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ON TRANSLATION PLANES WITH AFFINE CENTRAL COLLINEATIONS, II

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1. Introduction. This article, as the name implies, is a continuation of [**9**]. In that article the second author investigates finite translation planes containing both affine elations and affine homologies. (See the beginning of Section 2 for definitions.) Such translation planes are called *EH*-planes. In [**9**] the author restricted himself to translation planes of characteristic $p \geqq 5$. The main reasons for this were that Ostrom's and Hering's theorem [**13**; **4**] on affine elations excluded the case $p = 3$ and the conclusions were easier to interpret geometrically when $p \geqq 5$ (as opposed to the case $p = 2$). Since then Ostrom [**17**] has settled the case $p = 3$.

In this article we extend the results of [**9**] to translation planes of characteristic $p = 2$ and $p = 3$. The classification for *EH*-planes of characteristic $p \geqq 5$ is extended from 3 to 7 possible types. (See the paragraph following Lemma 1 and the beginnings of Sections 4 and 5 for definitions.) There are many examples for each of types 1, 2a, 3a, 3d but none are known for the other three types. However, O. Prohoska and M. Walker have discovered a plane of order 81 which seems to be of type 2b. In addition we show that an *EH*-plane of non-square order is of type 1, 2a or 3d.

We also investigate the action on l_∞ of the collineation group generated by the affine central collineations (Theorems 4, 5 and 7). Finally, in Section 6 we combine our results with a theorem of Ostrom's to obtain information about translation planes of non-square order with affine homologies of prime order $u > 5$.

As this article was being prepared, the authors received a preprint of an article [**5**] written by Hering. In [**5**] Hering investigates collineation groups generated by affine elations in arbitrary finite affine planes (not necessarily translation planes). He obtains results (geometrical and group-theoretic) essentially the same as those obtained for translation planes in [**4**; **13**; **16**].

Since many of our results are refinements of results in [**9**], we will often appeal to [**9**] for portions of, or even entire, proofs. Thus we expect the reader to be familiar with [**9**]. Also we expect the reader is familiar with the general theory of projective and affine planes as given in Dembowski [**3**] and Hughes and Piper [**7**]. Finally, knowledge of Ostrom's theory of Desarguesian nets as given in [**13**; **15**] is assumed.

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2. ***EH*-planes.** We shall think of an affine plane sometimes as a projective plane in which a special line, called the line at infinity and denoted by l_∞, is distinguished. The affine points are the points not on this line and the affine lines are the lines distinct from l_∞. The points in l_∞ will be called points at infinity. As in [**9**] the collineations we shall consider are defined as follows:

Definition. Let π be a translation plane of order p^r, p a prime. An *affine central collineation* of π is a central collineation σ of π whose axis is an affine line; σ is an *affine elation* if it is an elation; σ is an *affine homology* if it is a homology.

Given a translation plane π we can coordinatize it with a quasi-field. Let O be the origin; i.e., the point with coordinates (0, 0). The lines through O are called the *components* of π (André [**1**]). Note that every affine line of π is the translate of a component. If T is the group of translations of π and G the full collineation group of π, then $G = T\,C(\pi)$, where $C(\pi) = G_O$, since T is transitive on the affine points of π. $C(\pi)$ is called the *translation complement* of π (Ostrom [**16**]). To determine the nature of G it is sufficient to determine the subgroup $C(\pi)$. In this article, generally, all collineations not translations will thus be elements of $C(\pi)$.

Definition. Let π be a translation plane of order p^r with affine elations. The *elation net* of π is the net, denoted by N, whose points are the points of π and whose lines are the components, and their translates, of π which are axes of affine elations. The *degree* of the elation net is the number of components of π in the elation net. $N \cap l_\infty$ will denote the points at infinity of the components of N.

Since in a translation plane a line is the axis of an affine elation if and only if it is the translate of a component which is the axis of an affine elation, the lines of an elation net are precisely the axes of affine elations. By Ostrom [**13**; **17**] and Hering [**4**; **5**] we have

THEOREM 1. *Let π be a translation plane of order p^r with an elation net N of degree m and let $E(\pi)$ be the collineation group of π generated by the affine elations of π in $C(\pi)$. One of the following holds:*

(i) *$m = 1$ and $E(\pi)$ is an elementary abelian p-group consisting of affine elations.*

(ii) *$m = p^s + 1$ for some integer $s|r$, N is an invariant Desarguesian net and each component of N is the axis of p^s affine elations. N contains a Desarguesian subplane $\bar{\pi}$ of order p^s upon which $E(\pi)$ acts faithfully and $\bar{E}$, the group induced on $\bar{\pi}$ by $E(\pi)$ is $SL(2, p^s)$; thus $E(\pi) \cong SL(2, p^s)$ and $E(\pi)$ is doubly transitive on the components of N.*

(iii) *$p = 3$, $m = 10$, $2|r$, N is an invariant Desarguesian net, and each component of N is the axis of 3 affine elations. N contains a Desarguesian subplane $\bar{\pi}$ of order 9 upon which $E(\pi)$ acts faithfully and $\bar{E}$, the group induced on $\bar{\pi}$ by $E(\pi)$,*

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is $SL(2, 5)$*; thus* $E(\pi) \cong SL(2, 5)$ *and* $E(\pi)$ *is transitive, but not doubly transitive, on the components of* N.

(iv) $p = 2$, $m = 2^{2s} + 1$, *s odd*, $s \geqq 3$, $2s|r$, *and each component of* N *is the axis of* 2^s *affine elations.* N *contains a Lüneburg-Tits plane* $\bar{\pi}$ *of order* 2^{2s} *upon which* $E(\pi)$ *acts faithfully and* $\bar{E}$, *the group induced on* $\bar{\pi}$ *by* $E(\pi)$, *is* $Sz(2^s)$*; thus* $E(\pi) \cong Sz(2^s)$.

(v) $p = 2$, *m is odd, and each component of* N *is the axis of* 2 *affine elations.* $E(\pi) = SM$, $|S| = 2$ *and* $|M| = m$ *is odd, and* $E(\pi)$ *induces faithfully a Frobenius group on* $N \cap l_\infty$ *with Frobenius kernel* M *and Frobenius complement* S.

Proof. Hering [**4**] shows that if $p \neq 3$ one of the following occurs: (1) $m = 1$ and $E(\pi)$ is an elementary abelian p-group consisting of affine elations, (2) $m = p^s + 1$ and $E(\pi) \cong SL(2, p^s)$ with all p-elements affine elations of π, (3) $p = 2$ and $E(\pi) \cong Sz(2^s)$, s odd and $s \geqq 3$, or (4) $p = 2$ and $E(\pi) = SM$ with $|S| = 2$ and $|M| = m$, m odd. If $p = 3$ in addition to (1) or (2) Ostrom [**17**] shows that one more possibility exists: (5) $m = 10$ and $E(\pi) \cong SL(2, 5)$.

Assume (2) occurs; since $SL(2, p^s)$ is generated by a pair of distinct Sylow p-groups, we can apply Theorem 4 of Ostrom [**13**] and obtain (ii). (In [**13**] Ostrom uses the hypothesis $p > 3$ in Theorem 4 only to show that he has $SL(2, p^s)$ with all its p-elements affine elations. This we have and his proof with that modification applies.) If (5) occurs, then (iii) of the theorem holds (Ostrom [**17**, Corollary 3.8]).

Assume (3) occurs. In $Sz(2^s)$ all involutions are conjugate, the order of a Sylow 2-subgroup S is 2^{2s} and $Z(S)$ consists of the involutions in S (Lüneburg [**12**]). This implies that in $E(\pi)$ all involutions are affine elations since one is and that there is a $(1 - 1)$ correspondence between the components of N and the Sylow 2-subgroups of $Sz(2^{2s})$. Since $Sz(2^s)$ has $2^{2s} + 1$ Sylow 2-subgroups (Lüneburg [**12**]) N has degree $2^{2s} + 1$. (Also see Hering [**5**, Theorem 3.9]). By Hering [**5**, first paragraph of Section 4] $2^{2r} = |T|$, T the group of translations of π, is divisible by 2^{2s}. By Theorem 5 of [**5**] N contains a Lüneburg-Tits plane $\bar{\pi}$ of order 2^{2s} upon which $E(\pi)$ acts faithfully. This gives (iv).

Assume (4) occurs. Then the number of Sylow 2-subgroups of $E(\pi)$ is m and, since these are disjoint, there is a $(1 - 1)$ correspondence between the Sylow 2-subgroups of $E(\pi)$ and the components of N. The statement about the action of $E(\pi)$ on $N \cap l_\infty$ follows.

Remark. The above five possibilities are essentially disjoint ones. Only (ii) and (v) can occur at the same time; namely, when $E(\pi) \cong SL(2, 2)$ and $m = 3$.

In this article we shall investigate the following type of translation plane:

Definition. An *EH-plane* is a translation plane π of finite order p^r, possessing both affine elations and affine homologies. If N is the elation net of π of degree m then the *dimension of* π *over* N, denoted by $\dim_N \pi$, is 0 if (i) of Theorem 1

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holds, r/s if (ii) or (iii) of Theorem 1 holds, $r/2s$ if (iv) of Theorem 1 holds, r if (v) of Theorem 1 holds.

The following theorem shows there are essentially three types of EH-planes. For an affine homology the *co-center* is the intersection of its axis with the line at infinity. Note that the center of an affine homology is always a point on the line at infinity.

THEOREM 2. *Let π be an EH-plane of order p^r, let P be the center and Q the co-center of an affine homology σ of π, and let R be the center of an affine elation. Exactly one of the following must occur:*

(1) *R is unique and either $R = P$ or $R = Q$.*

(2) *R is not unique and both P and Q are centers of affine elations.*

(3) *R is not unique and neither P nor Q are centers of affine elations.*

Proof. Assume first that R is unique. σ fixes only the points P and Q on l_∞ and moves the other points. But σ also fixes R since R is the only center of affine elations. Hence either $R = P$ or $R = Q$.

Assume now that R is not unique. Assume also that the number of affine elation centers on l_∞ is $p^t + 1$ for some integer $t|r$. If P is the center of an affine elation and Q is not, then σ fixes P and permutes the other p^t affine elation centers among themselves in nontrivial orbits. Thus $|\sigma||p^t$. But also $|\sigma||p^r - 1$. This implies $|\sigma| = 1$—a contradiction. Hence P an elation center implies Q is an elation center. A similar proof shows that Q an elation center implies P is an elation center.

If R is not unique and $m \neq p^t + 1$ for some integer t then by Theorem 1 $p = 2$, $E(\pi) \cong SM$, $|M| = m$ is odd. We show that in this case neither P nor Q can be centers of affine elations. Since we will need this result later, we state it as a lemma.

LEMMA 1. *If in addition to the hypothesis of Theorem* 2, *statement* (v) *of Theorem* 1 *holds, then neither P nor Q are centers of nontrivial affine elations.*

Proof. Since $p = 2$, $|\sigma|$ is odd. If P is a center of a nontrivial affine elation τ then $\bar{\sigma} = \tau^{-1}\sigma\tau$ is an affine homology with center P and co-center $Q\tau \neq Q$. Then π has affine homologies with center P and distinct co-centers Q, $Q\tau$, $Q\tau\sigma$. By André [**1**] P is the center of at least 2 nontrivial affine elations. This contradicts the fact that if (v) of Theorem 1 holds then P is the center of at most one nontrivial affine elation. This proves the lemma and also Theorem 2.

Theorem 2 gives us the following simple classification of EH-planes:

Type 1. Statement (1) of Theorem 2 holds.

Type 2. Statement (2) of Theorem 2 holds.

Type 3. Statement (3) of Theorem 2 holds.

Note than an EH-plane of type 1 cannot be of type 2 or of type 3, but an EH-plane could possibly be of type 2 and of type 3. Johnson and Ostrom [**8**] have constructed translation planes of order 3^{2r}, possesssing affine elations of order 3 and affine homologies of order 2. These planes satisfy statements (2) and (3) of Theorem 2. The planes constructed by Ostrom in [**14**, Theorem 2.1] are EH-planes of type 2, but not of type 3, in which statement (ii) of Theorem 1 holds. Desarguesian planes will always be EH-planes of type 2 in which statement (ii) of Theorem 1 holds. Semi-field planes are EH-planes of type 1. Hall planes of characteristic two are EH-planes of type 3 in which statement (v) of Theorem 1 holds. The Lüneburg-Tits planes are not EH-planes of any type since they do not possess affine homologies (see Lemma below).

As in [**9**] we make the following definitions:

Definition. Let π be an EH-plane and let H be a group of affine homologies of π in the translation complement with center P and co-center Q (on l_∞). π is of *type k with respect to* H, $k = 1, 2$, or 3, if P and Q satisfy statement (k) of Theorem 2.

Remark. Note that with respect to a specific group H π is of one type only.

Let π be an EH-plane of order p^r, let $C = C(\pi)$ be the translation complement of π, $E = E(\pi)$ the group generated by the affine elations of π in C, H a group of affine homologies in C with center $P \in l_\infty$ and co-center $Q \in l_\infty$ (and hence axis OQ). Let $L = L(H) \equiv \langle E, H \rangle$. Since E is generated by all the affine elations in C, $E \lhd C$ and therefore $L = EH$. Note that in all three types of EH-planes, $H \cap C_L(E)$ consists only of the identity ($C_L(E)$ is the centralizer of E in L) and thus H induces under conjugation a nontrivial group of automorphisms of E isomorphic to itself. Thus we have the obvious

THEOREM 3. *Let π be an EH-plane with order p^r of type k with respect to a group H of affine homologies in $C(\pi)$. If $E(\pi)$ is the group generated by the affine elations of π in $C(\pi)$, then $L(H) \equiv \langle E(\pi), H\rangle = E(\pi)H$ and H is isomorphic to a group of automorphisms of $E(\pi)$.*

The notation $E(\pi)$, $C(\pi)$, $L(H)$ will be used throughout the rest of this article in the sense used above.

3. EH-planes of type 1. In this section we determine the action of $L(H)$ on the line l_∞ for an EH-plane of type 1.

THEOREM 4. *Let π be an EH-plane of order p^r, and assume π is of type* 1 *with respect to a group of affine homologies H in $C(\pi)$. $L(H)$ is a group of central collineations which induces a permutation group $\bar{L}$ on the points of l_∞ with $L \cong \bar{L}$. $\bar{L}$ has the following action:*

(i) *$\bar{L}$ fixes the point R, the center of the affine elations in $E(\pi)$;*

(ii) *$\bar{L}$ has one orbit $\mathscr{U}$ of order $|E|$ on which it acts as a Frobenius group with Frobenius kernel $\bar{E}$ and Frobenius complement $\bar{H}$;*

(iii) *every other orbit has length $|L(H)|$ and $\bar{L}$ acts as a regular group on each of these orbits.*

Furthermore $E(\pi) \cap H$ *consists only of the identity,* $|L(H)| = |E(\pi)|\,|H|$ *and* $|H|\,|(E(\pi)| - 1)$.

Proof. The proof of Theorem 4 in [**9**] applies since the hypothesis $p \geqq 5$ is not used there.

A divisor of $p^r - 1$, p a prime, is a *p-primitive divisor* if it is relatively prime to $p^s - 1$ for every positive integer s with $s|r$. It is easy to see that if u is a prime p-primitive divisor of $p^r - 1$, then u does not divide $p^s - 1$ for all $s < r$, $s \geqq 0$. As in [**9**] we have the following corollary:

COROLLARY 4.1. *Let* π *be an EH-plane of order* p^r *and assume* π *has a homology* σ *of order* u *which is a prime p-primitive divisor of* $p^r - 1$. *Then* π *is not of type* 1.

Proof. The proof of Corollary 4.1 in [**9**] carries over without any change.

4. *EH*-planes of type 2. In this section we look at EH-planes of type 2 and determine the nature of $L(H) = E(\pi)H$. To do this we need the following results:

LEMMA 2. *Let* π *be a Luneburg-Tits plane of order* 2^{2s}, $s \geqq 3$. *Then* π *has no nontrivial affine homologies.*

Proof. If π has an affine homology $\sigma \neq 1$ with center P, co-center Q, and axis OQ then for every point $R \in l_\infty$, $R \neq Q$, π has an affine homology $\sigma_R \neq 1$ with center R and co-center Q. This follows from the fact that the stabilizer in $C(\pi)$ of Q is transitive on the other q^2 points of l_∞ (Lüneburg [**12**, p. 89]). By André [**1**] π is (Q, Q)-transitive which is impossible (Lüneburg [**12**, p. 95]). This proves σ does not exist.

By Lemmas 1 and 2, if π is an EH-plane of type 2, then either statement (ii) or statement (iii) of Theorem 1 holds. We therefore make the following definition.

Definition. An EH-plane π of type 2 is of type 2a if statement (ii) of Theorem 1 holds; π is of type 2b if statement (iii) of Theorem 1 holds.

Remark. We define the notion "π is of type 2a (or 2b) with respect to a group H of affine homologies" in the obvious way.

For planes of type 2 we have the following comprehensive theorem:

THEOREM 5. *Let* π *be an EH-plane of type* 2 *with respect to the group H of affine homologies in* $C(\pi)$ *and let N be the elation net of* π *with degree* $p^s + 1$. $E(\pi) \cap H$ *consists only of the identity,* $L(H)$ *is the semi-direct product of* $E(\pi)$ *and H and is isomorphic to a subgroup of* $GL(2, p^s)$, *H is cyclic, and* $|H|\,|p^s - 1$. $\bar{L}$, *the permutation group induced on* l_∞ *by* $L(H)$, *has one orbit of length* $p^s + 1$ *consisting of the centers of the affine central collineations in* $L(H)$ *and all other orbits of* $\bar{L}$ *have length divisible by* $p^s|H|$. *Furthermore, either* (i) π *is of type* 2a *with* *respect to* H *and* $E(\pi) \cong SL(2, p^s)$ *or* (ii) π *is of type* 2*b with respect to* H, $p = 3$, $s = 2$, $2|r$, $|H||8$, *and* $E(\pi) \cong SL(2, 5)$.

Proof. Either π is of type 2a or π is of type 2b. If π is of type 2a, the proof of Theorem 5 in [**9**] carries over without any change. If π is of type 2b, the same proof can be used since in this case N is a Desarguesian net of degree 10 containing a Desarguesian subplane of order 9.

Remark. Note that in Theorem 5, if $|H| > 8$ or if $|H|$ is divisible by any odd integer, then π is of type 2a.

Using this remark, the proof of Corollary 5.1 in [**9**] gives

COROLLARY 5.1. *Let* π *be an EH-plane of order* p^r *and assume* π *has a homology* σ *of order* u *which is a prime* p*-primitive divisor of* $p^r - 1$. *If* π *is of type* 2 *with respect to* $H = \langle \sigma \rangle$, *then* π *is Desarguesian.*

Examples. As already remarked, Desarguesian planes and the planes of Ostrom [**14**] are *EH*-planes of type 2. Recently O. Prohoska and M. Walker have discovered a translation plane of order 81 possessing a set of affine elations which generate $SL(2, 5)$, but it is not known (to the present authors) whether or not this plane possesses affine homologies.

5. *EH*-planes of type 3. Just as for *EH*-planes of type 2, we can divide the class of *EH*-planes of type 3 into several subclasses. For this purpose we make the following definition:

Definition. Let π be an *EH*-plane of type 3. π is of type 3a (3b, 3c, 3d) if statement (ii) ((iii), (iv), (v)) of Theorem 1 holds. If H is a group of affine homologies of π, π is of type 3a with respect to H if π is of type 3a; similarly, for type 3b, 3c, 3d with respect to H.

Remark. The planes of Johnson and Ostrom [**8**] are of type 3a (and also of type 2a).

LEMMA 3. *Let* π *be an EH-plane of type* 3 *with respect to a group* H *of affine homologies. If* π *is not of type* 3d, *then* $E(\pi) \cap H$ *consists only of the identity.*

Proof. If π is not of type 3d, then it is of type 3a, 3b, or 3c. If the type is 3a or 3b let $\bar{\pi}$ be the Desarguesian subplane contained in the elation net N, while if π is of type 3c, let $\bar{\pi}$ be the Lüneburg-Tits subplane contained in the elation net N. In all three cases the group $E(\pi)$ fixes $\bar{\pi}$. If $\sigma \in E(\pi) \cap H$ and $\sigma \neq 1$ then σ is an affine homology fixing $\bar{\pi}$. Lemma 4.24 of Hughes and Piper [**7**, p. 102] says that the center and axis of σ are in $\bar{\pi}$—a contradiction. Thus $\sigma = 1$.

Remark. Lemma 3 is not true if π is of type 3d. See the examples after Corollary 6.1.

We turn now to obtaining some necessary conditions for π to be of type 3. We start with:

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LEMMA 4. *Let π be a translation plane of order p^r with a Desarguesian net N of degree $p^s + 1$, $s|r$, such that N contains a Desarguesian subplane $\bar{\pi}$ of order p^s. N is the union of $(p^r - 1)(p^s - 1)^{-1}$ Desarguesian subplanes of order p^s, any two of which intersect in only one point O common to all of them. Furthermore, if σ is a collineation of π fixing N then σ permutes these subplanes among themselves.*

Proof. Let π' be the Desarguesian plane of order p^r containing N and assume π' is coordinatized by a field K of order p^r. We may choose our coordinate system such that $\bar{\pi}$ is coordinatized by a subfield F of K having order p^s and the origin $O = (0, 0)$ is in $\bar{\pi}$. Let π_i, $i = 1, \ldots, d = (p^r - 1)(p^s - 1)^{-1}$ be the distinct images of $\bar{\pi}$ under the (O, l_∞)-homologies $(x, y) \to (x\beta, y\beta)$, $\beta \in K^* = K - \{0\}$. (Thus $\bar{\pi}_i$ correspond to the distinct cosets of $F^* = F - \{0\}$ in K^*.) Then

$$N = \bigcup_{i=1}^{d} \bar{\pi}_i \quad \text{and} \quad \bar{\pi}_i \cap \bar{\pi}_j = \{0\} \quad \text{if } i \neq j.$$

(See Exercise 4.23 on p. 105 of Hughes and Piper [**7**]).

Let σ be a collineation of π fixing N and consider the Desarguesian subplane $\bar{\pi}$. $\bar{\pi}$ is a subplane of π of order p^s contained in N. Let $P \in \bar{\pi}_i$, $P \neq O$. Then $P\sigma \in \bar{\pi}_j$ for some j. $\bar{\pi}_j$ and $\bar{\pi}_i\sigma$ have the same points on l_∞ and have the points P and O in common. Every other point of $\bar{\pi}_j$ is the intersection of a line through O and a point $Q \in \bar{\pi}_j \cap l_\infty$ and a line through P and a point $R \in \bar{\pi}_j \cap l_\infty$. This implies $\bar{\pi}_i\sigma$ contains all the points of $\bar{\pi}_j$ and hence $\bar{\pi}_j\sigma = \bar{\pi}_j$. Thus σ permutes these subplanes among themselves.

We have the following theorem which is a generalization of Theorem 6 in [**9**]. The proof is a modification of a proof for the same Theorem 6 communicated to the second author by C. Hering.

THEOREM 6. *If π is an EH-plane of type 3a, 3b, or 3c having order p^r and an elation net N of degree $p^s + 1$, then $\dim_N \pi = r/s$ is even.*

Proof. Assume $\dim \pi_N = r/s = t$ is odd, and let σ be an affine homology in $C(\pi)$ with axis l, l not a line of N. Let $u = |\sigma|$. Since σ is a homology with axis l not in N, the center of σ is a point at infinity of π which is not in N. Thus $u|p^s + 1$, the number of such points. Also, $u|p^r - 1$ since σ fixes two points at infinity of π and is fixed-point-free on the other points at infinity. If $t = \dim_N \pi = r/s$, then $p^s(p^{(t-1)s} + 1) = p^s(p^{r-s} + 1) = (p^r - 1) + (p^s + 1)$ implies $u|(p^{(t-1)s} + 1)$. Hence $u|(p^{(t-2)s} - 1)$ since $p^s(p^{(t-2)s} - 1) = (p^{(t-1)s} + 1) - (p^s + 1)$. An easy induction shows that for $1 \leqq k \leqq t$, $u|p^{(t-k)s} + 1$ if k is odd and $u|p^{(t-k)s} - 1$ if k is even. Thus t odd (our hypothesis) implies $u|(p^{(t-t)s} + 1) = 2$. Thus $u = 2$.

Note that $u = 2$ implies $p \neq 2$ and hence in particular π is not of type 3c. Assume π is of type 3a or 3b. N is a Desarguesian net containing a Desarguesian subplane $\bar{\pi}$ of order p^s. By Lemma 4 N is the union of $(p^r - 1)(p^s - 1)^{-1}$ Desarguesian subplanes and σ permutes these subplanes among themselves. If σ fixes one of these subplanes, then the center and axis of σ are in N (Lemma

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4.24 of [**7**, p. 102]). This is a contradiction. Hence σ acts semi-regularly on the set of these subplanes and therefore $2 = |\sigma|$ divides $(p^r - 1)(p^s - 1)^{-1} = p^{s(t-1)} + p^{s(t-2)} + \ldots + p^s + 1$. t odd implies $(p^r - 1)(p^s - 1)^{-1}$ is odd (since p is odd) and hence 2 does not divide $(p^r - 1)(p^s - 1)^{-1}$. This gives a contradiction. The theorem is proved.

COROLLARY 6.1. *Let π be an EH-plane having non-square order. One of the following holds:*

(i) *π is of type* 1

(ii) *π is of type 2a.*

(iii) *π is of type 3d.*

Proof. Follows from Theorem 5 and Theorem 6.

Examples. Translation planes coordinatized by semi-fields (of either square or non-square order) are EH-planes of type 1. The only known EH-planes of type 2a are the Desarguesian planes, the planes discovered by Ostrom [**14**], and the planes of characteristic 3 discovered by Johnson and Ostrom [**8**]. The planes in the last two classes all have square order, however.

Consider a translation plane π coordinatized by a proper nearfield Q of order 2^r, $r \geqq 6$. Then every component of π, except for the x- and y-axis, is the axis of an elation of order 2. Specifically, the component $y = ax$, $a \neq 0$, is the axis of the elation ϕ_a: $(x, y) \rightarrow (a^{-1}y, ax)$. Choose $a, b \in Q - \{0\}$ such that $ab \neq ba$ and let $c = a^{-1}b$. Then $\rho_n = \phi_a\phi_b\phi_c\phi_1$: $(x, y) \rightarrow (x, dy)$ where $d = b^{-1}aba^{-1} \neq 1$. Thus ρ is a nontrivial affine homology with axis $y = 0$ and center (∞). This examples shows that Lemma 3 is not true for planes of type 3d in general.

We also remark that the Hall planes of characteristic 2 as well as the translation planes derived from planes coordinatized by regular nearfields of characteristic 2 are EH-planes of type 3d.

We turn now to investigating the nature of the group $L(H)$ for EH-planes of type 3.

THEOREM 7. *Let π be an EH-plane of type* 3 *with respect to a group H of affine homologies having center P and co-center Q, let π have order p^r, let N be the elation net in π, and let K be the subgroup of $L(H)$ fixing each component of N.*

(i) *If π is of type* 3*a and N has degree $p^s + 1$, then $L(H)/K$ is isomorphic to a subgroup of $P\Gamma L(2, p^s)$ containing $PSL(2, p^s)$.*

(ii) *If π is of type* 3*b then $L(H)/K$ is isomorphic to a subgroup of $P\Gamma L(2, 5)$ containing $PSL(2, 5)$.*

(iii) *If π is of type* 3*c and N has degree $2^{2s} + 1$, then $L(H)K$ is isomorphic to a subgroup of* $\mathrm{Aut}(\mathrm{Sz}(2^s))$ *containing* $\mathrm{Sz}(2^s)$.

(iv) *If π is of type* 3*d and N has degree m, then $L(H)/K$ is isomorphic to a subgroup of* Aut *M, M the Frobenius kernel of $E(\pi)$.*

Proof. In all four cases there is a natural $(1 - 1)$ correspondence between the components of N and the Sylow p-subgroups of $E(\pi)$ such that $\tau \in K$

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if and only if the automorphism $\rho \to \tau^{-1}\rho\tau$ of $E(\pi)$ induced by τ fixes all the Sylow p-subgroups of $E(\pi)$.

The proof of Theorem 7 (iii) in [**9**] gives (i). If π is of type 3b then $E(\pi) \cong SL(2, 5)$. If $\tau \in L(H)$ then τ induces by conjugation an automorphism $\bar{\tau}$ of $E(\pi)$. If $\tau \in K$, then $\bar{\tau}$ fixes the Sylow 3-subgroups of $E(\pi)$. By a proof similar to the proof of Lemma 4 in [**9**], $\bar{\tau}$ fixes the Sylow 3-subgroups of $SL(2, 5)$ if and only if $\bar{\tau}$ is the identity automorphism. Thus K consists of the collineations of $L(H)$ which induces the identity automorphism by conjugation on $SL(2, 5)$. Hence $L(H)/K$ is isomorphic to the automorphism group induced on $E(\pi) \cong SL(2, 5)$ by $L(H)$ and (ii) follows since $\mathrm{Aut}(SL(2, 5)) \cong P\Gamma L(2, 5)$ (Hua [**6**]) and $SL(2, 5)$ induces $PSL(2, 5)$ by conjugation.

Assume now that π is of type 3c. Let $\tau \in K$. The automorphism $\bar{\tau}$ induced by conjugation on $E(\pi) \cong \mathrm{Sz}(2^s)$ fixes all the Sylow 2-subgroups of $E(\pi)$. By Lemma 11 of [**18**] $\bar{\tau} = \gamma_\delta\beta_\alpha$, where $\delta \in E(\pi)$ and $\gamma_\delta : \rho \to \delta^{-1}\rho\delta$ is the inner automorphism induced by δ, $\alpha \in$ Aut $(GF(2^s))$ and β_α is the outer automorphism induced by the semi-linear transformation α defined in 4-dimensional vector space over $GF(2^s)$. For every Sylow 2-subgroup S of $E(\pi)$, $S\gamma_\delta\beta_\alpha = S$ or $\delta^{-1}S\delta = S\beta_{\alpha^{-1}}$. Now $\beta_{\alpha^{-1}}$ fixes at least 3 Sylow 2-subgroups of Sz $(2^s) \cong E(\pi)$. This implies δ is in the normalizer of three distinct Sylow 2-subgroups of $E(\pi)$ and this is only possible if $\delta = 1$. Hence $\tau = \beta_\alpha$ and β_α fixes all the Sylow 2-subgroups of $S_z(2^s)$. This can only happen if $\alpha = 1$. Hence $\bar{\tau} = 1$.

Thus if π is of type 3c, $L(H)/K$ is isomorphic to the group of automorphisms induced on $E(\pi) \cong \mathrm{Sz}(2^s)$ by $L(H)$. Thus $L(H)/K$ is a subgroup of Aut $(\mathrm{Sz}(2^s))$ containing $\mathrm{Sz}(2^s)$ (since $L(H) \geqq E(\pi) \cong \mathrm{Sz}(2^s)$).

If π is of type 3d, then the elements of K induce (by conjugation) automorphisms of $E(\pi)$ fixing each Sylow 2-subgroup of $E(\pi)$. Since a Sylow 2-subgroup of $E(\pi)$ has only one non-identity element, this implies the automorphisms on $E(\pi)$ induced by elements of K fix the 2-elements of $E(\pi)$. Hence these automorphisms fix $E(\pi)$ pointwise since $E(\pi)$ is generated by its 2-elements. Thus $L(H)/K$ is isomorphic to the automorphism group induced on $E(\pi)$ by $L(H)$. Since M is characteristic in $E(\pi)$, (iv) follows.

The last result of this section is a continuation of Corollary 4.1 and Corollary 5.1.

THEOREM 8. *Let π be an EH-plane of order p^r with an affine homology of order u, u a prime p-primitive divisor of $p^r - 1$. One of the following holds:*

(i) *π is Desarguesian.*

(ii) *π is of type 3b, $u = 5$, and $p^r = 3^4$.*

(iii) *π is of type 3c with $p^r = 2^{4s}$, $2^{2s} + 1$ the degree of the elation net N, and $u | 2^{2s} + 1$.*

(iv) *π is of type 3d with $p = 2$ and $u | m$, m the degree of the elation net N.*

Proof. By Corollary 4.1 π is not of type 1 with respect to $\langle\rho\rangle$ and by Corollary 5.1 π is Desarguesian if it is of type 2 with respect to $\langle\rho\rangle$. If π is of type 3d with respect to $\langle \rho \rangle$ then $u|m$, the degree of N, since ρ acts semi-regularly on the points of $N \cap l_\infty$.

Assume π is of type 3 with respect to $\langle \rho \rangle$ but not of type 3d. Let N have degree $p^t + 1$. If $2t < r$, define d = g.c.d. $(2t, r)$. Then $d = 2tx + ry$ for integers x, y, and

$$p^d - 1 = p^{2tx}(p^{ry} - 1) + (p^{2tx} - 1) = p^{2tx}(p^r - 1)x_1 + (p^t + 1)y_1$$

for integers x_1, y_1. Therefore $u|p^d - 1$ and $d|r$—a contradiction to the fact that u is a p-primitive divisor of $p^r - 1$. Hence $2t = r$, since $t|r$ (Theorem 1) and $2|r/t$ (Corollary 6.1).

If π is of type 3a with respect to $\langle \rho \rangle$, the proof of Theorem 8 in [**9**] gives a contradiction since that proof depends only on the properties of $SL(2, p^s)$ and not on p being greater than 3.

Assume π is of type 3b. Then N has degree 10 and this implies $u = 5$ since $u|10$ and $u \neq 2$. The only power of 3 having 5 as a 3-primitive divisor is 3^4.

Assume π is of type 3c. Then $p = 2$ and $r = 4s$ since N has degree $2^{2s} + 1$ in this case. The fact that $u|2^{2s} + 1$ follows from the fact that ρ acts semi-regularly on the points of $N \cap l_\infty$. This proves the theorem.

Examples. The nearfield planes of characteristic 2 as well as the Hall planes of characteristic 2 are EH-planes fulfilling the hypothesis of Theorem 8 and they satisfy statement (iv). For the nearfield planes $m = p^r - 1$ and for the Hall planes $m = p^s + 1$, $s = r/2$. The authors do not know any planes satisfying the hypothesis of Theorem 8 and either statements (ii) or statement (iii). Possibly the plane recently observed by O. Prohoska and M. Walker satisfies statement (ii).

6. Applications. In this section some implications of the results derived above will be presented. As is true of our other results, the results given here will be generalizations of results in [**9**]. Our starting point is the following result of Ostrom [**16**, Theorem 3.12].

THEOREM 9. *Let π be a translation plane of non-square order and let σ be an affine homology of prime order u with center P and co-center Q. One of the following holds:*

(i) *P and Q are fixed by every collineation of π.*

(ii) *$\{P, Q\}$ is an orbit of the collineation group of π on l_∞.*

(iii) *π has affine elations.*

(iv) *$u \leqq 5$.*

Combining with the results of the previous sections, Ostrom's theorem gives the following:

THEOREM 10. *Let π be a translation plane of non-square order p^r with an affine homology σ of prime order $u \neq 2, 3, 5$. If P is the center and Q the co-center* *of σ and $C = C(\pi)$, the translation complement of π, then one of the following holds:*

(i) *P and Q are fixed or interchanged by C and C is a solvable group.*

(ii) *P is fixed by C but Q is moved by C; π is an EH-plane of type 1 and C is solvable.*

(iii) *Q is fixed by C but P is moved by C; π is an EH-plane of type 1 and C is solvable;*

(iv) *π is an EH-plane of type 2a with an elation net N of degree $p^s + 1$, $s|r$. If A is the set of points at infinity of N then A is the orbit of P under C and the center and co-center of every affine collineation of π is in A. C is non-solvable if $p^s > 3$ and C/K, K the subgroup of C fixing A pointwise, is isomorphic to a subgroup of $P\Gamma L(2, p^s)$ containing $PSL(2, p^s)$.*

(v) *$p = 2$, π is an EH-plane of type 3d with an elation net N of degree m, m odd, and the center and co-center of every affine homology of π lies outside of A, the set of points at infinity of N.*

Proof. By Theorem 9 the points P, Q are fixed by C, interchanged by C, or π has affine elations. If one of the first two possibilities occurs, then $C_{P,Q}$, the stabilizer of P and Q, has index 1 or 2 in C and by Burmeister and Hughes [**2**] $C_{P,Q}$ is solvable. Hence C is solvable.

Assume π has affine elations, π is an EH-plane and by Corollary 6.1 one of the three possibilities can occur: (1) π is of type 1, (2) π is of type 2a, (3) π is of type 3d. If π is of type 3d then (v) follows. (See Lemma 1 for the last part of (v).)

Assume π is of type 1. Then either P is fixed and Q is moved, or Q is fixed and P is moved. Assume P is fixed and let B be the orbit of Q under C. By a classical result of André [**1**], B is the orbit of Q under $E(\pi)$ and hence $C = E(\pi)C_Q$. Since C fixes P, by Burmeister and Hughes [**2**] C_Q is solvable and therefore C is also since $E(\pi)$ is elementary abelian. Thus (ii) holds. If Q is moved and P is fixed, a similar proof shows (iii) holds.

Assume π is of type 2a. Then (iv) holds. C is non-solvable if $p^s > 3$ since C contains a subgroup $E(\pi) \cong SL(2, p^s)$. The fact that π has no affine central collineation with center and co-center not in A follows from Theorem 6. The proof of the statement concerning C/K is essentially the same as the proof of Theorem 7(iii) in [**9**].

By $T(p^r)$ we mean the group of all mappings on $GF(p^r)$ of the form $x \rightarrow x^\sigma\alpha$, where $\alpha \in GF(p^r)$, $\alpha \neq 0$, and σ is an automorphism of $GF(p^r)$. $T(p^r)$ is solvable of order $r(p^r - 1)$. As an immediate consequence of Theorem 10 we have:

COROLLARY 10.1. *Let π be a translation plane of non-square order p^r and assume it has an affine homology of order u, u a prime p-primitive divisor of $p^r - 1$, with center P and co-center Q. One of the following holds:*

(i) *P and Q are fixed or interchanged by the translation complement $C(\pi)$ of π,*

$C(\pi)$ is solvable, and the permutation group induced on OP by $C(\pi)$ is isomorphic to a subgroup of $T(p^r)$.

(ii) *π is Desarguesian.*

(iii) *$p = 2$ and π is an EH-plane of type 3d.*

Proof. Clearly $u \neq 2$ and the beginning of the proof of Corollary 10.1 in [**9**] shows $u \neq 3, 5$. Then Theorem 10 applies. If (i) of Theorem 10 holds, then statement (i) above holds by the proof of Theorem 10 in [**9**]. Thus by Theorem 10, π is an *EH*-plane of type 1, type 2a, or type 3d. By Corollary 5.1, π is not of type 1. By Corollary 5.1, if π is of type 2a then it is Desarguesian. This proves the Corollary.

COROLLARY 10.2. *Let π be an affine plane of non-square order n in which every affine line is the axis of a non-trivial homology of prime order $u \neq 2, 3, 5$. π is a translation plane of order p^r for some prime p and some integer $r \geqq 1$; furthermore, one of the following holds:*

(i) *π is a semi-field plane.*

(ii) *π is Desarguesian.*

If u is a p-primitive divisor of $p^r - 1$ then (ii) *holds.*

Proof. The first conclusion follows from Theorem 11 of [**9**]. We can therefore apply Theorem 10. Clearly (i) and (v) of Theorem 10 cannot hold. If either (ii) or (iii) of Theorem 10 holds, then the first part of the proof of Theorem 11 in [**9**] can be applied to show that π is a semi-field plane. If (v) of Theorem 10 holds, then π is an *EH*-plane of type 2a and every affine line must be the axis of an affine elation. Hence the elation net of π has degree $p^r + 1$ and π has a collineation group isomorphic to $SL(2, p^r)$. Hence π is Desarguesian (Lüneburg [**11**]).

Remarks. As remarked in [**9**], the results of this section are close to being best possible. It is necessary to exclude the possibility $u = 2$. For in every nearfield plane and in every Hall plane of odd order, each component is the axis of a homology of order 2. Also in planes derived from regular nearfield planes of square order as well as certain generalized André planes there are components which are axes of involuntary homologies. Johnson and Ostrom [**8**] have discovered André translation planes possessing affine homologies of order 3 with distinct centers and co-centers but no affine elations.

If an attempt is made to remove the hypothesis "non-square order," at least one additional class will have to be admitted as a possibility. For the Hall planes of order q, which are not *EH*-planes if q is odd, have $q(q - 1)$ components that are axes of homologies of order $q + 1$.

REFERENCES

1. J. André, *Über nicht-Desarguessche Ebenen mit transitiver Translationgruppe*, Math. Z. *60* (1954) 156–186.

2. M. V. D. Burmeister and D. R. Hughes, *On the solvability of autotopism groups*, Arch. Math. *16* (1965), 178–183.
3. P. Dembowski, *Finite geometries* (Springer-Verlag, Berlin-Heidelberg, 1968).
4. C. Hering, *On shears of translation planes*, Abh. Math. Sem. Univ., Hamburg, *37* (1972), 258–268.
5. C. Hering, *On projective planes of type VI*, to appear.
6. L. K. Hua, *On the automorphisms of the symplectic group over any field*, Ann. of Math. *49* (1948), 739–759.
7. D. R. Hughes and F. C. Piper, *Projective planes* (Springer-Verlag, New York-Heidelberg-Berlin, 1973).
8. N. L. Johnson and T. G. Ostrom, *Translation planes with several homology or elation groups of order 3*, Geometriae Dedicata *2* (1973), 65–81.
9. M. J. Kallaher, *On translation planes with affine central collineations*, to appear in Geometriae Dedicata *4* (1975), 71–90.
10. M. J. Kallaher and T. G. Ostrom, *Fixed point free groups, rank three planes, and Bol quasi-fields*, J. Algebra *18* (1971), 159–178.
11. H. Lüneburg, *Charakterisierungen der endlichen desarguesschen projektiven Ebenen*, Math. Z. *85* (1964), 419–450.
12. ——— *Die Suzukigruppen und ihre geometrien*, Lecture Notes in Mathematics (Springer-Verlag, Berlin-Heidelberg-New York, 1965).
13. T. G. Ostrom, *Linear transformations and collineations of translation planes*, J. Algebra *14* (1970), 405–416.
14. ——— *A class of translation planes admitting elations which are not translations*, Arch. Math, *21* (1970), 214–217.
15. ——— *Finite translation planes*, Lecture Notes in Mathematics (Springer-Verlag, Berlin-Heidelberg, 1970).
16. ——— *Homologies in translation planes*, Proc. London Math. Soc. *26* (1973), 605–629.
17. ——— *Elations in finite translation planes of characteristic 3*, Abh. Math. Sem. Hamb. *41* (1974), 179–184.
18. M. Suzuki, *On a class of doubly transitive groups*, Ann. Math. *75* (1962), 105–145.

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