Some Geometrical Theorems.

By R. E. ALLARDICE, M.A.

1. The greatest line joining two points in the perimeter of a polygon is a side or a diagonal of the polygon.

For (fig. 2) PS is obviously less than one or other of the lines PQ, PR. Hence if AB, CD (fig. 3) are any two finite lines, L and M any points, one in each of these lines, LM is less than LC, or LD and LC is less than CA or CB. From this the theorem follows at once. The theorem and the above proof apply to crossed and gauche polygons.

2. Let ABC (fig. 4) be any triangle; CQ equal to CB and perpendicular to it; CP equal to CA and perpendicular to it; then the locus of O, their intersection, is a circle on AB.

The proof of this theorem is very simple; but what is remarkable about the theorem is that the point C has two degrees of freedom; and it would seem at first sight as if O ought to have two degrees of freedom also.

Since to a two-fold infinity of positions of C corresponds a one-fold infinity of positions of O, it is to be expected that to every position of O will correspond a one-fold infinity of positions of C; that is, C will have a locus. This locus is a straight line passing through O and through the middle point of the semicircle described on AB on the side remote from C; as may easily be shown.

3. A PROPERTY OF THE ELLIPSE.

If (x, y, z) be the trilinear co-ordinates of a point P, it may easily be shown that $x^2+y^2+z^2$ is a minimum when P is at (a, b, c), which is the symmedian point.

Now if P (fig. 5) move so that y^2+z^2 is constant, the locus of P is an ellipse of which AB and AC are the equi-conjugate diameters. But for the minimum position PD must be normal to the conic; and thus the tangent at P, which is the symmedian point, is parallel to BC.

Hence we get the following theorem :—

If the tangent to an ellipse at the point T (fig. 6) meets the equi-conjugate diameters in Q and Q′, then CT is a symmedian of the triangle CQQ′.

This theorem may also be proved directly as follows:—Let the tangent at T meet the axes in P and P′. Now it is known that pairs of conjugate diameters trace out on the tangent at T an involution, of which T is the centre.

Let R be the middle point of QQ′, and O the middle point of PP′. We have to show that P′TPR is a harmonic range; that is, (P′TPR) = −1.

We have $\quad TP.TP' = TQ.TQ'$ and $(P'QPQ') = -1$.

Now
$$\begin{aligned} TQ.TQ' &= (OT - OQ)(OQ' - OT) \\ &= OT(OQ + OQ') - OQ.OQ' - OT^2 \\ &= 2OT.OR - OP^2 - OT^2. \end{aligned}$$

But $\quad TQ.TQ' = TP.TP' = OP^2 - OT^2.$

Hence $\quad OP^2 - OT^2 = 2OT.OR - OP^2 - OT^2$

$$\therefore \quad OP^2 = OT.OR.$$

Hence P′TPR is a harmonic range; and, since P′OP is a right angle, OP bisects TCR.

The above proof contains a proof of the following property of an involution:—

If PP′, QQ′ are pairs of corresponding points in an involution of which T is the centre; and if P and P′ are harmonically conjugate with reference to Q and Q′; then, if O is the middle point of PP′ and R the middle point of QQ′, O is harmonically conjugate to T with reference to Q and Q′, and R is harmonically conjugate to T with reference to P and P′.

4. The following simple geometrical proof of a proposition given by Mr Pressland in his paper *On the triangle and its escribed parabolas,* contained in this volume, may be worth recording.

Let ABC (fig. 7) be the triangle; BP equal to CQ; to show that the envelope of PQ is a parabola touching all the sides of the triangle. (BP and CQ may be taken on the same side or on opposite sides of BC, and may be taken in any constant ratio, and the following proof, with some slight modification, will still apply).

The proof depends on the following converse of a well-known property of the parabola:—

If there be two straight lines OX and OY and a fixed point F, and if a circle through O and F meets OX and OY in P and Q, then PQ envelopes a parabola which touches OX and OY and of which F is the focus.

Let O be the circumcentre of the triangle ABC, and R the circumcentre of APQ. Then RO has equal projections on AB and AC (each being $\frac{1}{2}$BP); and hence RO is parallel to the bisector of the angle BAC. Therefore the locus of R is a straight line through O, parallel to the bisector of the angle BAC; and thus the circumcircle of APQ always passes through a fixed point A′, which is the image of A in RO. Hence, by the proposition stated above, PQ envelopes a parabola of which A′ is the focus, and which touches AB and AC.

It is obvious from the above also that if any circle through A and A′ meets AB and AC in P and Q, then BP = BQ; and the following more general theorem may also be proved very easily:—

If a series of circles pass through two fixed points A and A′, and if two straight lines APQR … and AP′Q′R′ … meet these circles in the points P, Q, R, … and P′, Q′, R′, …, then PQ : QR : … = P′Q′ : Q′R′ : ….

This theorem may also be got by inversion; it may in fact be got by inverting the theorem that the anharmonic ratio of a pencil is constant, by taking the particular case of a number of concurrent transversals.

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R. E. Allardice, Esq., President, in the Chair.

On the Heating of Conductors by Electric Currents, and the Electric Distribution in Conductors so heated.

By John M'Cowan, M.A., B.Sc.

The solution of the equation.

$$\sqrt{\{(x-a)(x-b)\}} + \sqrt{\{(x-c)(x-d)\}} = e.$$

By J. D. Hamilton Dickson, M.A.

1. If $P = a + b - c - d$, and $K = cd - ab$, the solution of the equation

$$\sqrt{\{(x-a)(x-b)\}} + \sqrt{\{(x-c)(x-d)\}} = e \qquad \ldots \qquad (1)$$

may be most readily found by putting $(x-a)(x-b) = z^2$; whence