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GLOBAL DIMENSION OF FACTOR RINGS

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Let R be a right Noetherian ring with right global dimension bounded by 2, which is integral over its centre, and let a be a regular non-unit element in R. Then $R/a \cdot R$ is right hereditary if and only if a is not in the square of any maximal ideal of R. More generally, we compare for a right Noetherian ring R which is integral over its center, the global dimension of R with the global dimension of $R/(a_1R + a_2R + \cdots + a_rR)$ for a regular R-sequence $\{a_i\}$, which will allow us to give a considerable extension of a result of Hillman.

1. INTRODUCTION

In this note we shall compare for a right Noetherian ring R which is integral over its center, the global dimension of R with the global dimension of $R/(a_1R + a_2R + \cdots + a_rR)$ (see Theorem 1) for a regular R-sequence $\{a_i\}$, which will allow us to give a unified treatment of the results of Hillman in [**7**] and those in [**12, 13**], see Theorem 2.

Moreover, the result also has consequences for Artinian algebras and for algebraic geometry:

PROPOSITION 1.

1. *Let K be an algebraically closed field and A a finite dimensional K-algebra. If A has finite global dimension, then every maximal ideal in A is idempotent.*
2. *Let K be an algebraically closed field and let P be the prime ideal of $R := K[x_1, x_2, \dots, x_n]$ generated by a regular R-sequence $f_1, f_2, \dots, f_r$ in R. Then the affine variety $V(f_1, f_2, \dots, f_r)$ is non-singular if and only if*

$$f_i \notin \mathcal{P}^2 + f_1 \cdot R + f_2 \cdot R + \cdots + f_{i-1} \cdot R$$

for $i = 1, 2, \dots, r$ for any maximal ideal $\mathcal{P}$ of R containing P (that is, P is a regular point).

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More precisely, for a ring R, we recall that an ordered sequence of normalising elements $a_1, a_2, \ldots, a_r$ in R (that is a_i is normal in $R/(a_1R + \cdots + a_{i-1}R)$) is called an R-**sequence** of R provided a_1 is neither a unit nor a zero-divisor in the ring R, the image of a_2 is neither a unit nor a zero-divisor in the ring R/a_1R, the image of a_3 is neither a unit nor a zero-divisor in the ring $R/(a_1R + a_2R)$, and so on. (For R-sequences in the case of commutative rings, see [**9**, Chapter 3] for details, for the case of non-commutative rings, see [**4**].) The main result is then the following.

Theorem 1. *Assume that R is a right Noetherian ring with finite global dimension, which is integral over its center.*

1. Let $a_1, a_2, \ldots, a_r$ be an R-sequence of normalising elements in $J(R)$ such that

$$a_i \notin \mathcal{P}^2 + a_1R + a_2R + \cdots + a_{i-1}R, \quad i = 1, 2, \ldots, r$$

for any maximal ideal $\mathcal{P}$ of R containing $a_1, a_2, \ldots,$ and a_{i-1}. Then

$$r.gl.dim.\, R/(a_1R + a_2R + \cdots + a_rR) = r.gl.dim.\, R - r\,.$$

(Note that in the commutative situation this is a regular R-sequence.)

2. Let $a_1, a_2, \ldots, a_r$ be an R-sequence of centralising elements such that

$$a_i \notin \mathcal{P}^2 + a_1R + a_2R + \cdots + a_{i-1}R, \quad i = 1, 2, \ldots, r$$

for any maximal ideal $\mathcal{P}$ of R containing $a_1, a_2, \ldots,$ and a_{i-1}. Then we have that

$$r.gl.dim.\, R/(a_1R + a_2R + \cdots + a_rR) \leqslant r.gl.dim.\, R - r\,.$$

Remark 1.

1. As an application, we can extend Hillman's result [**7**] to the case of commutative Noetherian rings of finite global dimension; thereby we can observe explicitly the relationship between the global dimension of the factor ring of a commutative Noetherian ring by an ideal generated by a certain R-sequence and the non-sigularity of affine varieties.
2. Examples 2 show that the conditions are not sufficent.

(1) In the commutative situation though, we prove a converse.

Proposition 2. *Let R be a commutative Noetherian ring with finite global dimension and let $a_1, a_2, \ldots, a_r$ in R be an R-sequence. Then*

$$gl.dim.\, R/(a_1R + a_2R + \cdots + a_rR) \leqslant gl.dim.\, R - r$$

if and only if

$$a_i \notin \mathcal{P}^2 + a_1R + \cdots + a_{i-1}R\,, \quad i = 1, 2, \ldots, r$$

for any maximal ideal $\mathcal{P}$ of R containing $a_1, a_2, \ldots,$ and a_r.

There is also one instance in the non commutative situation, where we get a necessary and sufficient condition, in case R has global dimension bounded by 2, which gives a unified treatment of the results in [**7, 12** and **13**].

THEOREM 2. *Assume that R is a right Noetherian ring with $r.gl.dim. R \leqslant 2$, which is integral over its center. Let a be a central regular non-unit element in R. Then the ring R/aR is right hereditary if and only if a is not in the square of any maximal ideal of R.*

From this we derive with the help of Corollary 4 in Section 2:

COROLLARY 1. *Assume that R is a right Noetherian right hereditary PI ring which is integral over its center. Let $f(x)$ be a central regular non-unit polynomial in $S := R[x]$ (and $T := R[[x]]$ respectively). Then the ring $S/f(x)S$ (and $T/f(x)T$ respectively) is right hereditary if and only if $f(x)$ is not in the square of any maximal ideal of S (and T respectively).*

If R is a prime PI right hereditary ring, then by [**16**] the center of R is a Dedekind domain and R is finitely generated as a module over its center. Therefore R is right Noetherian which satisfies the hypotheses of Theorem 2 and Corollary 1. This easily leads to:

COROLLARY 2. ([**7,** Theorem 2], [**12,** Theorem] and [**13,** Proposition 1]) *Assume that R is a prime PI right hereditary ring and $f(x)$ in S (and T respectively) (the same notation as above) is a non-zero central non-unit polynomial. Then $S/f(x)S$ (and $T/f(x)T$ respectively) is right hereditary if and only if $f(x) \notin \mathcal{P}^2$ for any maximal ideal $\mathcal{P}$ of S (and T respectively).*

One should compare Corollaries 1, 2 with [**7,** Theorem 2], [**12,** Theorem] and [**13,** Proposition 1] in which $f(x)$ is assumed to be a non-zero central polynomial with "$f(x)S$ **a prime ideal** of S".

2. PROOFS AND FURTHER COMMMENTS

Recall that a ring R is called **semi-local** if $R/J(R)$ is a semi-simple Artinian ring, where $J(R)$ denotes the Jacobson radical of R. In particular, if $R/J(R)$ is a simple Artinian ring, R is called a **local** ring. A semi-local ring R is called **semi-perfect** if every idempotent of $R/J(R)$ can be lifted to an idempotent of R.

The idea for the proof of the following lemma is essentially taken from [**12,** Lemma 3].

LEMMA 1. *Let R be a right Noetherian semi-local ring with Jacobson radical J and $J = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_n$, where $\mathcal{P}_i$ are the maximal ideals of R, $1 \leqslant i \leqslant n$. Let a be* *a normal element in* J, *with* $a \notin \mathcal{P}_i^2$ *for any* i, *then* J/aR *is isomorphic to a direct summand of* J/aJ *and moreover* $aJ = Ja$.

PROOF: Let $a^* = a + J^2$ in J/J^2. Then since R/J is a semi-simple Artinian ring, $J/J^2 = a^*R \oplus N^*$ for some R-submodule N of J. Hence $J = aR + N + J^2$ and so $J = aR + N$ by Nakayama's Lemma. Let us set $T = aJ + N$. Then we have $J = T + aR$. To prove $T \cap aR = aJ$, let t be an element in $T \cap aR$. Write $t = ar = aj + n$, where $r \in R, j \in J$ and $n \in N$. Then $t^* = (ar)^* = n^* \in a^*R \cap N^* = 0$. Thus $ar \in J^2$ and so $ar \in \mathcal{P}_i^2$. Let $A_i = \{b \in R \mid ab \in \mathcal{P}_i^2\}$. Then since a is normal, A_i is a two-sided ideal of R. If A is not contained in $\mathcal{P}_i$, then $A_i + \mathcal{P}_i = R$ and so $a \in aR = a(A_i + \mathcal{P}_i) \subseteq \mathcal{P}_i^2$, which is a contradiction. Hence $A_i \subseteq \mathcal{P}_i$. In particular, $r \in \mathcal{P}_i$ for all i. Therefore $r \in J$ and hence $t = ar \in aJ$. By this fact $T \cap aR = aJ$ and it follows that $J/aJ = (T/aJ) \oplus (aR/aJ)$. By the way,

$$J/aR = (T + aR)/aR \cong T/(T \cap aR) = T/aJ\,.$$

Hence J/aR is isomorphic to a direct summand of J/aJ.

Similarly, we can prove that $aJ = aR \cap J^2 = Ra \cap J^2 = Ja$. ∎

A ring R is said to be **integral** over a subring S if every element of R satisfies a monic polynomial with coefficients in S. A two-sided ideal A of R is said to have the right **Artin-Rees** property if for every right ideal I of R, there exists a positive integer n such that $A^n \cap I \subseteq IA$.

The next lemma is quoted from [**5**, Lemma 2.1].

LEMMA 2. *Assume that* R *is a right Noetherian ring which is integral over its center* C.

1. *If* R *is semi-local, then* $J(R)$ *has the right Artin-Rees property.*
2. R *is semi-local if and only if* C *is semi-local.*

For a ring R, we write $r.gl.dim.\,R$ to denote the **right global dimension** of R. Also for a right R-module M we write $pr.dim.\,M_R$ to denote the projective dimension of M.

Assume that C is a commutative local domain with finite global dimension and R is a local C-algebra, which is finitely generated as a C-module. Let $\wp$ (respectively $\mathcal{P}$) be the unique maximal ideal of C (respectively R) and assume that $a \in \wp \backslash \mathcal{P}^2$. In [**14**, Theorem 6.4] Ramras showed that

$$r.gl.dim.\,R/aR = n - 1 \text{ if } r.gl.dim.\,R = n < \infty\,.$$

This result can be extended to:

Lemma 3. *Let R be a right Noetherian ring, which is integral over its center C, and let a be a regular normal non-unit element of $J(R)$ satisfying $a \notin \mathcal{P}^2$ for any maximal ideal $\mathcal{P}$ of R. If*

$$r.gl.dim.\, R = n < \infty, \text{ then } r.gl.dim.\, R/aR = n - 1.$$

Proof: If $n = 0$, then R is a semi-simple Artinian ring and so $J(R) = 0$. Thus we may assume that $n > 0$. Let $\wp$ be a maximal ideal of C. Then the localisation $R_\wp$ of R at $\wp$ is also integral over $C_\wp$ and so $R_\wp$ is semi-local by Lemma 2. Clearly, $a/1$ is a regular normal non-unit element of $J(R_\wp)$. Now for our convenience let $S = R_\wp$ and $J = J(S)$. By [**10**, Proposition 7.3.6(b)] we have

$$pr.dim. J/(JaS)_{S/aS} = pr.dim. J_S \leqslant n.$$

By Lemma 1, J/aS is isomorphic to a direct summand of J/Ja and hence $pr.dim.(J/aS)_{S/aS} \leqslant n$.

On the other hand, since S/aS is semi-local and S/aS is integral over its center, the Jacobson radical J/aS of S/aS has the right Artin-Rees property by Lemma 2. Hence

$$r.gl.dim.\, S/aS = pr.dim.((S/aS)/(J/aS))_{S/aS} < \infty$$

by [**3**, Corollary]. Therefore we have

$$r.gl.dim.\, S = (r.gl.dim.\, S/aS) + 1$$

by [**10**, Theorem 7.3.7].

Finally, let $\overline{R} = R/aR$ and $\overline{C} = \{\overline{c} \in \overline{R} \mid c \in C\}$. Then we have

$$r.gl.dim.\, \overline{R} = sup\{r.gl.dim.\, \overline{R}_\wp \mid \wp \text{ a maximal ideal of } \overline{C}\}$$

by [**11**, Theorem 9.22]. Obviously, a maximal ideal $\overline{\wp}$ of $\overline{C}$ is the homomorphic image of a maximal ideal $\wp$ of C in $\overline{R}$. Then $\overline{R}_{\overline{\wp}} \cong R_\wp/aR_\wp$ and hence we conclude that $r.gl.dim.\, R/aR = n - 1$. ∎

Recall that a ring R is called **right hereditary** if every right ideal of R is projective as a right R-module. R is right hereditary if and only if $r.gl.dim.\, R \leqslant 1$, since submodules of free modules are projective and conversely.

As we shall see later, the converse of Lemma 3 does not hold in general, not even in the case of local rings. However, we still have:

Corollary 3. *Assume that R is a right Noetherian local ring, which is integral over its center with $r.gl.dim.\, R \leqslant 2$. Let a be a regular normal non-unit element, then*

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the ring R/aR is right hereditary if and only if $a \notin \mathcal{P}^2$, where $\mathcal{P}$ is the maximal ideal of R.

PROOF: The sufficiency follows from Lemma 3. As for the necessity, assume that R/aR is right hereditary. Then the exact sequence

$$0 \longrightarrow Ra/\mathcal{P}a \longrightarrow \mathcal{P}/\mathcal{P}a \longrightarrow \mathcal{P}/Ra \longrightarrow 0$$

of right R/aR-modules splits. So it splits also as a sequence of R-modules. Therefore $\mathcal{P}/\mathcal{P}a = (T/\mathcal{P}a) \oplus (Ra/\mathcal{P}a)$ for some R-submodule T of $\mathcal{P}$. Thus

$$\mathcal{P} = T + Ra \text{ and } \mathcal{P}a = T \cap Ra\,.$$

Now if $a \in \mathcal{P}^2$, then $Ra \subseteq \mathcal{P}^2$ and so

$$\mathcal{P} = \mathcal{P}^2 + T = J(R)\mathcal{P} + T\,.$$

By Nakayama's Lemma, $\mathcal{P} = T$. Hence $Ra = \mathcal{P}a$ and therefore $R = \mathcal{P}$, which is a contradiction. ∎

We next give a criterion of when (in Lemma 3) there is an element $a \in J(R)$ with $a \notin \mathcal{P}_i^2$, $1 \leqslant i \leqslant n$.

If R is a semi-perfect ring, we may assume that R is basic, and we **denote** by S_i, $1 \leqslant i \leqslant n$, the simple R-modules, and by $\mathcal{P}_i$, $1 \leqslant i \leqslant n$ the corresponding maximal ideals, that is, $R/\mathcal{P}_i = S_i$. Moreover P_i denotes the projective cover of S_i and e_i is a primitive idempotent in R with $P_i = Re_i$.

Then we have the following

PROPOSITION 3. *For a ring R the following conditions are equivalent for $1 \leqslant r \leqslant n$:*

1. *There is an element $a \in \mathcal{P}_i \setminus \mathcal{P}_i^2$ for every $1 \leqslant i \leqslant r$.*
2. *$\mathcal{P}_i \neq \mathcal{P}_i^2$ for every $1 \leqslant i \leqslant r$.*
3. *$Ext_R^1(S_i, S_i) \neq 0$ for $1 \leqslant i \leqslant r$.*

PROOF: Obviously (1) implies (2), and (2) is equivalent to (3).

In order to show that (2) implies (1) we shall use induction. We first observe that without loss of generality we may assume that $J^2 = 0$. For $r = 1$ there is nothing to prove. For the inductive step we note:

CLAIM 1. For $i \neq j$ we have $e_iRe_i \subset \mathcal{P}_j^2$.

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PROOF: By using Peirce decomposition, we have

$$\mathcal{P}_i = \begin{pmatrix} J(e_iRe_i) & e_iR(1-e_i) \\ (1-e_i)Re_i & (1-e_i)R(1-e_i) \end{pmatrix}$$

and

$$\mathcal{P}_i^2 = \begin{pmatrix} 0 & e_iR(1-e_i) \\ (1-e_i)Re_i & (1-e_i)R(1-e_i) \end{pmatrix}$$

Therefore

$$J(e_iRe_i) = \mathcal{P}_i/\mathcal{P}_i^2 \cong S_i^{(n_i)},$$

where n_i is the multiplicity of S_i in $\operatorname{rad}(P_i)$, the radical of P_i. But from this the statement of the claim follows. □

By induction we have found an element

$$b \in \mathcal{P}_i \setminus \mathcal{P}_i^2 \text{ for every } 2 \leqslant i \leqslant r.$$

If $b \in \mathcal{P}_1 \setminus \mathcal{P}_1^2$ then we are done. Otherwise we choose – according to Claim 1 – an element $0 \neq b_0 \in J(e_1Re_1)$. The element $a = b_0 + b$ will do. □

As an application we can now **prove Proposition 1 (1)** from the introduction:

Assume that R is a finite dimensional algebra over an algebraically closed field. If $a \in J(R)$ with $a \notin \mathcal{P}_i^2$, $1 \leqslant i \leqslant n$, then R can not have finite global dimension. In fact it follows from Proposition 2, that $\operatorname{Ext}_R^1(S,S) \neq 0$ for every simple R-module S. But it was shown by Igusa [**8**, Theorem 4.5] that if R has finite global dimension, then there are no loops in the quiver of R, equivalently $\operatorname{Ext}_R^1(S,S) = 0$ for every simple R-module S. Therefore R can not have finite global dimension. A diffrerent way of phrasing this is to say that for R of finite global dimension, all maximal ideals must be idempotent. □

From Lemma 3, we derive the following fact about the global dimension of certain factor rings of a Noetherian ring.

LEMMA 4. *Let R be a right Noetherian ring, which is integral over its center C, and let a be a central regular non-unit element in R. Suppose $a \notin \mathcal{P}^2$ for any maximal ideal $\mathcal{P}$ of R. If*

$$r.gl.dim.\, R = n < \infty, \text{ then } r.gl.dim.\, R/aR \leqslant n-1.$$

PROOF: We may assume that $n > 0$. Let $\wp$ be a maximal ideal of C containing

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a. Then $a \in J(R_\wp)$, and so by Lemma 3 we have that

$$r.gl.dim.\, R_\wp/aR_\wp = (r.gl.dim.\, R_\wp) - 1\,.$$

By [**11**, Theorem 9.2.2],

$$r.gl.dim.\, R = sup\{r.gl.dim.\, R_\wp \mid \wp \text{ a maximal ideal of } C\}$$

and

$$r.gl.dim.\, R/aR = sup\{r.gl.dim.\, R_\wp/aR_\wp \mid a \in \wp \text{ a maximal ideal of } C\}.$$

Hence we conclude $r.gl.dim.\, R/aR \leqslant (r.gl.dim.\, R) - 1$. ∎

For the sequel we **define** S to be either the polynomial ring $R[x]$ or the ring of formal power series $R[[x]]$.

COROLLARY 4. *Assume that R is a right Noetherian PI ring which is integral over its center C. Let $f(x)$ be a central regular non-unit polynomial of S satisfying $f(x) \notin \mathcal{P}^2$ for any maximal ideal $\mathcal{P}$ of S. If $r.gl.dim.\, R = n < \infty$, then $r.gl.dim.\, S/f(x)S \leqslant n$.*

PROOF: By [**10**, Theorem 13.8.12], S is integral over its center, and by [**10**, Theorem 7.5.3(iii)] $r.gl.dim.\, S = n + 1$. Now the assertion follows from Lemma 4. ∎

We now come to the **proof of Theorem 1**. The proof is in a similar spirit to that of [**1**, Proposition 1.5].

(1) By Lemma 3, $r.gl.dim.\, R/a_1R = (r.gl.dim.\, R) - 1$. We have

$$r.gl.dim.\, \overline{R}/\overline{a}_2\overline{R} = r.gl.dim.\, (R/(a_1R + a_2R))\,,$$

where $\overline{R} = R/a_1R$ and $\overline{a}_2 = a_2 + a_1R \in \overline{R}$. Note that $\overline{a}_2$ is a regular normal non-unit element of $\overline{R}$ and $\overline{a}_2 \in J(\overline{R})$. For a maximal ideal $\mathcal{P}$ of R containing $a_1R + a_2R$ we have $\overline{a}_2 \notin \overline{\mathcal{P}}^2$, since $a_2 \notin (\mathcal{P}^2 + a_1R)$. So again by Lemma 3,

$$r.gl.dim.\, \overline{R}/\overline{a}_2\overline{R} = (r.gl.dim.\, \overline{R}) - 1 = (r.gl.dim.\, R) - 2\,.$$

By continuing this process, finally we have

$$r.gl.dim.\, R/(a_1R + a_2R + \cdots + a_rR) = (r.gl.dim.\, R) - r\,.$$

(2) This can be proved in a similar way as in (1) by using Lemma 4 iteratively. ∎

REMARK 2.

1. Theorem 1 is a kind of non-commutative analogue of one direction of a result in commutative algebra (see also [**17**, Chapter VIII, Section 11, Corollary 2]).
2. A commutative Noetherian local ring is regular (that is, it has a regular R-sequence) if and only if it has finite global dimension (see [**2**, Theorem 1.10]). In this case we now establish the converse of Theorem 1.

We are now in the situation to **prove Proposition 2**:

Assume that

$$gl.dim.\, R/(a_1R + a_2R + \cdots + a_rR) \leqslant (gl.dim.\, R) - r$$

and let $\mathcal{P}$ be a maximal ideal of R containing $a_1, a_2, \ldots,$ and a_r. Then we see that $gl.dim.\, R_{\mathcal{P}}/(a_1R_{\mathcal{P}} + a_2R_{\mathcal{P}} + \cdots + a_rR_{\mathcal{P}})$ is finite and moreover, $a_1/1, a_2/1, \ldots, a_r/1$ is an R-sequence in $R_{\mathcal{P}}$. So without loss of generality, we may assume that R is a local ring with the unique maximal ideal $\mathcal{P}$. Let $T = R/(a_1R + a_2R + \cdots + a_{r-1}R)$ and put $t = a_r + (a_1R + a_2R + \cdots + a_{r-1}R) \in T$. Then T is a local ring,

$$t \in \overline{\mathcal{P}} =: \mathcal{P}/(a_1R + a_2R + \cdots + a_{r-1}R)$$

and it is a regular non-unit element in T. Since $T/tT = R/(a_1R + a_2R + \cdots + a_rR)$ has finite global dimension, T has also finite global dimension by [**9**, Theorem 10, p.180].

We now claim that $t \notin \overline{\mathcal{P}}^2$. Assume the contrary. Since T is local with finite global dimension, it is a regular local domain. Let $n = gl.dim.\, T$. Then the Krull dimension of T, $dim.\, T$, is also n, and equals the vector space dimension of $\overline{\mathcal{P}}/\overline{\mathcal{P}}^2$ over the field $T/\overline{\mathcal{P}}$. Since s is regular, $dim.\, T/tT \leqslant n - 1$. But $t \in \overline{\mathcal{P}}^2$, and so the dimension of the vector space $(\overline{\mathcal{P}}/tT)/(\overline{\mathcal{P}}/tT)^2$ over the field $(T/tT)/(\overline{\mathcal{P}}/tT) \cong T/\overline{\mathcal{P}}$ is n. This implies that T/tT is not a regular local domain, a contradiction. Therefore $t \notin \overline{\mathcal{P}}^2$ and hence $a_r \notin \mathcal{P}^2 + a_1R + a_2R + \cdots + a_{r-1}R$. In a similar way, $a_i \notin \mathcal{P}^2 + a_1R + a_2R + \cdots + a_{i-1}R$ for $i = 1, 2, \ldots, r$. This proves one direction.

Conversely assume that $\mathcal{P}$ is a maximal ideal of R containing $a_1, a_2, \ldots, a_r$. Then by assumption,

$$a_i \notin \mathcal{P}^2 + a_1R + a_2R + \cdots + a_{i-1}R \text{ for } i = 1, 2, \ldots, r.$$

In the localisation $R_{\mathcal{P}}$ of R at maximal ideal $\mathcal{P}$, we have

$$a_i/1 \notin \mathcal{P}_{\mathcal{P}}^2 + a_1R_{\mathcal{P}} + a_2R_{\mathcal{P}} + \cdots + a_{i-1}R_{\mathcal{P}}, \quad i = 1, 2, \ldots, r.$$

In fact, if

$$a_i/1 \in \mathcal{P}_{\mathcal{P}}^2 + a_1R_{\mathcal{P}} + a_2R_{\mathcal{P}} + \cdots + a_{i-1}R_{\mathcal{P}},$$

then there is

$$c \in R\backslash\mathcal{P} \text{ such that } a_ic \in \mathcal{P}^2 + a_1R + a_2R + \cdots + a_{i-1}R.$$

Let

$$A_i = \{b \in R \mid a_ib \in \mathcal{P}^2 + a_1R + a_2R + \cdots + a_{i-1}R\}.$$

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Then A_i is an ideal of R. Since $c \in R\backslash\mathcal{P}$, we have $A_i \not\subseteq \mathcal{P}$ and so $A_i + \mathcal{P} = R$. Therefore

$$a_i \in a_i(A_i + \mathcal{P}) \subseteq \mathcal{P}^2 + a_1R + a_2R + \cdots + a_{i-1}R,$$

which is a contradiction. So by Theorem 1

$$gl.dim.\, R_\mathcal{P}/(a_1R_\mathcal{P} + a_2R_\mathcal{P} + \cdots + a_rR_\mathcal{P}) \leqslant (gl.dim.\, R_\mathcal{P}) - r.$$

Hence we have the desired result. ☐

Observe that the necessary condition of Proposition 2 is more refined than the assumption in Theorem 1. On the other hand, one may suspect that the necessary condition in Proposition 2 can be replaced by somewhat more simple form "$a_i \notin \mathcal{P}^2$, $i = 1, 2, \ldots, r$ for any maximal ideal $\mathcal{P}$ of R containing $a_1, a_2, \ldots, a_r$". But this is not possible as the following example shows.

EXAMPLE 1. Let $R = C[[x, y]]$, the formal power series ring over the complex number field C. Then $gl.dim.\, R = 2$ and $\mathcal{P} = xC[[x,y]] + yC[[x,y]]$ is the unique maximal ideal of R. In this case the elements $a_1 = x$, $a_2 = x + y^2$ form an R-sequence. Moreover $x \notin \mathcal{P}^2$ and $x + y^2 \notin \mathcal{P}^2$. However, the ring $R/(a_1R + a_2R)$ can not have finite global dimension. Indeed, $gl.dim.\, R/a_1R = 1$. So if $R/(a_1R + a_2R)$ has finite global dimension, then its global dimension is 0, a contradiction.

REMARK 3. 1. If in Proposition 2 the elements $a_1, a_2, \ldots, a_r$ lie in $J(R)$, then the global dimension of $R/(a_1R + a_2R + \cdots + a_rR)$ is $(gl.dim.\, R) - r$.

2. If K is an algebraically closed field and P is the prime ideal generated by an R-sequence $f_1, f_2, \ldots, f_r$ in $K[x_1, x_2, \ldots, x_n]$, then an application of Proposition 2 and [**6**, Theorem 5.1., p.32] yields the statement of Proposition 1 (2).

From Proposition 2 we immediately get the following corollary, generalising [**7**, Theorem 2].

COROLLARY 5. *Assume that R is a commutative Noetherian ring with finite global dimension and $f(x)$ is a regular non-unit polynomial in S. Then we have the inequality $gl.dim.\, S/f(x)S \leqslant gl.dim.\, R$ if and only if $f(x) \notin \mathcal{P}^2$ for any maximal ideal $\mathcal{P}$ of S.*

To elaboreate on the assumption of Theorem 1 we next consider some examples of Noetherian rings, in which there is a central regular non-unit element not in the square of any maximal ideal. Two typical examples – up to Morita equivalence – are:

1. Ω an order in a division ring.
2. The polynomial ring $R[x]$ or the formal power series ring $R[[x]]$ over a ring R with finite global dimension.

EXAMPLE 2.

1. The ring $C[[x,y,z]][1,\alpha,\beta,\alpha\beta]$ with $\alpha^2 = x + z^2$, $\beta^2 = y$, $\alpha\beta = -\beta\alpha$ is a local ring with global dimension 3, where $C[[x,y,z]]$ is the formal power series ring over the complex number field C. In this case the maximal ideal $\mathcal{J}$ is generated by $\{\alpha, \beta, z\}$. Note that z is a central regular non-unit element which is not in $\mathcal{J}^2$ (see [**15**, Example (b), p.351]).

2. The ring $R = C((x,y))[[z]][1,\alpha,\beta,\alpha\beta]$ is the localisation of the ring in (1) with respect to the ideal (x,y). In R the element z is a central regular element and the ring R/zR is simple Artinian. So the global dimension of R is 1 by [**14**, Proposition 5.6]. Moreover R is a local prime PI hereditary ring with center $C((x,y))[[z]]$. In $R[t]$ the polynomial $f(t) = t^3 - x \in R[t]$ is a central regular non-unit, which generates is a prime ideal of $R[t]$, and it is easily checked that the factor ring $R[t]/f(t)R[t]$ is hereditary. Therefore $f(t)$ is not in the square of any maximal ideal of $R[t]$ (see [**12, 13**]).

3. In [**4**, Example 7.1] a local ring T is constructed with is a central regular non-unit element u, which is not in the square of the maximal ideal.

4. In [**15**] M. Ramras constructs for every positive integer $n \geqslant 3$ a ring R satisfying the following conditions:

(a) R is a prime right Noetherian ring which is finitely generated as a module over its center.

(b) $r.gl.dim.\, R = n$.

(c) There is a central regular non-unit element a such that aR is a prime ideal of R.

(d) $r.gl.dim.\, R/aR = n - 1$, but $a \in \mathcal{P}^2$ for some maximal ideal $\mathcal{P}$ of R.

5. Let $A = C[[x,y,z]][1,\alpha,\beta,\alpha\beta]$ with $\alpha^2 = x, \beta^2 = y$ and $\beta\alpha = -\alpha\beta + z$, where $C[[x,y,z]]$ is the formal power series ring over the complex number field C. Then A is a prime local Noetherian ring, which is finitely generated as a module over its center. In this case the ideal $\mathcal{J}$ generated by α and β is the maximal ideal of A, and z is a central regular non-unit element of A, and zA is a prime ideal. By [**15**, Example 2, p.351], $r.gl.dim.\, A = 3$ and $r.gl.dim.\, A/zA = 2$. However $z \in \mathcal{J}^2$. (This is an example from [**15**].)

Let $k = n - 3$ and let $R = A[x_1, x_2, \ldots, x_k]$ be the polynomial ring over A. Then R is a prime Noetherian ring with global dimension n, which is finitely generated over its center and $r.gl.dim.\, R/zR = n - 1$. The element z is a central regular non-unit such that zR is a prime ideal of R. However z is in the square of the maximal ideal

$$\mathcal{P} = \mathcal{J}[x_1, x_2, \ldots, x_k] + x_1R + x_2R + \cdots + x_kR.$$

If we take R to be the formal power series ring $A[[x_1, x_2, \ldots, x_k]]$, then R is a local ring satisfying the above conditions (a), (b), (c), and (d).

The converse of Theorem 1 holds however, if $r.gl.dim. R \leqslant 2$, as stated in the introduction as Theorem 2, **which we shall now prove.**

By Lemma 4 and Theorem 1 we only need to prove the necessary condition.

Assume that R/aR is right hereditary and let $\mathcal{P}$ be a maximal ideal of R. If $a \notin \mathcal{P}$, then $a \notin \mathcal{P}^2$ and so we are done.

Suppose $a \in \mathcal{P}$ Then $\mathcal{P}/aR$ is projective as a right R/aR-module, since R/aR is right hereditary. Thus the exact sequence

$$0 \longrightarrow R/\mathcal{P} \longrightarrow aR/a\mathcal{P} \subseteq \mathcal{P}/a\mathcal{P} \longrightarrow \mathcal{P}/aR \longrightarrow 0$$

splits. Therefore it also splits as a sequence of R-modules. Hence there is an R-submodule T of $\mathcal{P}$ such that

$$\mathcal{P} = aR + T\,,\ a\mathcal{P} = aR \cap T \text{ and } \mathcal{P}/a\mathcal{P} \cong (aR/a\mathcal{P}) \oplus (T/a\mathcal{P})\,.$$

So we have $aR/a\mathcal{P} \cong \mathcal{P}/T$ as R-modules because of the relations $\mathcal{P} = aR + T$ and $a\mathcal{P} = aR \cap T$. Since $R/\mathcal{P} \cong aR/a\mathcal{P}$, we have an isomorphism $R/\mathcal{P} \cong \mathcal{P}/T$. Because $(R/)\mathcal{P} \cdot \mathcal{P} = 0$ we also have $(\mathcal{P}/T) \cdot \mathcal{P} = 0$; that is, $\mathcal{P}^2 \subset T$.

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