

QUOTIENT GROUPS AND REALIZATION OF TIGHT RIESZ GROUPS

Dedicated to the memory of Hanna Neumann

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1. Introduction

Let (G, \ll) be an l -group having a compatible tight Riesz order \leq with open-interval topology U , and H a normal subgroup. The first part of the paper concerns the question: Under what conditions on H is the structure of $(G, \ll, \wedge, \vee, \leq, U)$ carried over satisfactorily to $G'_H \equiv G/H$ by the canonical homomorphism; and its answer (Theorem 8°): H should be an l -ideal of (G, \ll) closed and not open in (G, U) . Such a normal subgroup is here called a tangent. An essential step is to show that \ll' is the associated order of \leq' .

If H is a maximal tangent then G'_H is fully ordered. The second part of the paper shows that there is a natural realization ρ of G as a subdirect product of the groups G'_H which is an order isomorphism for \leq as well as \ll , if for example (G, \ll) is lattice-complete and \leq is non-androgynous, and all maximal tangents are replete. The lattice-completeness requirement can be relaxed to weak projectability. But if \leq is androgynous then ρ fails to be one-one. Extra conditions ensure that ρ is also a topological embedding of G in $A = \prod_H G'_H$ and that ρ is concordant. The topology used on A is the open-interval topology. The main results are Theorems 15° and 18°.

A realization theory for androgynous groups — those for which not every element $a > 0$ is a weak unit of (G, \ll) — remains an open question. We postpone to a later paper the use of the present realization to construct a Gelfand theory by topologizing the set of maximal tangents.

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2. Preliminaries

A poset (X, \leq) is said to have the tight Riesz (m, n) property (to be $TR(m, n)$, for short) when

$$a_i < b_j \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n$$

for elements $a_1, \dots, a_m, b_1, \dots, b_n$ implies the existence of an element x such that $a_i < x < b_j$.

The loose Riesz (m, n) property ($= LR(m, n)$) is defined by replacing $<$ by \leq at all occurrences. It is easily shown that $TR(2, 2) \Leftrightarrow TR(m, n)$, and $LR(2, 2) \Leftrightarrow LR(m, n)$ for all $m \geq 2, n \geq 2$; and $TR(1, 2) \Rightarrow TR(2, 2) \Rightarrow LR(2, 2)$. Also $[TR(1, 2) \text{ and } TR(2, 1) \text{ and } LR(2, 2)] \Leftrightarrow TR(2, 2)$. $TR(1, 2)$ implies order-denseness. For some properties of these interpolation axioms on posets see Cameron and Miller [1]. A poset (X, \leq) is said to be an *antilattice* if only the trivial meets and joins exist.

A *tight Riesz (1, 2) group* ($= TR(1, 2)$ group) is a directed partially ordered group (G, \leq) with the $TR(1, 2)$ (equivalently, with the $TR(2, 1)$) property. (In [4], [5] and [6] it was assumed that G is commutative. Here we do not insist on commutativity.) The *open-interval topology* U is the topology on G having as subbase (in fact, as base) the set of all open intervals $(a, b), a < b$. For any $x \in G$, the sets $(x - a, x + a), a > 0$, form a base for U at x . For the positive cone and strict positive cone write

$$P = \{x \in G : x \geq 0\}, \quad P^* = P \setminus \{0\};$$

these are normal subsets of the group. The topological boundary ∂P of P with respect to U consists of 0 together with the *pseudopositives* of (G, \leq) , i.e. the elements $p \not\geq 0$ such that $x > 0 \Rightarrow x + p > 0$, or equivalently, such that $x > 0 \Rightarrow p + x > 0$. The elements of $\partial P \cap (-\partial P) = \bar{P} \cap (-\bar{P})$ are 0 together with the *pseudozeros*. The set $\bar{P} = P \cup \partial P$ is the positive wedge of the *associated preordering* on G , written \preceq :

$$\bar{P} = \{x \in G : x \succcurlyeq 0\} = \{0\} \cup P^* \cup \{\text{pseudopositives}\}.$$

Note that

$$(2.1) \quad \begin{aligned} x > 0 &\Rightarrow x \succ 0; \\ x < y \prec z &\Rightarrow x < z; \quad x \prec y < z \Rightarrow x < z. \end{aligned}$$

The topological lemma Section 2, 6° of Loy and Miller [4] is valid for all not-necessarily-commutative $TR(1, 2)$ groups. The associated preordering for a poset is discussed in Cameron and Miller [1].

1° THEOREM. *Let (G, \leq) be a $TR(1, 2)$ group. Then with the above notation*

- (i) (G, U) is a topological group, and U is not discrete;

- (ii) (G, \leq) is an order-dense antilattice;
- (iii) U is Hausdorff $\Leftrightarrow (G, \leq)$ has no pseudozeros $\Leftrightarrow \preceq$ is a partial order.
- (iv) If U is Hausdorff, then (G, \preceq) is a partially ordered group.

PROOF. see Fuchs [3], p 20, and Loy and Miller [4].

We are concerned chiefly with the case where (G, \preceq) is an l -group; then \wedge, \vee denote the lattice operations with respect to \preceq . Given a priori a partially ordered group (G, \preceq) , any TR(1,2) partial order \leq on G having \preceq as its associated order (and therefore having no pseudozeros) will be called a *compatible tight Riesz order* for (G, \preceq) (CTRO, for short). Tight Riesz groups looked at in this light have been discussed by Wirth [11] and Reilly [7].

2°. Let (G, \preceq) be an l -group with compatible tight Riesz order \leq . Then

- (i) (G, \preceq) is LR(2,2) and (G, \leq) is TR(2,2);
- (ii) (G, \wedge, \vee, U) is a topological lattice;
- (iii) \preceq and \leq are both isolated orders (i.e. $nx \succ 0$ for some positive integer n implies $x \succ 0$; and similarly for $>$).

PROOF. (i) Cameron and Miller [1], p. 10; (ii) [4], p. 236; (iii) [4], p. 236, Reilly [7], p. 11.

We are interested in carriers and l -ideals. Let (G, \preceq) be an l -group. An l -ideal of G is a convex directed normal subgroup; it is necessarily also a sublattice. Given a subset $Q \subseteq G$, $\text{lid}(Q)$ denotes the l -ideal generated by Q , i.e. the intersection of all l -ideals containing Q . An o -ideal of a partially ordered group is a convex directed normal subgroup.

Again, write

$$Q^\perp = \{x \in G : |x| \wedge |q| = 0 \text{ for all } q \in Q\},$$

and abbreviate $\{a\}^\perp$ to a^\perp . Q^\perp is a convex subgroup and a sublattice; however, Q^\perp is an l -ideal for every $Q \subseteq G$ if and only if all carriers of G are invariant (Fuchs [2], p. 82). Always $Q \subseteq Q^{\perp\perp}$.

The relation $a^\perp = b^\perp$ is an equivalence relation on G ; the intersections of its classes with the positive cone are called the *carriers* of G ; for $a \succ 0$ the carrier of a is

$$\hat{a} = \{b \succ 0 : a \wedge x = 0 \text{ if and only if } b \wedge x = 0\}.$$

The partial order \preceq induces on \mathfrak{C} , the set of all carriers, a partial order

$$(2.2) \quad \hat{a} \preceq \hat{b} \text{ if and only if } b \wedge x = 0 \Rightarrow a \wedge x = 0;$$

(\mathfrak{C}, \preceq) is a distributive lattice.

By a *weak unit* of G we mean a weak unit of the l -group (G, \preceq) i.e. an element $w \succ 0$ such that $w \wedge x = 0 \Rightarrow x = 0$. Let \mathfrak{w} denote the set of weak units; it may be

empty, but if not empty then it is a carrier of G , to wit, the greatest carrier. In fact (\mathbb{C}, \leq) has a greatest element, \mathfrak{w} , if and only if $\mathfrak{w} \neq \emptyset$.

Now suppose that \leq is a compatible tight Riesz order of the l -group (G, \leq) , and consider the following property involving \leq and \leq :

(A) For all $x, y \in G$,

$$x > x \wedge y \Rightarrow x > y,$$

and its dual formulation

(A') For all $x, y \in G$,

$$x < x \vee y \Rightarrow x < y.$$

Each implies the other. If (A) (equivalently, (A')) fails to hold in G , we call (G, \leq) (or perhaps \leq) *androgynous*, otherwise it is called *non-androgynous*. For an example of an androgynous group one can take $G = \mathbb{R} \times \mathbb{R}$, with $\langle x_1, x_2 \rangle > 0$ if and only if $x_1 > 0, x_2 \geq 0$; here $\langle x_1, x_2 \rangle \geq 0$ if and only if $x_1 \geq 0, x_2 \geq 0$. There are a number of equivalent formulations of property (A).

3°. Let (G, \leq) be an l -group with compatible tight Riesz order \leq . Each of the following properties is separately equivalent to (A): for all x, y :

(i) If $x \wedge y$ is neither x nor y , then both x, y belong to $x \wedge y + \partial P$.

(ii) If $x \vee y$ is neither x nor y , then $x \vee y$ belongs to both $x + \partial P$ and $y + \partial P$.

(iii) $x > 0, y > 0 \Rightarrow x \wedge y > 0$.

(iv) $P^* \subseteq \mathfrak{w}$.

PROOF. The pairwise equivalence of (A), (A'), (i), (ii) and (iii) is easily checked. Consider (A) \Leftrightarrow (iv). Assume (A) and let $a > 0$. Then $x \wedge a = 0 \Rightarrow a > x \wedge a \Rightarrow x = x \wedge a \Rightarrow x = 0$, so $a \in \mathfrak{w}$. Thus $P^* \subseteq \mathfrak{w}$. Conversely assume (iv). Let $x > x \wedge y$; the element $z = x - x \wedge y = 0 \vee (x - y)$ being in P^* is a weak unit. Let $a = y - x \wedge y$; we have

$$\begin{aligned} z \wedge a &= [0 \vee (x - y)] \wedge [0 \vee (y - x)] \\ &= 0 \vee [(x - y) \wedge (y - x)] = 0 \end{aligned}$$

so $a = 0$. Thus (A) holds.

Thus when G is non-androgynous it has weak units. If G is androgynous it may still happen that $\mathfrak{w} \neq \emptyset$, but then necessarily $P^* \not\subseteq \mathfrak{w}$. An l -group (G, \leq) can possess both androgynous and non-androgynous compatible tight Riesz orders.

4°. Let the G in 3° be non-androgynous. A necessary and sufficient condition for $P^* = \mathfrak{w}$ is: for every $x \in \partial P$ there exists $y \in \partial P, y \neq 0$, with $x \wedge y = 0$.

More generally, for $x \geq 0$ we have

$$(2.3) \quad x \in \bar{P} \setminus \mathfrak{w} \Leftrightarrow x^\perp \neq (0) \Leftrightarrow \exists y \in \partial P, y \neq 0, x \wedge y = 0.$$

PROOF. (2.3) is easily verified, and implies the first statement.

3. Quotient groups

Throughout this section let (G, \leq) be a nontrivial $TR(1,2)$ group without pseudozeros, with open-interval topology U and associated ordering \leq ; let H be a normal subgroup, $H \neq (0)$, $G' = G/H$, and let $\theta: G \rightarrow G'$, $a \mapsto a' = a + H$, denote the canonical homomorphism.

To ensure that a reasonable sufficiency of the structure of (G, \leq, \ll, U) (and \wedge, \vee when they exist) can be carried over to G' , some restrictions need to be placed on H . This section is devoted to finding suitable circumstances. First, for the quotient orders \leq', \ll' to exist we need H to be \leq -convex and \ll -convex. Since \ll -convexity implies \leq -convexity, both orders exist if H is \ll -convex, and then

$$(3.1) \quad \begin{aligned} a' > '0' &\Leftrightarrow a + h > 0 \text{ for some } h \in H, a \notin H \\ &\Leftrightarrow a - P^* \text{ meets } H, a \notin H, \end{aligned}$$

$$(3.2) \quad \begin{aligned} a' \gg '0' &\Leftrightarrow a + h \gg 0 \text{ for some } h \in H \\ &\Leftrightarrow a - \bar{P} \text{ meets } H; \end{aligned}$$

and θ is order-preserving, i.e. $a > 0 \Rightarrow a' \geq '0'$ and $a \gg 0 \Rightarrow a' \gg '0'$. Hence (G', \leq') and (G', \ll') are partially ordered groups.

5°. Under the assumptions of the first paragraph:

(i) G' is directed with respect to \leq' and \ll' .

(ii) H is \leq -directed if and only if every coset $a + H$ is \leq -directed, and likewise for \ll .

(iii) If H is \leq -directed then H meets P^* . If H is \ll -directed then H meets $\bar{P} \setminus \{0\}$.

(iv) If H is \leq -convex then H is open in (G, U) if and only if H meets P^* . So if H is \leq -directed and \leq -convex then H is open.

Suppose H is \leq -convex. Then:

(v) If H is not open then (G', \leq') is a $TR(1,2)$ group.

(vi) If (G, \leq) is $LR(2,2)$ and H is \ll -directed then (G', \leq') is $LR(2,2)$.

(vii) If (G, \leq) is $TR(2,2)$, and H is closed not open and \ll -directed, then (G', \leq') is a $TR(2,2)$ group.

(viii) If (G, \ll) is $LR(2,2)$ and H is \ll -directed and \ll -convex (i.e. H is an o -ideal), then (G', \ll') is $LR(2,2)$.

PROOF. (i)–(iv) are straightforward. (v) Let $a' < 'b'_1, b'_2$ in G' . Then $a < b_{11}, b_{21}$ for some $b_{11} \in b'_1, b_{21} \in b'_2$; since (G, \leq) is $TR(1,2)$ there exists c such that

$$a < c < b_{11}, b_{21}.$$

Because H does not meet P^* , $x > 0$ implies $x' > '0'$.

Thus

$$a' < 'c' < 'b'_1, b'_2.$$

Thus (G', \leq') is $\text{TR}(1, 2)$ and being \leq' -directed is a $\text{TR}(1, 2)$ group.

(vi) The proof of Fuchs [3], p. 14 can be adapted to this case.

(vii) The conditions and (v), (vi) ensure that (G', \leq') is $\text{TR}(1, 2)$ and $\text{LR}(2, 2)$ and these together imply $\text{TR}(2, 2)$.

(viii) This is the result of Fuchs referred to in the proof of (vi).

EXAMPLE. Let $G = \mathbb{R} \times \mathbb{Z}$, with $\langle x, m \rangle > 0$ if and only if $x > 0, m \geq 0$. (G, \leq) is a $\text{TR}(2, 2)$ group without pseudozeros. If $H = \mathbb{R} \times (0)$ then H is \leq -convex, \leq -directed, closed and open; but $G' \cong \mathbb{Z}$ is not $\text{TR}(1, 2)$.

Proposition 5° and the preceding remarks show that for a desirable theory we should require at least that H be $\neq (0)$, \leq -convex, so \leq -convex, closed and not open, and \leq -directed. (Since H does not meet P^* , it cannot be \leq -directed, and \leq -convexity is satisfied vacuously.) In other words, when (G, \leq) is an l -group, H should be a closed l -ideal not meeting P^* . Since it must meet $\partial P \setminus (0)$, we call such an H a *tangent*.

Topological as well as interpolation considerations require that H be closed not open. For let \mathcal{Q} denote the *quotient topology* on G' , i.e. the strongest topology making $\theta: (G, U) \rightarrow G'$ continuous; a subset $Q \subseteq G'$ belongs to \mathcal{Q} if and only if $Q = \theta(U)$ for some $U \in \mathcal{U}$, and here one can take in particular $U = \theta^{-1}(Q)$; θ is an open mapping. It is known that (G', \mathcal{Q}) is a topological group, is Hausdorff if and only if H is closed in (G, U) , and is discrete if and only if H is open. If U' denotes the open-interval topology of (G', \leq') , and this is a TR group, then U' is not discrete, so certainly $U' \neq \mathcal{Q}$ if H is open.

Again, $U' \neq \mathcal{Q}$ if (G', \leq') has pseudozeros while H is closed. Suppose it is known that (G', \leq') has no pseudozeros: then there is on G' the associated order \prec of \leq' , as well as \leq' , the quotient order of \leq ; and in general it is not clear that \prec and \leq' coincide, as one might hope.

We proceed to show that these troubles do not arise, and there are other benefits, when (G, \leq) is an l -group and H is a tangent. The main result is Theorem 8°, which we reach by two lemmas.

6°. *If H is \leq -convex and closed not open, then the map $\theta: (G, U) \rightarrow (G', U')$ is continuous, so $U' \subseteq \mathcal{Q}$; and $U' = \mathcal{Q}$ if and only if this map is also open.*

PROOF. By 5° (iv), H does not meet P^* , so

$$(3.3) \quad x > 0 \Rightarrow x' > '0'.$$

Consider a base open set $V \equiv (a', b')$, $a' < 'b'$, of U' , and $x_0 \in \theta^{-1}(V)$. There exist $h_1, h_2 \in H$ such that $a + h_1 < x_0 < b + h_2$; then $U = (a + h_1, b + h_2)$ is

open in G , contains x_0 , and $U \subseteq \theta^{-1}(V)$ because of (3.3). So $\theta^{-1}(V)$ is open, θ is continuous, $U' \subseteq Q$.

If $U' = Q$ then $U \in U \Rightarrow \theta(U) \in Q \Rightarrow \theta(U) \in U'$ so θ is open. Conversely if θ is open and $V \in Q$, then $V = \theta(W)$ for some $W \in U$, and so $V \in U'$.

7°. If H is \leq -convex and closed not open, then the property

$$(3.4) \quad \left[\begin{array}{l} x < y; \text{ there exist } h_1, h_2 \in H \\ \text{such that } x < h_1 \text{ and } h_2 < y \end{array} \right] \Rightarrow \left[\begin{array}{l} \text{there exists } h_3 \in H \\ \text{such that } x < h_3 < y \end{array} \right]$$

implies

$$(3.5) \quad \theta(a, b) = (a', b') \text{ for all } a < b,$$

and hence $U' = Q$.

PROOF. Let $a < b$. Since $a < x < b \Rightarrow a' < x' < b'$, necessarily $\theta(a, b) \subseteq (a', b')$. Let $\xi \in (a', b')$, say $\xi = x'$; there exist $h_1, h_2 \in H$ such that $a - x < h_1$ and $h_2 < b - x$, also $a - x < b - x$, so by (3.4) there exists $h_3 \in H$ for which $a < x + h_3 < b$, and here $x + h_3 \in \xi$. Therefore $\xi \in \theta(a, b)$. This proves (3.5) which in turn implies that θ is open, so $U' = Q$.

8°. THEOREM. Let (G, \leq) be an l -group with a compatible tight Riesz order \leq ; let H be a tangent, and $G' = G/H$. Then (G', \leq') is an l -group with compatible tight Riesz order \leq' ;

$$\theta: (G, U) \rightarrow (G', U'), \quad x \mapsto x'$$

is a continuous open map as well as a group and lattice homomorphism; and $U' \equiv Q$ is Hausdorff.

PROOF. Since (G, \leq) is a lattice, it is LR(2, 2). We verify (3.4) of 7°. Let $x < y$, $x < h_1$ and $h_2 < y$ with $h_1, h_2 \in H$. If $h_1 \wedge h_2 = h_1 \vee h_2$ then $h_1 = h_2$ and (3.4) is verified trivially. Assume $h_1 \wedge h_2 < h_1 \vee h_2$. By the tight Riesz property of (G, \leq) there exists x_1 such that $x < x_1 < y, h_1$ and then also y_1 such that $x_1, h_2 < y_1 < y$. The LR(2, 2) property for (G, \leq) then implies the existence of k such that

$$x_1, h_1 \wedge h_2 \leq k \leq h_1 \vee h_2, y_1.$$

Since (H, \leq) is an l -ideal, $k \in H$; and $x < k < y$. This proves (3.4).

By 7°, $U' = Q$; since H is closed, this topology is Hausdorff, so (G', \leq') has no pseudozeros. Since H is an l -ideal, (G', \leq') is an l -group and θ is a lattice homomorphism. By 5° (vii), (G', \leq') is a TR(2, 2) group.

It remains to prove that \leq' coincides with \prec , the associated order of \leq' . The cones of \leq', \leq' and \prec are respectively $\theta(P), \theta(\bar{P})$ and $\overline{\theta(P)}$; since θ is continuous, $\theta(\bar{P}) \subseteq \overline{\theta(P)}$, so

$$(3.6) \quad a' > '0' \Rightarrow a' \succ '0' \Rightarrow a' \succneq '0'.$$

Next, note that

$$(3.7) \quad a' \prec b' < c' \Rightarrow a' < c',$$

and

$$(3.8) \quad \text{if } a' > '0' \Rightarrow a' + x' \succneq '0', \text{ then } x' \succneq '0'.$$

Here (3.7) follows immediately from (3.6) and the fact that \prec is the associated order of \leqq . Assume the premise of (3.8), and $a' > '0'$. Since (G', \leqq') is TR, there exists b' such that $a' > b' > '0'$; then $a' + x' > b' + x' \succneq '0'$ so (3.7) gives $a' + x' > '0'$. This proves $x' \succneq '0'$.

Let $a', b' \in G'$. By (3.6), $a' \wedge b' \leqq' a', b'$ gives

$$a' \wedge b' \prec a', b'.$$

Suppose $x' \prec a', b'$. Then $u' > '0'$ implies $x' < a' + u'$, and $x' < b' + u'$, so

$$x' \leqq' (a' + u') \wedge (b' + u') = a' \wedge b' + u'.$$

Thus $u' > '0' \Rightarrow u' + a' \wedge b' - x' \succneq '0'$, so (3.8) gives

$$x' \prec a' \wedge b'.$$

Therefore $a' \wedge b'$ is also the infimum of a', b' with respect to \prec ; and similarly for $a' \vee b'$. Take $a' \prec b'$, to get $a' = a' \wedge b' \leqq' b'$. Recalling (3.6) we deduce that \leqq' coincides with \prec .

The coincidence of these two orders implies for G a separation property of some independent interest, namely

9°. Under the conditions of 8°, if $x + \bar{P}$ does not meet H then there exists $y < x$ such that $y + \bar{P}$ does not meet H .

PROOF. If $x + \bar{P}$ does not meet H then not $x' \leqq' '0'$, i.e. not $x' \prec '0'$, so there exists $a > 0, a' - x' \not\prec '0'$, i.e. $x - a + P^*$ does not meet H . Take $x - a < y < x$.

For a \leqq -convex normal subgroup H of G not meeting P^* , the two orders have the following descriptions:

$$x' \succneq '0' \text{ if and only if } (\exists h \in H) (\forall a \in G, (a > 0 \Rightarrow x + h + a > 0),$$

$$x' \succneq '0' \text{ if and only if } (\forall a \in G) (\exists h \in H) (a > 0 \Rightarrow x + h + a > 0).$$

For a study of those convex l -subgroups of (G, \leqq) which do meet P^* , see Reilly [7], sections 3 and 4.

4. Realization of G using maximal tangents

Theorem 8° of the previous section clears the way for the use of maximal tangents to represent G . We proceed to do this, for somewhat special circumstances; namely when (G, \leq) is weakly projectable (and so commutative), and \leq is non-androgynous. The main results are Theorem 15°, 18°. Throughout this section it is assumed that (G, \leq) is a nontrivial l -group with compatible tight Riesz order \leq ; U is the open-interval topology of \leq and $\bar{}$ denotes closure with respect to U (except where it refers to filets). Here ‘non-trivial’ means that $G \neq (0)$ and (G, \leq) is not trivially or fully ordered.

10°. *If H is an l -ideal of (G, \leq) , then so is \bar{H} .*

PROOF. \bar{H} is a normal subgroup of G ; it is also a sublattice since (2°) the lattice operations on G are continuous.

It remains to prove that \bar{H} is \leq -convex. Let $u < x < v$ with $u, v \in \bar{H}$. For $a > 0$, there exist $h_1 \in (u - a, u + a) \cap H$ and $h_2 \in (v - a, v + a) \cap H$, and here

$$h_1 \wedge h_2 \in (u - a, u + a) \cap H, \quad h_1 \vee h_2 \in (v - a, v + a) \cap H,$$

$$h_1 \wedge h_2 \leq h_1 \vee h_2,$$

so without loss of generality we can assume $h_1 < h_2$.

Let $(x - b, x + b)$ be any base neighbourhood of x ; take any a such that $0 < a < b$, and construct h_1, h_2 for this a as above. Write

$$k = h_1 \vee (x - a).$$

We have $k \in (x - b, x + b)$, and $h_1 \leq k \leq h_2$ so $k \in H$. Thus $x \in \bar{H}$.

Let H_0 be any l -ideal not meeting P^* . By Zorn’s lemma there exists a maximal element in the class, ordered by \subseteq , of all l -ideals containing H_0 and not meeting P^* . By 10°, such a maximal element is closed. Thus

11°. *Every l -ideal maximal with respect to not meeting P^* is a maximal tangent, and conversely. Every l -ideal not meeting P^* is contained in a maximal tangent.*

Let \mathfrak{S} denote the set of maximal tangents.

12°. *If H is a maximal tangent and G is commutative then for $G' = G/H$, \leq' coincides with \leq' , these being order-dense full orders.*

PROOF. First we show that every non-trivial l -ideal W of the l -group (G', \leq') meets $Q^* = \theta(P) \setminus \{0\}$, the strict cone of \leq' . Write

$$M = \{x: x' \in W\};$$

M is a \llcorner -convex normal subgroup of G , and since θ is a lattice homomorphism, M is also a sublattice, and therefore an l -ideal. Moreover $H \subseteq M$. If M meets P^* in x , then $x' \in W \cap Q^*$. Suppose W does not meet Q^* ; then \bar{M} is a tangent, so $H = M$ and hence $W = (0)$, contradiction.

Now let $a' \succ' 0'$ in G' . Then

$$\text{lid}(a') = \{x' : 0 \llcorner' |x'| \llcorner' na' \text{ for some positive integer } n\}$$

being a non-trivial l -ideal, meets Q^* ; i.e. there exists $x' \in G'$, $0 < x' \llcorner' na'$. By 2° (iii) and 8° , $a' \succ' 0'$. Thus \leqq' and \llcorner' coincide. Since \leqq' is an antilattice order (5° (vi)) and \llcorner' is a lattice order, the common order must be full. Since \leqq' is TR, it is order-dense.

We shall need to use replete maximal tangents. In any l -group (G, \llcorner) write

$$\bar{a} = \{x \in G : |x| \in |a|^\wedge\}$$

and call the sets \bar{a} the *filets* of G . Clearly for $a \succ 0$, $\hat{a} = \bar{a} \cap \bar{P}$. (For any subset A , \bar{A} continues to mean the closure of A .) The filets form a lattice isomorphic with (\mathcal{C}, \llcorner) when ordered by

$$\bar{a} \llcorner \bar{b} \Leftrightarrow |a|^\wedge \llcorner |b|^\wedge \Leftrightarrow a^\perp \supseteq b^\perp.$$

For some standard properties of filets in commutative l -groups see Ribenboim [8], pp 31–38. A subgroup K of G will be called *replete* if it is a union of filets, i.e. $x \in K \Rightarrow \bar{x} \in K$.

For the groups G at present under consideration, the maximal tangents may not be replete. For example, if $G = C[0, 1]$ (the continuous functions on $[0, 1]$) with $f > 0$ if and only if $f(t) > 0$ for $0 \leqq t \leqq 1$, then $f \succ 0$ if and only if $f(t) \geqq 0$ for $0 \leqq t \leqq 1$ (i.e. \leqq and \llcorner are the tight and loose pointwise orderings respectively), and (G, \llcorner) is an l -group with non-androgynous CTRO \leqq , and U is the metric topology of the sup norm. The filets \bar{f} can be identified with the closed supports $\text{supp}(f)$ (Ribenboim [8], pp 42, 43), and the maximal tangents are the maximal l -ideals

$$H_{t_0} = \{f : f(t_0) = 0\}, \quad 0 \leqq t_0 \leqq 1.$$

It is easily seen that they are not replete. In fact, if $g(t_0) = 0$ and $g(t) > 0$ for $t \neq t_0$, there exists no replete tangent containing g .

On the other hand $B[0, 1]$ (the bounded functions on $[0, 1]$) with the same orders is again an l -group with non-androgynous CTRO, but now the filets \bar{f} can be identified with the zero sets

$$Z(f) = \{t : f(t) = 0\},$$

while the maximal tangents are precisely the subsets of the form $\{f: Z(f) \in J\}$ where J is a maximal filter of subsets of $[0, 1]$; so the maximal tangents are all replete.

13°. If (G, \leq) is a non-trivial commutative l -group with non-androgynous compatible tight Riesz order \leq , then \mathfrak{S} is not empty.

PROOF. The equation $\mathfrak{w} = \bar{P} \setminus \{0\}$ would imply that $x \succ 0, y \succ 0 \Rightarrow x \wedge y \succ 0$, hence that (G, \leq) is an antilattice as well as a lattice, and so fully ordered, contrary to assumption. Hence there exists $c \succ 0, c \notin \mathfrak{w}$. Write

$$K = \bigcup \{ \bar{a}: \bar{a} \leq \bar{c} \} = c^{\perp\perp}.$$

K is an l -ideal; and K cannot meet P^* since distinct carriers are disjoint subsets, $\bar{c} \prec \mathfrak{w}$ and $P^* \subseteq \mathfrak{w}$ (by 3°). By 11°, K is contained in some maximal tangent.

In the extreme case $P^* = \mathfrak{w}$ of 13°, the maximal tangents are all replete. For let $H \in \mathfrak{S}$ and write

$$L \equiv L_H = \bigcup \{ \bar{a}: \bar{a} \leq \bar{h} \text{ for some } h \in H \}.$$

L is an l -ideal. If L meets P^* , in p say, we have $\bar{p} \leq \bar{h}$ for some $h \succ 0, h \in H$, so $P^* = \mathfrak{w} = \bar{p} = \bar{h}$, whence $h \in H \cap P^*$, contradiction. Since $H \subseteq L$ and H is maximal we have $H = L$, so H is replete. $B[0, 1]$ is a case where $P^* = \mathfrak{w}$.

Generally, note that a maximal tangent H is replete if and only if $L_H = H$.

We proceed to discuss the realization of G , assumed commutative. For an arbitrary choice of $H \in \mathfrak{S}$ write (when emphasis or clarity requires it) $G'_H = G/H, x'_H, U'_H, \dots$ for the corresponding entities, and form the full direct product

$$A = \prod_{H \in \mathfrak{S}} G'_H.$$

Let p_H denote the projection $\xi \mapsto \xi_H, A \rightarrow G'_H$ onto the H th factor. We make A partially ordered group in two ways, writing

$$\xi \succ 0 \text{ if and only if } \xi_H \succ '0'$$

$$\xi \succcurlyeq 0 \text{ if and only if } \xi_H \geq '0'$$

(recall 12°). It is easily verified that

14°. (A, \leq) is an l -group with $(\xi \wedge \eta)_H = \xi_H \wedge \eta_H, (\xi \vee \eta)_H = \xi_H \vee \eta_H$, so that each p_H is a lattice homomorphism; \leq is a compatible tight Riesz order for \leq

Let N denote the open-interval topology of (A, \leq) , and write

$$Q^* = \{ \xi \in A: \xi \succ 0 \}, Q = Q^* \cup \{0\},$$

so that $\bar{Q} = \bar{Q}^* = \{ \xi \in A: \xi \succcurlyeq 0 \}$. For a subset $F \subseteq A$ let N_F denote N relativized to F .

Write ρ for the natural group homomorphism of G into A , and for $x \in G$ write $\tilde{x} = (x_H)_{H \in \mathfrak{H}}$ for the element $\xi \in A$ for which $p_H(\xi) = x'_H$, so that $\rho: x \mapsto \tilde{x}$. Clearly ρ preserves order: $x > 0 \Rightarrow \rho(x) > 0$, and $x \geq 0 \Rightarrow \rho(x) \geq 0$. Also $p_H[\rho(G)] = G'_H$. We look for conditions on G making ρ one-one and also making ρ^{-1} order-preserving. (If ρ^{-1} exists and ρ and ρ^{-1} are both order-preserving we call ρ *isotone*, for the order in question.) If ρ has these properties, it is a *realization* of G , in the sense of Ribenboim [8], p. 61.

We can note straight away that, without further conditions on (G, \preceq) , if ρ is one-one then in fact it is isotone for both orderings. For let ρ be one-one, and suppose $x \in G, \rho(x) \geq 0$. Then for every $H \in \mathfrak{H}$ we have $x'_H \geq 0'$, and since the canonical homomorphism θ_H is a lattice homomorphism, also $0 \preceq (x \wedge 0)'_H \preceq 0$. Therefore $\rho(x \wedge 0) = 0, x \geq x \wedge 0 = 0$. Thus ρ is \preceq -isotone. Now suppose $y \in G, \rho(y) > 0$. By what has just been proved, $y > 0$. If $y \not\triangleright 0$ then $\text{lid}(y)$ is an l -ideal not meeting P^* (by 2° (iii)), so there exists $K \in \mathfrak{H}, y \in \text{lid}(y) \subseteq K$; whence $y'_K = 0'$ whereas in fact $y'_H > 0'$ for all $H \in \mathfrak{H}$. Thus $y > 0$; ρ is \leq -isotone.

An l -group (G, \preceq) is called *Stone* if for every $a \in G$

$$(4.2) \quad G = a^\perp \oplus a^{\perp\perp}.$$

Every lattice-complete l -group is a Stone l -group, and commutative (cf. Fuchs [2] Theorems 16, 18, p. 91).

Strzelecki has introduced the concept of weak projectability: a commutative l -group (G, \preceq) is called *weakly projectable* if for every $a, b \in G$ there exists $z \in a^\perp$ such that $b \in (|a| + |z|)^{\perp\perp}$. Every commutative Stone l -group is weakly projectable. On the other hand, $C[0, 1]$ with the loose pointwise ordering is weakly projectable but not Stone (Spirason and Strzelecki [10], Speed and Strzelecki [9]).

15° THEOREM. *If (G, \preceq) is a weakly projectable l -group with a non-androgynous compatible tight Riesz order \leq , whose maximal tangents are all replete, then ρ is isotone for \leq and \preceq , and is thus a realization of (G, \leq, \preceq) as a subdirect product of fully ordered groups.*

PROOF. By 13°, \mathfrak{H} is nonempty (we assume that \preceq is not a full order). By previous remarks it suffices to show that

$$(4.3) \quad \ker(\rho) = \bigcap_{H \in \mathfrak{H}} H = (0).$$

We prove that if $b > 0$ in G then there exists $H_b \in \mathfrak{H}$ with $b \notin H_b$. Then given any $c \neq 0$ we have $|c| > 0$ and so $c \notin \bigcap H$.

So let $b > 0$. Now b^\perp is an l -ideal; since (G, \leq) is non-androgynous b^\perp does not meet P^* , for otherwise there would exist $x > 0, x \wedge b = 0$, contradicting 3°(iii). By 11° there exists $H_b \in \mathfrak{H}, H_b \supseteq b^\perp$. Suppose that $b \in H_b$: then we prove $H_b = G$ as follows. Let $a \in G$; by weak projectability there exists $z \in b^\perp$ such

that $a \in (b + |z|)^{\perp\perp}$. In any commutative l -group, $y^{\perp\perp}$ is the smallest replete l -ideal containing y . Since $b + |z| \in H_b$ and H_b is replete, $a \in (b + |z|)^{\perp\perp} \subseteq H_b$. Thus $H_b = G$, contradiction; so $b \notin H_n$, as required.

If (G, \leq) is androgynous then ρ cannot be one-one. For suppose (G, \leq) is androgynous; by 3°(iii) there exist $x > 0, y > 0$ with $x \wedge y = 0$. If $H \in \mathfrak{H}$ then H is a prime l -ideal by 12°, so either $x \in H$ or $y \in H$, i.e. $y \in H$ since $H \cap P^* = \emptyset$. Therefore $0 \neq y \in \bigcap_{H \in \mathfrak{H}} H$, so ρ is not one-one. (This argument is due to Davis.)

Topologically the situation for ρ is somewhat less satisfactory. As well as the open-interval topology N on A , there is the product topology T , which in this context is less relevant than N . We shall also need to consider the penetration of A by the image under ρ : the following possibilities suggest themselves.

(α) $\rho(P^*)$ is dense in (Q^*, N) : given $0 \leq \eta < \xi$ in A there exists $a > 0$ in G such that $\eta < \rho(a) < \xi$.

(β) $0 \in \overline{\rho(P^*)}$: given $\xi > 0$ in A there exists $a > 0$ in G such that $0 < \rho(a) < \xi$.

(γ) $\rho(G)$ is dense in (A, N) : given $\eta < \xi$ in A there exists $x \in G$ such that $\eta < \rho(x) < \xi$.

It is clear that $(\alpha) \Rightarrow (\beta)$; it is less obvious, but true, that $(\alpha) \Rightarrow (\gamma)$. Sherman has shown, by using a result of Reilly's, that $(\gamma) \Rightarrow (\alpha)$ (personal communication). However, (β) appears to be a weaker condition. For example, let G be $C[0, 1]$, ordered as before; \mathfrak{H} can be identified with $[0, 1]$, as we saw, and each f under ρ with itself. It follows easily that (β) does not hold.

16°. Let (G, \leq) be an l -group with a compatible tight Riesz order \leq . Then

(i) $T \subseteq N$, so that $T_{\rho(G)} \subseteq N_{\rho(G)}$.

If \mathfrak{H} is infinite then $T_{\rho(G)} \neq N_{\rho(G)}$.

(ii) If ρ is one-one, then for all $a < b$ in G ,

$$\rho(a, b) = \rho(G) \cap \prod_{H \in \mathfrak{H}} (a'_H, b'_H) = \rho(G) \cap (\tilde{a}, \tilde{b}),$$

so $\rho: (G, U) \rightarrow (\rho(G), N_{\rho(G)})$ is an open map.

(iii) The map $\rho: (G, U) \rightarrow (A, T)$ is continuous. If ρ is one-one and (β) holds then

$$\rho: (G, U) \rightarrow (A, N)$$

is continuous.

The proof of 16° is straightforward.

17° COROLLARY. Let (G, \leq) be a commutative l -group with compatible tight Riesz order \leq , and let ρ be one-one. If (β) holds, then ρ is a topological embedding of (G, U) in (A, N) . If (γ) holds, then ρ is a concordant realization, i.e. $\rho: (G, \leq) \rightarrow (A, \leq)$ is a lattice isomorphism into.

PROOF. By 16° (ii) and (iii), $\rho: (G, U) \rightarrow (\rho(G), N_{\rho(G)})$ is a homeomorphism, so ρ embeds G .

To prove ρ concordant, note that $\rho(x \wedge y) \leq \rho(x) \wedge \rho(y)$ since ρ preserves \leq , and let $\xi \leq \rho(x), \rho(y)$ in A . Let $\eta > 0$ in A ; then $\xi - \eta < \rho(x), \rho(y)$, and by assumption (y) there exists $z \in G$ with $\xi - \eta < \rho(z) < \rho(x), \rho(y)$, whence $z < x \wedge y, \xi < \rho(z) + \eta < \rho(x \wedge y) + \eta$, so $\xi \leq \rho(x \wedge y)$. Thus $\rho(x \wedge y) = \rho(x) \wedge \rho(y)$, and similarly for \vee .

Putting 15° and 17° together we obtain

18° THEOREM. *Let (G, \leq) be a commutative l -group with a non-androgynous compatible tight Riesz order \leq , whose maximal ideals are all replete. If (G, \leq) is weakly projectable and (β) holds then ρ is a topological embedding of (G, U) in (A, N) as well as a realization of (G, \leq, \leq) in A . If also (y) holds, then ρ is concordant for \leq .*

Theorem 15° applies only for non-androgynous CTRO's; for these we know that $P^* \subseteq \mathfrak{w}$, so that \mathfrak{w} is the maximal non-androgynous CTRO, provided it is a CTRO. The final lemma deals with this point. Let (G, \preceq) be any l -group whose set of weak units \mathfrak{w} is not empty, and write Q for the positive cone of \preceq . Since \mathfrak{w} is a subsemigroup and $\mathfrak{w} \subseteq Q^*$, \mathfrak{w} is the strict cone of a partial ordering on G , call it \leq , making (G, \leq) a partially ordered group. Let \leq denote the associated preorder of \leq . It is easily proved that $x \succ 0 \Rightarrow x \succ 0$. Let comparison of orders refer to comparison (with respect to \subseteq) of their positive cones. We have

19°. *Let (G, \preceq) be an l -group with $\mathfrak{w} \neq \emptyset$. If \leq , the order having \mathfrak{w} as strict positive cone, is $TR(1,2)$ without pseudozeros, then it is the largest non-androgynous compatible tight Riesz order for \preceq .*

PROOF. We have only to show that \leq is a CTRO for \preceq . Now

$$a > 0 \Rightarrow \alpha > 0 \Rightarrow a > 0,$$

and here \leq is order-dense, \preceq is a lattice order, and \leq is the associated order of \preceq . This is the basis on which the argument following (3.6) depends; that argument shows here that \preceq and \leq coincide. Therefore \leq is compatible for \preceq .

In general \mathfrak{w} does not give the largest CTRO: Wirth [11] has shown that for an abelian divisible l -group (G, \leq) , there is a largest CTRO if and only if (G, \leq) is fully ordered.

At the other extreme, let

$$\sigma = \{s: \text{for each } x \succ 0, \text{ there exists a positive integer } n \text{ such that } x \prec ns\}$$

denote the set of strong units of (G, \leq) . Wirth has shown that for an abelian divisible para-archimedean l -group (G, \leq) , there is a smallest CTRO if and only if $\sigma \neq \emptyset$, and then σ is the strict positive cone of that CTRO.

In the circumstances of Theorem 15°, the carrier lattice \mathfrak{C} has greatest element w . So if \bar{c} is covered in \mathfrak{C} by w , then it determines by

$$H_{\bar{c}} = \bigcup \{ \bar{a} : \bar{a} \preceq \bar{c} \} = c^{\perp\perp}$$

an element of \mathfrak{H} . If G has only finitely many carriers, so that \mathfrak{C} is a Boolean lattice (Fuchs [2], p. 82), every (replete) maximal tangent is of this form, and \mathfrak{H} is in one-one correspondence with the set of atoms of \mathfrak{C} .

A final incidental remark: the maximal tangents H need not be lattice-closed, in the sense

$$\left. \begin{array}{l} X \subseteq H \\ \bigvee X \text{ exists in } (G, \preceq) \end{array} \right\} \Rightarrow \bigvee X \in H.$$

For a counter-example take $G = B(0, 1)$ with \preceq and \preceq the tight and loose point-wise ordering respectively; (G, \preceq) is lattice-complete and \preceq is a non-androgynous CTRO. Any maximal tangent containing

$$\text{gr}\{f \succeq 0 : f(x) = 0 \text{ for } 0 < x < \alpha_f, \text{ for some } \alpha_f \in (0, 1)\}$$

is not lattice-closed.

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