SMALL FORCING MAKES ANY CARDINAL SUPERDESTRUCTIBLE

JOEL DAVID HAMKINS

Abstract. Small forcing always ruins the indestructibility of an indestructible supercompact cardinal. In fact, after small forcing, any cardinal κ becomes *superdestructible*—any further $<\kappa$-closed forcing which adds a subset to κ will destroy the measurability, even the weak compactness, of κ. Nevertheless, after small forcing indestructible cardinals remain resurrectible, but never strongly resurrectible.

Arthur Apter, motivated by issues arising in his recent paper [1] with Saharon Shelah, asked me the following question: "Does small forcing preserve the inde-structibility of a supercompact cardinal after the Laver preparation?" While it is tempting to believe that all large cardinal properties are preserved by small forcing, the fact is that the answer to his question is no. Even adding a Cohen real ruins the indestructibility of any cardinal. What's more, it is ruined in a very strong way. In this paper I will prove that small forcing makes any cardinal superdestructible.

Before stating my theorem, let me make some definitions. In one of my favorite arguments, Laver [5] proved that with the proper preparation, now called the Laver preparation, a supercompact cardinal κ can be made *indestructible* in the sense that any $<\kappa$-directed closed forcing preserves the supercompactness of κ. We say that κ is *destructible*, therefore, when some $<\kappa$-directed closed poset destroys the supercompactness of κ. Going beyond this, define that κ is *superdestructible* when every $<\kappa$-closed forcing which adds a subset to κ destroys the measurability of κ, and that κ is *superdestructible at* λ when any $<\kappa$-closed forcing which adds a subset to λ destroys the λ-supercompactness of κ. Define κ to be *resurrectible* if and only if whenever a $<\kappa$-directed closed forcing $\mathbb{Q}$ happens to destroy the supercompactness of κ, it can nevertheless be restored with further $<\kappa$-distributive forcing $\mathbb{R}$; and κ is *strongly resurrectible* when $\mathbb{R}$ can be made actually $<\kappa$-closed (this resembles the notion for huge cardinals in [2]). Finally, a poset $\mathbb{P}$ is *small* relative to κ when $|\mathbb{P}| < \kappa$. Throughout I consider only nontrivial posets—forcing with them must add some new set. Now I am ready to state my main theorem.

MAIN THEOREM. *Small forcing makes any cardinal superdestructible. Indeed, after small forcing, any $<\kappa$-closed forcing which adds a subset to κ will destroy the weak compactness of κ. What's more, after small forcing, κ becomes superdestructible at κ^+, κ^{++}, etc. Nevertheless, after small forcing an indestructible cardinal remains resurrectible, but never strongly resurrectible.*

I will actually prove a better theorem: after forcing of size $\beta < \kappa$, any $\leq\beta$-closed forcing which adds a subset but no bounded subset to κ will destroy the measurability and weak compactness of κ. After adding a Cohen real, for example, any countably closed poset which adds a subset but no bounded subset to κ will destroy the measurability of κ.

This theorem is related to my Fragile Measurability theorems in [3]. There, I show how to force from a model in which κ is strong, supercompact, or I_1, while preserving this large cardinal property, to a model in which the measurability of κ is *fragile* in the sense that it is destroyed by any forcing which preserves $\kappa^{<\kappa}$, κ^+, but not $P(\kappa)$. To get superdestructibility from fragility we drop the requirement that κ^+ is preserved, but require the poset to be a little closed. The two properties are similar in that if κ is fragile or superdestructible, the measurability of κ is easily destroyed by forcing. In my fragile measurability models [3], κ is both fragile and superdestructible.

What is perhaps the first theorem in this line is due to W. Hugh Woodin [6], who forced to a model of a supercompact cardinal κ whose measurability and weak compactness is destroyed by the poset $\text{Add}(\kappa, 1) = \kappa^{<\kappa}$. Woodin used a reverse Easton κ-iteration, adding a system of coherent clubs. Later, he simplified his argument to add just a subset of δ at certain stages δ. My theorems here show that his entire κ-iteration may be replaced by any small forcing, such as adding a Cohen real. But certainly Woodin's argument is the inspiration for both my fragile measurability result [3] and also this paper.

Because in the inner models like $L[\mu]$ the large cardinal property is fragile and superdestructible, all these theorems—Woodin's theorem, my Fragile Measurability theorem, and the Superdestruction theorem—tend to show how one may obtain inner-model-like properties by forcing. For superdestructibility this is interesting; it has the consequence that large cardinals, in principle, cannot automatically have any amount of indestructibility.

Before beginning the proof, I would like to point out that in response to Apter's question Saharon Shelah has proved, independently, that small forcing makes κ destructible. His technique is to code the small generic g into the continuum function above κ. If λ is above all this coding, then a reflection argument shows that since the continuum function below κ cannot code the new set, κ cannot be still λ-supercompact. Since it relies, however, on building a particular $<\kappa$-closed poset which will destroy the supercompactness of κ, this technique seems not to show superdestructibility. My argument establishes the stronger result that essentially *all* such posets kill the supercompactness of κ.

Let's now begin my proof. I will rely on the following fact. Woodin based the theorem I mentioned above on a similar fact concerning his reverse Easton κ-iterations.

KEY LEMMA. *If $|\mathbb{P}| = \beta$, $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\leq\beta$-closed, and $\text{cof}(\lambda) > \beta$, then $\mathbb{P} * \dot{\mathbb{Q}}$ adds no new subset of λ all of whose initial segments are in the ground model V.*

PROOF. Such sets, which are not in V but all of whose initial segments are in V, I will say are *fresh* over V. If the lemma fails for some $\mathbb{P}$ and $\dot{\mathbb{Q}}$, then we may assume

there is a name τ for the characteristic function of the fresh set, so that

$$\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \tau \in 2^{\check{\lambda}} \,\&\, \tau \notin \check{V} \,\&\, \forall \gamma < \check{\lambda} \, \tau{\restriction}\gamma \in \check{V}.$$

By refining to a condition if necessary, we may assume that $\mathbb{P}$ adds a fresh subset to some minimal $\delta \leq \beta$, so

$$1 \Vdash_{\mathbb{P}} \dot{h} \in 2^{\check{\delta}} \,\&\, \dot{h} \notin \check{V} \,\&\, \forall \alpha < \check{\delta} \, \dot{h}{\restriction}\alpha \in \check{V}.$$

(I will actually only use that $\dot{\mathbb{Q}}$ is $\leq\delta$-closed.) The basic idea of this proof will be to use the small set h added by $\mathbb{P}$ to define a path through an initial segment of the tree of attempts to decide more and more of τ, using the $\leq\delta$-closure of $\dot{\mathbb{Q}}$. Since all the initial segments of τ are in V we will find a set b in V which reveals to us the path determined by h, and this will contradict the fact that h is not in V. A bit of notation: if $\langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$, then let $b_{\langle p,\dot{q}\rangle}$ be the longest sequence b such that $\langle p, \dot{q} \rangle \Vdash \check{b} \subseteq \tau$. Also, write $\langle p, \dot{q} \rangle \parallel \tau{\restriction}\gamma$ to mean that $\langle p, \dot{q} \rangle$ decides $\tau{\restriction}\gamma$, i.e., $\langle p, \dot{q} \rangle \Vdash \tau{\restriction}\gamma = \check{b}$ for some $b \in V$. The crucial aspect of the following claim is that the first coordinate p does not vary.

CLAIM. *There is $\langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ such that whenever $g * G$ is V-generic below $\langle p, \dot{q} \rangle$ then for every $\gamma < \lambda$ there is a condition $\langle p, \dot{r} \rangle \in g * G$ such that $\langle p, \dot{r} \rangle \parallel \tau{\restriction}\gamma$.*

PROOF. Let $g * G$ be V-generic for $\mathbb{P} * \dot{\mathbb{Q}}$. In $V[g][G]$ pick for every $\gamma < \lambda$ a condition $\langle p_\gamma, \dot{q}_\gamma \rangle \in g * G$ such that $\langle p_\gamma, \dot{q}_\gamma \rangle \parallel \tau{\restriction}\gamma$. Thus $\langle\, p_\gamma \mid \gamma < \lambda \,\rangle$ is a sequence of conditions from the poset $\mathbb{P}$. Since $\mathrm{cof}(\lambda) > \beta$, and this is preserved by $\mathbb{P}$ and $\mathbb{Q}$, there must be some condition $\bar{p}$ which is repeated cofinally. In fact, we could have used $\bar{p}$ in every choice. So assume that $\langle \bar{p}, \dot{q}_\gamma \rangle$ decides $\tau{\restriction}\gamma$ for every $\gamma < \lambda$. This fact must be forced by some $\langle p, \dot{q} \rangle$, where $p \leq \bar{p}$. Thus, any generic $g * G$ containing $\langle p, \dot{q} \rangle$ satisfies

$$\forall \gamma < \lambda \, \exists \dot{r} \, \langle \bar{p}, \dot{r} \rangle \in g * G \,\&\, \langle \bar{p}, \dot{r} \rangle \parallel \tau{\restriction}\gamma.$$

Now replace $\bar{p}$ with the stronger condition p to conclude the claim. $\dashv$

Fix $\langle p, \dot{q} \rangle$ as in the claim.

CLAIM. *For any $\langle p, \dot{r} \rangle \leq \langle p, \dot{q} \rangle$ there are $\dot{r}_0$ and $\dot{r}_1$ such that $\langle p, \dot{r}_0 \rangle, \langle p, \dot{r}_1 \rangle \leq \langle p, \dot{r} \rangle$ and $b_{\langle p,\dot{r}_0\rangle} \perp b_{\langle p,\dot{r}_1\rangle}$.*

PROOF. If not, then some $\langle p, \dot{r} \rangle$ fails to split in that sense. Force below $\langle p, \dot{r} \rangle$ to obtain V-generic $g * G$ with $\langle p, \dot{r} \rangle \in g * G$. Because of the splitting failure, all $b_{\langle p,\dot{s}\rangle}$ with $\langle p, \dot{s} \rangle \leq \langle p, \dot{r} \rangle$ must cohere. But by the property of the first claim, they also decide more and more of τ. Thus,

$$\tau_{g*G} = \bigcup \{\, b_{\langle p,\dot{s}\rangle} \mid \langle p, \dot{s} \rangle \leq \langle p, \dot{r} \rangle \,\},$$

which contradicts that $\Vdash \tau \notin \check{V}$. $\dashv$

Iterating the claim transfinitely, I define $\dot{q}_t$ by induction on $t \in 2^{<\delta}$, so that $\dot{q}_\emptyset = \dot{q}$ and

(1) $t \subseteq \bar{t} \implies \langle p, \dot{q}_{\bar{t}} \rangle \leq \langle p, \dot{q}_t \rangle$
(2) $b_{\langle p,\dot{q}_{t^\frown 0}\rangle} \perp b_{\langle p,\dot{q}_{t^\frown 1}\rangle}.$

At successor stages, simply apply the claim. At limit stages, when $\dot{q}_t$ is defined for all $t \subsetneq \bar{t}$, then $p \Vdash_{\mathbb{P}} \langle \dot{q}_t \mid t \subsetneq \bar{t} \rangle$ is descending, and so by the closure of $\dot{\mathbb{Q}}$ we obtain $\dot{q}_{\bar{t}}$.

Now force below $\langle p, \dot{q} \rangle$ so that $\langle p, \dot{q} \rangle \in g * G$ for some V-generic $g * G$. Let $h = (\dot{h})_g$ be the new δ-sequence which was added by $\mathbb{P}$. Thus every initial segment $t \subsetneq h$ is in V. Let $q_t = (\dot{q}_t)_g$. By condition (1) it follows that $\langle q_t \mid t \subsetneq h \rangle$ is a δ-descending sequence in $\mathbb{Q} = \mathbb{Q}_g$, and so by closure there is a condition r such that $r \leq q_t$ for all $t \subsetneq h$. Let $b = \bigcup_{t \subsetneq h} b_{\langle p, \dot{q}_t \rangle}$. Thus, $r \Vdash_{\mathbb{Q}} b \subsetneq \tau$, and therefore $b \in V$. But this is impossible, since b will decode for us in V the generic set h: by construction, $b_{\langle p, \dot{q}_t \rangle} \subseteq b$ only when $t \subseteq h$, since condition (2) ensures that whenever $t \hat{\ } i$ first deviates from h, then $b_{\langle p, \dot{q}_{t\hat{\ }i} \rangle}$ will deviate from b. We therefore conclude that $h \in V$, contrary to our choice. $\dashv$

Now I am ready to prove the main theorem in parts.

SUPERDESTRUCTION THEOREM I. *Small forcing makes any cardinal superdestructible.*

PROOF. It suffices to show that if card $\mathbb{P} < \kappa$, and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $<\kappa$-closed, and adds a new subset of κ, then κ is not measurable after forcing with $\mathbb{P} * \dot{\mathbb{Q}}$. Let's suppose this fails for some $\mathbb{P} * \dot{\mathbb{Q}}$, and that $V[g][G]$ is a forcing extension by $\mathbb{P} * \dot{\mathbb{Q}}$ in which κ is measurable. Since κ is measurable, there is an embedding $j : V[g][G] \to N$ for some transitive N with $\mathrm{cp}(j) = \kappa$. By elementarity we may decompose N into its forcing history and write the embedding as $j : V[g][G] \to M[g][j(G)]$ for some transitive M. One should not assume that $M \subseteq V$, since the embedding j is not neccessarily the lift of an embedding in V. Nevertheless, we have the following claim:

CLAIM. $P(\kappa)^M \subseteq V$.

PROOF. First note that $M_\kappa = V_\kappa$ since $\mathrm{cp}(j) = \kappa$. Now suppose that $B \subseteq \kappa$ and $B \in M$. Thus, $B \cap \alpha$ is in V for every $\alpha < \kappa$, and so every initial segment of B is in V. It follows by the Key Lemma that $B \in V$. $\dashv$

Let $A \subseteq \kappa$ be the new set added by $\mathbb{Q}$, so $A \in V[g][G] \setminus V[g]$. Since $A = j(A) \cap \kappa$ it follows that $A \in M[g][j(G)]$. But the $j(G)$ forcing was $<j(\kappa)$-closed, and so actually $A \in M[g]$. Therefore, $A = \dot{A}_g$ for some name $\dot{A} \in M$. We may view $\dot{A}$ as a function from κ to the set of anti-chains in $\mathbb{P}$, and this can be coded with a subset of κ. So, by the claim, $\dot{A} \in V$, and thus $A = (\dot{A})_g \in V[g]$. This contradicts the choice of A. $\dashv$

Before going on to the improved versions of the Superdestruction Theorem, let me just point out the following corollary.

COROLLARY. *One can force to make every large cardinal superdestructible.*

PROOF. Just add a Cohen real and apply the Superdestruction Theorem. $\dashv$

That it is so easy to make cardinals superdestructible is surprising, since in [3] a very great effort is made to make a single supercompact cardinal have fragile measurability. This corollary also shows that Woodin's entire reverse Easton iteration—the one which makes the measurability of a supercompact cardinal κ

destructible by $\text{Add}(\kappa, 1)$—can be replaced by the forcing to add a Cohen real or indeed any small forcing, with the result that every cardinal κ becomes destructible by $\text{Add}(\kappa, 1)$, among many other posets.

SUPERDESTRUCTION THEOREM II. *After small forcing, any $<\kappa$-closed forcing which adds a subset to κ will destroy the weak compactness of κ.*

PROOF. We will follow the proof of the previous theorem, but use instead only a weakly-compact embedding. Let $V[g][G]$, etc., be as in the earlier proof. Now suppose only that κ is weakly compact in $V[g][G]$. Pick $\lambda \gg \kappa$ very large, and let $X \prec V_\lambda[g][G]$ be an elementary submodel of size κ with $V_\kappa \subseteq X$ and g, G, $\mathbb{P}$, $\dot{\mathbb{Q}}$, $A \in X$. The Mostowski collapse of X will be a structure $N[g][G^*]$ of size κ, where N is transitive. By the weak-compactness of κ there is an embedding $j: N[g][G^*] \to M[g][j(G^*)]$ for some transitive M with $\text{cp}(j) = \kappa$. Since again by the critical point we know that $M_\kappa = V_\kappa$, it follows by the Key Lemma that $P(\kappa)^M \subseteq V$. Now argue again that $A \in M[g]$ and so $A = \dot{A}_g$ for some name $\dot{A} \in M$. But again $\dot{A}$ can be thought of as a function from κ to the antichains of $\mathbb{P}$, and so it may be coded as a subset of κ. Thus, again $\dot{A} \in V$, and so $A = \dot{A}_g \in V[g]$, contrary to the choice of A. ⊣

Next, I push the previous arguments up to the case where the new sets are added by $\mathbb{Q}$ perhaps only above κ.

SUPERDESTRUCTION THEOREM III. *After small forcing any cardinal κ becomes superdestructible at κ, at κ^+, at κ^{++}, etc. In fact, if the small forcing is $<\delta$-distributive, then κ becomes superdestructible at every λ below $\aleph_{\kappa+\delta}$.*

PROOF. Suppose that card $\mathbb{P} < \kappa$, that $\mathbb{P}$ is $<\delta$-distributive, but, using the notation of the previous proofs, that κ remains λ-supercompact in $V[g][G]$, where $G \subseteq \mathbb{Q}$ adds a new subset $A \subseteq \lambda$, and $\lambda = \aleph_{\kappa+\beta}$ for some $\beta < \delta$. We may assume that $\mathbb{Q}$ adds no new subsets of any smaller ordinal. In $V[g][G]$ there is a λ-supercompact embedding $j: V[g][G] \to M[g][j(G)]$.

CLAIM. $P(\lambda)^M \subseteq V$.

PROOF. I will show by induction that $P(\aleph_{\kappa+\alpha})^M \subseteq V$ for all $\alpha \leq \beta$. To begin, we know by the argument in the previous theorems that $P(\kappa)^M \subseteq V$, since there are no new subsets of κ in $V[g][G]$ all of whose initial segments are in V. Also, I claim that

$$(\aleph_{\kappa+\alpha})^M = (\aleph_{\kappa+\alpha})^{M[g][j(G)]} = (\aleph_{\kappa+\alpha})^{V[g][G]} = (\aleph_{\kappa+\alpha})^V.$$

The first equality holds because of the smallness of $\mathbb{P}$ and the closure of $j(\mathbb{Q})$. The second equality holds because of the closure of the embedding j. The last equality holds by the smallness of $\mathbb{P}$ and the minimality of λ. Now suppose that $P(\aleph_{\kappa+\alpha})^M \subseteq V$, that $B \subseteq \aleph_{\kappa+(\alpha+1)}$, and that $B \in M$. Every initial segment of B is coded with a subset of $\aleph_{\kappa+\alpha}$ in M, and therefore lies in V by the induction hypothesis. Since $\aleph_{\kappa+(\alpha+1)}$ is regular, it follows by the Key Lemma that B is in V. This completes the successor stage. Now suppose that $P(\aleph_{\kappa+\alpha})^M \subseteq V$ for all $\alpha < \gamma$ where $\gamma \leq \beta$ is a limit ordinal. If $B \subseteq \aleph_{\kappa+\gamma}$ and $B \in M$ then again every initial segment of B is in V by the induction hypothesis. But the forcing $\mathbb{P}$ is $<\delta$-distributive, and $\gamma \leq \beta < \delta$, so $\mathbb{P}$ cannot add B (this is where the limitation on λ

is used). Similarly, the highly closed $\mathbb{Q}$ cannot add B, so it must be that $B \in V$. This establishes the limit case, and so the claim is proved.   ⊣

Since the embedding is closed under λ-sequences, it follows that $A \in M[g][j(G)]$. But $j(G)$ is $<j(\kappa)$-closed and $\lambda < j(\kappa)$, so $A \in M[g]$, and thus $A = \dot{A}_g$ for some name $\dot{A} \in M$. Again, we may view $\dot{A}$ as a function from λ to the set of antichains of $\mathbb{P}$. Thus, $\dot{A}$ may be coded with a subset of λ in M. By the claim it follows that $\dot{A} \in V$, and so $A = \dot{A}_g \in V[g]$, contrary to our choice of A.   ⊣

SUPERDESTRUCTION THEOREM IV. *Suppose that* $\mathbb{P}$ *has cardinality* β, *adds a new subset to* $\bar{\delta}$, *and is* $<\delta$-*distributive. Suppose also that* $\beta < \kappa \le \lambda < \aleph_{\kappa+\delta}$. *Then any further* $\le \bar{\delta}$-*closed forcing which preserves* $2^{<\kappa}$ *but adds a subset to* λ *will destroy the* λ-*supercompactness of* κ.

PROOF. Just apply the full power of the Key Lemma to the previous proofs. We never used full $<\kappa$-closure—rather, we used $\le\bar{\delta}$-closure to apply the Key Lemma, and we used the preservation of $j(2^{<\kappa})$ by $j(\mathbb{Q})$ to know that $A \in M[g]$. So the proofs go through for the broader class of posets in this theorem.   ⊣

This last version of the Superdestruction Theorem is actually an enormous improvement, reducing $<\kappa$-closure to something much less. If, for example, $\mathbb{P}$ is the forcing to add a Cohen real, then we obtain the following corollary.

COROLLARY. *After adding a Cohen real, the measurability of any cardinal* κ *is destroyed by any countably-closed poset which adds a new subset, but no bounded subset, to* κ. *Similarly, the* λ-*supercompactness of* κ *is destroyed by any countably-closed poset which adds a new subset to* λ, *but no bounded subset to* κ, *for* $\lambda = \kappa$, κ^+, κ^{++}, *etc.*

Finally, I will show that indestructible cardinals are not too severely wounded when they are made superdestructible; they remain resurrectible (this was proved, independently, by James Cummings). My proof uses the instrumental Term Forcing Lemma, a part of mathematical folklore, which allows us in a sense to reverse the order of an iteration $\mathbb{P} * \dot{\mathbb{Q}}$.

TERM FORCING LEMMA. *If* $\mathbb{P} * \dot{\mathbb{Q}}$ *is a forcing iteration, then there is a poset* $\mathbb{Q}_{\text{term}}$ *such that forcing with the product* $\mathbb{Q}_{\text{term}} \times \mathbb{P}$ *produces canonically a generic for the poset* $\mathbb{P} * \dot{\mathbb{Q}}$. *Hence, forcing with* $\mathbb{Q}_{\text{term}} \times \mathbb{P}$ *is equivalent to forcing with* $\mathbb{P} * \dot{\mathbb{Q}} * \ddot{\mathbb{R}}$ *for some (name of a) poset* $\ddot{\mathbb{R}}$. *Finally, if* $1 \Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ *is* $<\kappa$-*directed closed, then* $\mathbb{Q}_{\text{term}}$ *is also* $<\kappa$-*directed closed.*

PROOF. We may assume, by using a better name if necessary, that $\dot{\mathbb{Q}}$ is a full name, in the sense that if $1 \Vdash_{\mathbb{P}} \sigma \in \dot{\mathbb{Q}}$ then there is a name $\tau \in \text{dom}(\dot{\mathbb{Q}})$ such that $1 \Vdash_{\mathbb{P}} \sigma = \tau$. Now let

$$\mathbb{Q}_{\text{term}} = \{ \sigma \in \text{dom}(\dot{\mathbb{Q}}) \mid 1 \Vdash \sigma \in \dot{\mathbb{Q}} \}.$$

Define the order $\sigma \le_{\text{term}} \tau$ if and only if $1 \Vdash_{\mathbb{P}} \sigma \le_{\mathbb{Q}} \tau$. Now suppose that $G_{\text{term}} \subseteq \mathbb{Q}_{\text{term}}$ is V-generic, and $g \subseteq \mathbb{P}$ is $V[G_{\text{term}}]$-generic. We must find in $V[G_{\text{term}}][g]$ a generic for $\mathbb{P} * \dot{\mathbb{Q}}$. Let $G = \{ \sigma_g \mid \sigma \in G_{\text{term}} \}$.

CLAIM. $g * G$ *is* V-*generic for* $\mathbb{P} * \dot{\mathbb{Q}}$.

PROOF. We know that $g \subseteq \mathbb{P}$ is V-generic, so it suffices to show that G is $V[g]$-generic for $\mathbb{Q} = \dot{\mathbb{Q}}_g$. First observe that G is truly a filter, since if σ_g, τ_g are in G, with $\sigma, \tau \in G_{\text{term}}$, then there must be some term $\eta \in G_{\text{term}}$ such that $\eta \leq_{\text{term}} \sigma, \tau$. It follows that $\eta_g \leq_{\mathbb{Q}} \sigma_g, \tau_g$. So G is a filter. Let's now check the genericity criterion. Suppose that $D \subseteq \mathbb{Q}$ is dense, where $D = \dot{D}_g$ for some name $\dot{D}$. We may assume that $1 \Vdash_{\mathbb{P}} \dot{D}$ is dense in $\dot{\mathbb{Q}}$. Now let

$$D_{\text{term}} = \{ \, \sigma \in \mathbb{Q}_{\text{term}} \mid 1 \Vdash_{\mathbb{P}} \sigma \in \dot{D} \, \}.$$

Observe that D_{term} is dense in $\mathbb{Q}_{\text{term}}$ since given any name $\sigma \in \mathbb{Q}_{\text{term}}$ we may find a name τ such that $1 \Vdash_{\mathbb{P}} \tau \leq_{\mathbb{Q}} \sigma \,\&\, \tau \in \dot{D}$. Thus there is a name $\sigma \in G_{\text{term}} \cap D_{\text{term}}$, and so $\sigma_g \in G \cap D$. $\dashv$

Since $\mathbb{Q}_{\text{term}} \times \mathbb{P}$ produces a generic $g * G$ for $\mathbb{P} * \dot{\mathbb{Q}}$, it follows that the regular open algebra of $\mathbb{P} * \dot{\mathbb{Q}}$ completely embeds into the regular open algebra of $\mathbb{Q}_{\text{term}} \times \mathbb{P}$ via the map

$$\langle p, \dot{q} \rangle \mapsto [\![\langle p, \dot{q} \rangle \in g * G]\!]^{\mathbb{Q}_{\text{term}} \times \mathbb{P}}.$$

By standard quotient forcing arguments (see, e.g., [4], p. 237, ex. 23.6), it follows that forcing with $\mathbb{Q}_{\text{term}} \times \mathbb{P}$ is equivalent to forcing with $\mathbb{P} * \dot{\mathbb{Q}} * \ddot{\mathbb{R}}$ for some (name of a) poset $\ddot{\mathbb{R}}$.

It remains to prove the last sentence of the lemma. Suppose that

$$1 \Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \text{ is } {<}\kappa\text{-directed closed},$$

and that $A \subseteq \mathbb{Q}_{\text{term}}$ is a ${<}\kappa$ size family with the FIP. With a slight abuse of name notation, it follows that $1 \Vdash_{\mathbb{P}} A \subseteq \dot{\mathbb{Q}}$ is a ${<}\kappa$ size family with the FIP. Using the directed closure of $\dot{\mathbb{Q}}$, we obtain a name σ such that $1 \Vdash \sigma \leq_{\mathbb{Q}} \tau$ for every $\tau \in A$. Thus, $\sigma \leq_{\text{term}} \tau$ for every $\tau \in A$, and the lemma is proved. $\dashv$

RESURRECTION THEOREM. *After small forcing an indestructible cardinal remains resurrectible, but never strongly resurrectible.*

PROOF. The implicit claim of this theorem, that indestructible cardinals are resurrectible, is clear: if κ is indestructible, and $\mathbb{Q}$ is ${<}\kappa$-directed closed, then κ is supercompact in $V^{\mathbb{Q}}$. So no further forcing needs to be done to recover the supercompactness of κ. Thus, indestructible cardinals are in fact strongly resurrectible.

Now suppose that $\mathbb{P}$ is small. I will show that κ remains resurrectible after forcing with $\mathbb{P}$. So suppose $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is ${<}\kappa$-directed closed. I want to recover the supercompactness of κ by further forcing after $\mathbb{P} * \dot{\mathbb{Q}}$. By the Term Forcing Lemma, forcing with $\mathbb{Q}_{\text{term}} \times \mathbb{P}$ is equivalent to forcing with $\mathbb{P} * \dot{\mathbb{Q}} * \ddot{\mathbb{R}}$, for some $\ddot{\mathbb{R}}$. Furthermore, $\mathbb{Q}_{\text{term}}$ is ${<}\kappa$-directed closed and therefore preserves the supercompactness of κ, since κ was indestructible in V. Small forcing by $\mathbb{P}$ then also preserves the supercompactness of κ. Thus, forcing with $\mathbb{Q}_{\text{term}} \times \mathbb{P}$, and hence also $\mathbb{P} * \dot{\mathbb{Q}} * \ddot{\mathbb{R}}$, preserves the supercompactness of κ. Therefore, the forcing $\ddot{\mathbb{R}}$ over $V^{\mathbb{P} * \dot{\mathbb{Q}}}$ must have recovered the supercompactness of κ.

It remains to check that $\ddot{\mathbb{R}}$ is sufficiently distributive. That is, we have to show that $\ddot{\mathbb{R}}$ adds no new γ-sequences for any $\gamma < \kappa$. Suppose that $G_{\text{term}} * g \subseteq \mathbb{Q}_{\text{term}} * \mathbb{P}$ is V-generic, and produced the generics $g * G * H \subseteq \mathbb{P} * \dot{\mathbb{Q}} * \ddot{\mathbb{R}}$, where $V[G_{\text{term}}][g] = V[g][G][H]$. Suppose that $s \in V[G_{\text{term}}][g]$ is a γ-sequence of ordinals for some $\gamma < \kappa$. So $s = \dot{s}_g$ for some $\dot{s} \in V[G_{\text{term}}]$, where $\dot{s}$ is a function from γ to a (small)

set of antichains in $\mathbb{P}$ matched with ordinals (i.e., the possible values of $\dot{s}(\gamma)$). It follows that $\dot{s} \in V$ since $\mathbb{Q}_{\text{term}}$ is $<\kappa$-directed closed. Thus, $s \in V[g]$. Thus, the only γ-sequences added by H must have been already in $V[g]$, and so $\ddot{\mathbb{R}}_{g*G}$ must be $<\kappa$-distributive. So κ is resurrectible in $V[g]$.

Finally, I will show that κ is not strongly resurrectible in $V[g]$. Let $\mathbb{Q}$ be the poset in $V[g]$ to add a Cohen subset to κ, or in fact any $<\kappa$-closed poset which adds a subset to κ. We know by the Superdestruction Theorem that $\mathbb{Q}$ will destroy the measurability of κ. If $\dot{\mathbb{R}}$ is the $\mathbb{Q}$-name of a $<\kappa$-closed poset in $V[g]^{\mathbb{Q}}$, then it follows that $\mathbb{Q}*\dot{\mathbb{R}}$ is $<\kappa$-closed in $V[g]$, since the $*$-iteration of closed posets is closed. Since it also adds a subset to κ it follows again by the Superdestruction Theorem that $\mathbb{Q}*\dot{\mathbb{R}}$ will destroy the measurability of κ. Thus, the supercompactness of κ cannot be recovered by $<\kappa$-closed forcing. Therefore, κ is not strongly resurrectible in $V[g]$. ⊣

Let me list, finally, two natural questions which remain unanswered in this paper. The first asks whether the limitation on λ in the Superdestruction Theorem III can be removed. The second asks more generally whether small forcing leads to a certain attractive complement of Laver indestructibility.

QUESTION. *After small forcing, does κ become superdestructible at λ for every λ?*

QUESTION. *After small forcing, does every $<\kappa$-closed forcing destroy the supercompactness of κ?*

Though I have not answered these questions in this paper, I nevertheless know that the answer to both of them is 'yes'. In a forthcoming paper which I am now writing with Saharon Shelah, we prove that after small forcing, any $<\kappa$-closed forcing will destroy even the strong compactness of κ. Thus, after small forcing, a supercompact cardinal κ has a dual version of Laver indestructibility. Namely, it is destroyed by any $<\kappa$-closed forcing.

REFERENCES

[1] ARTHUR APTER and SARAHON SHELAH, *Menas' result is best possible*, to appear in **Transactions of the American Mathematical Society**.

[2] JULIUS B. BARBENEL, *Making the hugeness of κ resurrectable after κ-directed closed forcing*, **Fundamenta Mathematicae**, vol. 137 (1991), pp. 9–24.

[3] JOEL DAVID HAMKINS, *Fragile measurability*, this JOURNAL, vol. 59 (1994), pp. 262–282.

[4] THOMAS JECH, **Set theory**, Academic Press, London, 1978.

[5] RICHARD LAVER, *Making the supercompactness of κ indestructible under κ-directed closed forcing*, **Israel Journal of Mathematics**, vol. 29 (1978), pp. 385–388.

[6] W. HUGH WOODIN, *Forcing to a model of a supercompact κ whose weak compactness is killed by* Add$(\kappa, 1)$, unpublished theorem.

MATHEMATICS 15-215
CITY UNIVERSITY OF NEW YORK, CSI
2800 VICTORY BLVD.
STATEN ISLAND, NY 10314, USA

E-mail: hamkins@integral.math.csi.cuny.edu