10
Dirac equation

Electrons are light, with a rest mass of

$$m_e c^2 = 0.5110\,\mathrm{MeV} \tag{10.1}$$

For the energies of interest here, electrons must be treated relativistically. Fortunately, for leptons, one knows how to do this with the Dirac equation [Bj65, Sc68]

$$\begin{aligned} (c\,\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m_0 c^2)\psi &= i\hbar\frac{\partial\psi}{\partial t} \\ \mathbf{p} &= \frac{\hbar}{i}\nabla \end{aligned} \tag{10.2}$$

Here ψ is a 4-component column vector and $\boldsymbol{\alpha}$ and β are 4×4 hermitian matrices satisfying the relations

$$\begin{aligned} \beta\alpha_k + \alpha_k\beta &= 0 \qquad\qquad ;k = 1,2,3 \\ \alpha_k\alpha_l + \alpha_l\alpha_k &= 2\delta_{kl} \\ \beta^2 &= 1 \end{aligned} \tag{10.3}$$

A specific (*standard*) representation of the Dirac matrices is given in 2×2 form by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \qquad\qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.4}$$

Introduce

$$\begin{aligned} \boldsymbol{\gamma} &\equiv i\boldsymbol{\alpha}\beta \\ \gamma_4 &\equiv \beta \end{aligned} \tag{10.5}$$

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It follows that these new matrices are also hermitian and satisfy the following algebra

$$
\begin{aligned}
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu &= 2\delta_{\mu\nu} \\
\gamma_\mu^\dagger &= \gamma_\mu \qquad ; \; \mu = 1, \ldots, 4
\end{aligned} \tag{10.6}
$$

In the standard representation, the gamma matrices are given by

$$
\boldsymbol{\gamma} = \begin{pmatrix} 0 & -i\boldsymbol{\sigma} \\ i\boldsymbol{\sigma} & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.7}
$$

Multiplication on the left by γ_4 and division by $\hbar c$ leads to the covariant form of the Dirac equation

$$
\begin{aligned}
\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{m_0 c}{\hbar} \right) \psi &= 0 \\
\gamma_\mu &= (\boldsymbol{\gamma}, \gamma_4) \\
x_\mu &= (\mathbf{x}, ict)
\end{aligned} \tag{10.8}
$$

Repeated Greek indices are summed from 1 to 4.

To include an *electromagnetic field* one makes the gauge invariant replacement $p_\mu \to p_\mu - (e/c)A_\mu$ or

$$
\begin{aligned}
\frac{\partial}{\partial x_\mu} &\to \frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c} A_\mu \\
A_\mu &= (\mathbf{A}, i\Phi)
\end{aligned} \tag{10.9}
$$

This yields the Dirac equation

$$
\left[\gamma_\mu \left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c} A_\mu \right) + \frac{m_0 c}{\hbar} \right] \psi = 0 \tag{10.10}
$$

The equivalent Dirac hamiltonian is obtained by working backwards

$$
\begin{aligned}
H &= c\,\boldsymbol{\alpha} \cdot \left(\mathbf{p} - \frac{e}{c}\mathbf{A} \right) + \beta m_0 c^2 + e\Phi \\
&= H_0 + H_1 \\
H_1 &= -e\boldsymbol{\alpha} \cdot \mathbf{A} + e\Phi
\end{aligned} \tag{10.11}
$$

Here $e = -|e| = -e_p$ is the charge on the electron.

The Dirac equation for the adjoint field is obtained from Eq. (10.10) by taking the adjoint and multiplying on the right with γ_4

$$
\begin{aligned}
\bar{\psi} \left[\gamma_\mu \left(\frac{\overleftarrow{\partial}}{\partial x_\mu} + \frac{ie}{\hbar c} A_\mu \right) - \frac{m_0 c}{\hbar} \right] &= 0 \\
\bar{\psi}(x) &\equiv \psi^\dagger(x) \gamma_4
\end{aligned} \tag{10.12}
$$

The Dirac electromagnetic current is given by

$$\begin{aligned} ej_\mu &= e\left(\frac{1}{c}\psi^\dagger \boldsymbol{\alpha} c\,\psi,\, i\psi^\dagger\psi\right) \\ &= ie\,\bar{\psi}(x)\gamma_\mu\psi(x) \end{aligned} \tag{10.13}$$

It then follows by direct differentiation and use of the equations of motion that the Dirac current is conserved

$$\frac{\partial j_\mu}{\partial x_\mu} = 0 \tag{10.14}$$

Note that this relation holds in the presence of the electromagnetic field, as it must.

One obtains stationary state, plane wave solutions to the free Dirac equation upon substitution of the form

$$\psi = e^{-iEt/\hbar} e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}\, u(\mathbf{p}) \tag{10.15}$$

The resulting equations only have solutions for eigenvalues of the energy. We denote these eigenvalues and the corresponding eigenfunctions by

$$\begin{aligned} E &= +E_p & &;\ \text{solution } u(\mathbf{p}) \\ E &= -E_p & &;\ \text{solution } v(\mathbf{p}) \\ E_p &\equiv \sqrt{\mathbf{p}^2c^2 + m_0^2c^4} \end{aligned} \tag{10.16}$$

That it yield the correct relativistic energy–momentum relation is one of the requirements used to derive the Dirac equation. A little algebra shows that the four eigenfunctions (u_1, u_2, v_1, v_2) can be exhibited explicitly as the columns of the following modal matrix, again expressed in 2×2 form,

$$\mathscr{M} = \left(\frac{E_p + m_0c^2}{2E_p}\right)^{1/2} \begin{bmatrix} 1 & -\dfrac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p + m_0c^2} \\ \dfrac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p + m_0c^2} & 1 \end{bmatrix} \tag{10.17}$$

They satisfy the orthonormality conditions

$$\begin{aligned} u_i^\dagger u_j &= v_i^\dagger v_j = \delta_{ij} \\ u_i^\dagger v_j &= v_i^\dagger u_j = 0 \end{aligned} \tag{10.18}$$

Evidently, from the Dirac equation, these solutions satisfy

$$\begin{aligned} (i\gamma_\mu p_\mu + m_0c)u(\mathbf{p}) &= 0 \\ (i\gamma_\mu p_\mu - m_0c)v(-\mathbf{p}) &= 0 \\ p_\mu &\equiv (\mathbf{p}, iE_p/c) \end{aligned} \tag{10.19}$$

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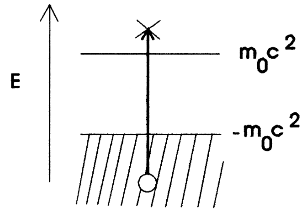

Fig. 10.1. Promotion of particle from a negative energy to a positive energy state in Dirac's hole theory.

Now $\mathbf{p}$ is the momentum eigenvalue. Note that the second equation is written for $v(-\mathbf{p})$. This solution can be interpreted with the aid of Dirac's *hole theory*.

A heuristic understanding of the role of the negative energy states was given by Dirac. Since particles in the positive energy states could just keep cascading down endlessly, he invoked the Pauli Exclusion Principle and assumed that in the vacuum the negative energy states are all filled. One always measures quantities with respect to the vacuum and the constant contribution of the filled states has no consequence for this theory.

This picture does have implications. A particle in one of the filled negative energy states can be promoted by some mechanism to one of the positive energy states as illustrated in Fig. 10.1. Since if one fills the negative energy state one recovers the vacuum, the hole (absence of a particle) must have the opposite properties of the particle. Dirac called these *antiparticles*. The antiparticle of the electron is the positron. If $v(-\mathbf{p}, \lambda)$ is the negative energy solution of a particle with charge $e = -|e|$, momentum $-\mathbf{p}$ and helicity λ with respect to $-\mathbf{p}$, then it represents a positron with charge $+|e|$, momentum $+\mathbf{p}$ and helicity λ with respect to $+\mathbf{p}$. (Since the spin also reverses, the helicity, or component of spin along the momentum, is unchanged.) Another immediate consequence of Dirac hole theory is that the vacuum becomes a dynamical quantity; it is polarizable for example, and relativistic quantum mechanics immediately confronts one with the relativistic quantum many-body problem.

The solutions in Eq. (10.17) can be combined to yield the projection operators

$$
\begin{aligned}
\sum_{\text{spins}, E>0} u(\mathbf{p}, s)_\alpha \bar{u}(\mathbf{p}, s)_\beta &= \left(\frac{m_0c^2 - i\gamma_\mu p_\mu c}{2E_p}\right)_{\alpha\beta} \\
\sum_{\text{spins}, E<0} v(-\mathbf{p}, s)_\alpha \bar{v}(-\mathbf{p}, s)_\beta &= \left(\frac{-m_0c^2 - i\gamma_\mu p_\mu c}{2E_p}\right)_{\alpha\beta}
\end{aligned}
\tag{10.20}
$$

The relativistic quantum field for a free electron can be expanded

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in terms of the normal-model solutions to the Dirac equation obtained above. The coefficients in the expansion become creation and destruction operators satisfying canonical anti-commutation relations (for fermions). After a canonical transformation to particles and holes, the field in the Schrödinger representation takes the form [Bj65, Fe71]

$$\psi(\mathbf{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k},\lambda} \left[a_{\mathbf{k},\lambda} u(\mathbf{k}\lambda) e^{i\mathbf{k}\cdot\mathbf{x}} + b^{\dagger}_{\mathbf{k},\lambda} v(-\mathbf{k}\lambda) e^{-i\mathbf{k}\cdot\mathbf{x}} \right] \tag{10.21}$$

We again quantize in a big box of volume Ω and use periodic boundary conditions. The Dirac current is given in terms of the field by

$$j_{\mu}(\mathbf{x}) = i\bar{\psi}(\mathbf{x})\gamma_{\mu}\psi(\mathbf{x}) \tag{10.22}$$

The hamiltonian in first quantization for a Dirac particle in an external, time-dependent field $A^{\text{ext}}_{\mu}(\mathbf{x}, t)$ is given by Eq. (10.11). In second quantization this hamiltonian takes the form [Bj65, Fe71]

$$\hat{H} = \int \hat{\psi}^{\dagger}(\mathbf{x}) \left\{ c\,\boldsymbol{\alpha} \cdot \left[\mathbf{p} - \frac{e}{c}\mathbf{A}^{\text{ext}}(\mathbf{x}, t) \right] + \beta m_0 c^2 + e\Phi^{\text{ext}}(\mathbf{x}, t) \right\} \hat{\psi}(\mathbf{x})\, d^3x \tag{10.23}$$

Here $\hat{\psi}(\mathbf{x})$ and $\hat{\psi}^{\dagger}(\mathbf{x})$ are field operators in the Schrödinger picture satisfying canonical anti-commutation relations [Eq. (10.21) provides a convenient representation]. The interaction hamiltonian in the external electromagnetic field is evidently

$$\hat{H}^{(1)} = -e\,\hat{j}_{\mu}(\mathbf{x}) A^{\text{ext}}_{\mu}(\mathbf{x}, t) \tag{10.24}$$

This hamiltonian can be used to determine the relativistic, quantum behavior of an electron in an arbitrary, time-dependent external electromagnetic field. It also governs pair production processes.

There are several readily established relations on the traces of the gamma matrices which are useful in the calculation of rates and cross sections [Bj65].

$$\begin{aligned}
\text{trace}\,\gamma_{\mu} &= 0 \\
\text{trace}\,\gamma_{\mu}\gamma_{\nu} &= 4\delta_{\mu\nu} \\
\text{trace}\,\gamma_{\mu}\gamma_{\nu}\gamma_{\rho} &= 0 \\
\text{trace}\,\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma} &= 4(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})
\end{aligned} \tag{10.25}$$

Other relations are given in appendix D.

Since $\hbar$ and c have now served their purpose, and we know where all the factors are, it is convenient to go over to a more common set of units used in nuclear and particle physics where

$$\hbar = c = 1 \tag{10.26}$$

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This simplifies the algebra considerably, and we shall henceforth assume this to be the case. All momenta and energies now become inverse lengths with the conversion factor

$$\hbar c = 197.3 \text{ MeV fm} \tag{10.27}$$

We shall take care to ensure that all final results are written in transparent and dimensionally correct form.