaced by:

(c′) $|\operatorname{grad} V_\omega(z)| = Q(z)$ for $z \varepsilon (\partial\omega - \alpha) \cap \Omega$

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where Q is a positive and continuous function in Ω. Such a region $\omega \in \mathcal{C}$ whose stream function V_ω satisfies (a), (b) and (c) or (a), (b) and (c′) will be called a solution (for the value λ or the function Q respectively). In particular, for a fairly general class of functions Q, Beurling proved the following theorem in [4].

THEOREM A. *If $\Omega = \mathfrak{D}$ and there exists a bounded $\omega_1 \in \mathcal{C}$ with stream function V_1 such that for ζ on the free boundary of ω_1 we have*

$$\limsup_{\substack{z \to \zeta \\ z \in \omega}} \frac{|\operatorname{grad} V_1(z)|}{Q(z)} < 1, \tag{1}$$

then there exists a solution $\omega_0 \subset \omega_1$. If, in addition, there exists $\omega_2 \in \mathcal{C}$ with stream function V_2 such that $\omega_2 \subset \omega_1$ and for ζ on the free boundary of ω_2 we have

$$\liminf_{\substack{z \to \zeta \\ z \in \omega}} \frac{|\operatorname{grad} V_2(z)|}{Q(z)} > 1, \tag{2}$$

then $\omega_2 \subset \omega_0 \subset \omega_1$.

We say α is starlike if for each $z \in \alpha$ we have $\mathfrak{D} \cap \{\rho z\colon 0 \leqslant \rho \leqslant 1\} = \varnothing$. For $\omega \in \mathcal{C}$, we say the free boundary of ω is starlike if for each $z \in \beta$, we have $\omega \cap \{\rho z\colon \rho \geqslant 1\} = \varnothing$. In [6] it is shown that if α is starlike and $\Omega = \mathfrak{D}$, then for each $\lambda > 0$, there exists a unique solution which has a starlike free boundary. Acker [1] generalized this result:

THEOREM B. *If $\Omega = \mathfrak{D}$, α is starlike and $\rho Q(\rho z)$ is a non-decreasing function of ρ for each $z \in \mathfrak{D}$, then there exists a unique solution which has a starlike free boundary.*

In Section 2, we prove that if both boundary components of Ω are starlike, then for each $\lambda > 0$ there exists a solution which has a starlike free boundary. We prove this by taking limits of solutions for a sequence of functions $\{Q_n\}_{n=1}^{\infty}$ where each Q_n satisfies Acker's monotonicity property. A similar idea is used in [3] to solve a different problem. We will require the following result whose proof is a simple consequence of Theorem A.

THEOREM C. *If Q_1 and Q_2 are continuous and positive functions in $\mathfrak{D}$ with $Q_1 \geqslant Q_2$ and if for $\Omega = \mathfrak{D}$ there are unique solutions for both Q_1 and Q_2 which we respectively denote ω_1 and ω_2, then $\omega_1 \subset \omega_2$.*

For the rest of this paper we let β be the free boundary of Ω and suppose both α and β are starlike. We observe that for any $\omega \in \mathcal{C}$, one of the following must hold:

(i) $(\partial\omega - \alpha) \cap \Omega = \varnothing$,

(ii) $(\partial\omega - \alpha) \cap \Omega$ is a proper subset of the free boundary of ω,

(iii) $(\partial\omega - \alpha) \subset \Omega$.

If $\omega \in \mathcal{C}$ and satisfies (i), then $\omega = \Omega$ and (c) is vacuously true; hence, Ω is a trivial solution. It follows from Theorem A that if λ is sufficiently large, there will be a solution satisfying (iii). In Section 3, we show when there are non-trivial solutions which satisfy (ii).

2. Existence

THEOREM 1. *For each $\lambda > 0$, there exists a solution ω which has a starlike free boundary.*

PROOF. We first suppose α and β are analytic curves and remove this condition at the end of the proof. For the case where $\Omega = \mathfrak{D}$, see [6]. If $\Omega \neq \mathfrak{D}$, then we let $w = f(z)$ be a schlicht mapping of Ω onto $\{w\colon 1 < |w| < R\}$ such that α corresponds to $\{w\colon |w| = 1\}$ and β corresponds to $\{w\colon |w| = R\}$. We note that if $g(w) = z$ is the inverse of f, then

$$\frac{\partial \arg g(w)}{\partial \arg w} = \operatorname{Re} \frac{w g'(w)}{g(w)} > 0 \tag{3}$$

for $1 < |w| < R$. This implies that

$$\frac{\partial |f(z)|}{\partial |z|} > 0. \tag{4}$$

Let V_Ω be the stream function of Ω and

$$\mu = \sup_{z \in \beta} |\operatorname{grad} V_\Omega(z)|\,. \tag{5}$$

If $\mu < \lambda$, then the result follows from Theorem A. If $\mu = \lambda$, then replace λ by $\lambda - \varepsilon$, apply Theorem A and then let $\varepsilon \to 0$. Therefore we must consider the case where $\mu > \lambda$. For integers $n > 1/(R - 1)$, we define:

$$\begin{aligned} &q_n(z) = 1, \quad \text{if } 1 < |f(z)| \leqslant R - 1/n, \\ &q_n(z) = 1/n(R - |f(z)|), \quad \text{if } R - 1/n \leqslant |f(z)| \leqslant R - \lambda/n\mu, \\ &q_n(z) = \mu/\lambda, \quad \text{if } R - \lambda/n\mu \leqslant |f(z)| < R. \end{aligned} \tag{6}$$

We then define

$$(7)\qquad \begin{aligned} Q_n(z) &= \lambda q_n(z), && \text{if } z \in \Omega, \\ Q_n(z) &= \mu, && \text{if } z \in \mathfrak{D} - \Omega. \end{aligned}$$

From (4) it follows that Q_n satisfies the monotonicity property of Theorem B. Therefore, for $n > 1/R - 1$, there exists a unique solution for the function Q_n which we denote by ω^n. By (6), we have $|\operatorname{grad} V_\Omega(z)| < \mu \leqslant Q_n(z)$ which implies by Theorem A tht $\omega^n \subset \Omega$ for all $n > 1/R - 1$. Furthermore, since $Q_{n+1} \leqslant Q_n$, by Theorem C we see that $\omega = \bigcup_n \omega^n$ is a solution.

In the general case where α and β are not analytic curves, we solve the free boundary problem for the sequence of regions

$$\Omega_m = \{z \in \Omega\colon 1/m < V_\Omega(z) < 1 - 1/m\}$$

and take the limit of the sequence of solutions as $m \to \infty$.

COROLLARY. *Suppose that ω is the solution found in Theorem 1 and V_ω is the stream function of ω. If ζ belongs to the free boundary of ω, then*

$$(8)\qquad \liminf_{\substack{z \to \zeta \\ z \in \omega}} \left(\frac{|\operatorname{grad} V_\omega(z)|}{\lambda} \right) \geqslant 1.$$

PROOF. In the proof of Theorem 1, if V_n is the stream function of ω^n and z belongs to the free boundary of ω^n, then $|\operatorname{grad} V_n(z)| \geqslant Q(z) \geqslant \lambda$. The result follows by taking limits.

3. Properties of the free boundary

If $\lambda > 0$, the solution found in Theorem 1 wiill be denoted ω_λ. For the case where $\Omega = \mathfrak{D}$, we denote the solution by $\hat{\omega}_\lambda$. We have the following theorem.

THEOREM 2. $\omega_\lambda \subset \hat{\omega}_\lambda \cap \Omega$.

Before proving this theorem we make the following remark. If $\Omega \neq \omega_\mu$ where μ is defined by (5), then there are non-trivial solutions which satisfy (ii). Furthermore, if Ω is unbounded, there will be non-trivial solutions for all values of λ and there exists λ_0 such that ω_λ satisfies (i) for all $\lambda \leqslant \lambda_0$.

PROOF OF THEOREM 2. Let $\sigma_R \in \mathcal{C}$ have for its free boundary the circle $|z| = R$. If V_R is the stream function of σ_R, then it is easy to show that for R sufficiently large, V_R will satisfy (1). By (8), if V_λ is the stream function of ω_λ, we see that V_λ satisfies (2). Hence by Theorem A, for large R we have $\omega_\lambda \subset \hat{\omega}_\lambda \subset \sigma_R$.

We shall omit the proof of the next theorem since it is essentially the same as for the case where $\Omega = \mathfrak{D}$ which is given in [5].

THEOREM 3. *If α and β are convex, then for each $\lambda > 0$, the free boundary of ω_λ is convex. Furthermore, if V_λ is the stream function of ω_λ and $z \in \omega_\lambda$, then*

$$|\operatorname{grad} V_\lambda(z)| \geqslant \lambda. \tag{9}$$

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