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A_1-TYPE SUBGROUPS CONTAINING REGULAR UNIPOTENT ELEMENTS

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Abstract

Let G be a simple exceptional algebraic group of adjoint type over an algebraically closed field of characteristic $p > 0$ and let $X = \mathrm{PSL}_2(p)$ be a subgroup of G containing a regular unipotent element x of G. By a theorem of Testerman, x is contained in a connected subgroup of G of type A_1. In this paper we prove that with two exceptions, X itself is contained in such a subgroup (the exceptions arise when $(G, p) = (E_6, 13)$ or $(E_7, 19)$). This extends earlier work of Seitz and Testerman, who established the containment under some additional conditions on p and the embedding of X in G. We discuss applications of our main result to the study of the subgroup structure of finite groups of Lie type.

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1. Introduction

Let G be a simple algebraic group of adjoint type over an algebraically closed field K of characteristic $p > 0$. Let $X = \mathrm{PSL}_2(q)$ be a subgroup of G, where $q \geqslant 4$ is a p-power, and let $x \in X$ be an element of order p. By the main theorem of [26], x is contained in a closed connected subgroup of G of type A_1, unless $G = G_2$, $p = 3$ and x belongs to the conjugacy class of G labelled $A_1^{(3)}$ as in [14]. With a view towards applications to the study of the subgroup structure of finite groups of Lie type, it is desirable to seek natural extensions of this result. In particular, under what conditions can one embed the full subgroup X in an A_1 subgroup of G?

As a special case of the main theorem of [29], this question has a positive answer when G is classical and X is not contained in a proper parabolic subgroup of G (for $G = \mathrm{SL}_n(K)$, this is a well-known theorem of Steinberg [32]). One can see that the condition on the embedding of X is necessary by considering indecomposable representations of X which do not arise as restrictions of indecomposable representations of an algebraic A_1. In [29], Seitz and Testerman also provide a positive answer if G is a simple exceptional algebraic group (of type G_2, F_4, E_6, E_7 or E_8) and p is large enough, still under the same assumption that X is not contained in a proper parabolic subgroup of G. More precisely, the approach in [29] requires $p > N(G)$ where

$$N(G_2) = 19,\ N(F_4) = 43,\ N(E_6) = 43,\ N(E_7) = 67,\ N(E_8) = 113. \quad (1)$$

More general results on the embedding of finite quasisimple subgroups in exceptional algebraic groups are established by Liebeck and Seitz in [18]. For instance, if $X = \mathrm{PSL}_2(q)$ and q is sufficiently large, then [18, Theorem 1] implies that X is contained in a proper closed positive-dimensional subgroup of G. Here 'sufficiently large' means that $q > t(G) \cdot (2, p-1)$ with

$$t(G_2) = 12,\ t(F_4) = 68,\ t(E_6) = 124,\ t(E_7) = 388,\ t(E_8) = 1312. \quad (2)$$

It is natural to seek an extension of [29, Theorem 2] by removing the conditions on p and the embedding of X in G when G is of exceptional type and $X = \mathrm{PSL}_2(q)$. In [30], Seitz and Testerman study the case where $x \in X$ is *semiregular* in G (that is, $C_G(x)$ is a unipotent group). Notice that if x is *not* semiregular then $x \in C_G(s)^0$ for some nontrivial semisimple element $s \in G$ and one can hope to answer the question in the proper reductive subgroup $C_G(s)^0$; so the semiregular case, where such a reduction is not possible, is particularly interesting. In this situation, the main result of [30] states that X is contained in a connected subgroup of type A_1 if either $q > p$, or if $q = p$ and $\mathrm{PGL}_2(q) \leqslant N_G(X)$.

In this paper, we extend the results in [30] by studying the remaining case where $X = \mathrm{PSL}_2(p)$ and $\mathrm{PGL}_2(p) \not\leqslant N_G(X)$. In order to do this, we assume $x \in X$ is *regular* in G, which means that $C_G(x)$ is an abelian unipotent group of dimension r, where r is the rank of G (equivalently, x is contained in a unique Borel subgroup of G). It is well known that regular unipotent elements exist in all characteristics and they form a single conjugacy class. Since the order of x is the smallest power of p greater than the height of the highest root of G (see [35, Order Formula 0.4]), our hypothesis implies that $p \geqslant h$, where h is the Coxeter number of G. (Recall that $h = (1/r)\dim G - 1 = \mathrm{ht}(\alpha_0) + 1$, where $\mathrm{ht}(\alpha_0)$ is the height of the highest root of G.)

Our main result is the following (in this paper, an A_1*-type subgroup* is a closed connected subgroup isomorphic to $\mathrm{SL}_2(K)$ or $\mathrm{PSL}_2(K)$).

THEOREM 1. *Let G be a simple exceptional algebraic group of adjoint type over an algebraically closed field of characteristic $p > 0$. Let $X = \mathrm{PSL}_2(p)$ be a subgroup of G containing a regular unipotent element of G. Then exactly one of the following holds:*

(i) *X is contained in an A_1-type subgroup of G;*

(ii) *$G = E_6$, $p = 13$ and X is contained in a D_5-parabolic subgroup of G;*

(iii) *$G = E_7$, $p = 19$ and X is contained in an E_6-parabolic subgroup of G.*

In all three cases, X is uniquely determined up to G-conjugacy.

REMARK 1. Let us make some comments on the statement of Theorem 1.

(a) To see the uniqueness of X in part (i), it suffices to show that every subgroup $Y = \mathrm{PSL}_2(p)$ of G containing x is conjugate to X. Write $X < A$ and $Y < B$, where A and B are A_1-type subgroups of G. By Proposition 2.11(ii), A and B are G-conjugate, say $A = B^g$, so $X, Y^g < A$. Finally, by applying [17, Theorem 5.1] and Lang's theorem, we deduce that X and Y^g are A-conjugate.

(b) The interesting examples arising in (ii) and (iii) were found by Craven [9] in his recent study of the maximal subgroups with socle $\mathrm{PSL}_2(q)$ in finite exceptional groups of Lie type. The action of such a subgroup X on the adjoint module $\mathrm{Lie}(G)$ is described in Theorem 8.1 (see Section 8) and its construction is explained in [9, Section 9]. Let us say a few words on the construction in (ii), where $G = E_6$ and $p = 13$. Let $P = QL$ be a D_5-parabolic subgroup of G and identify the unipotent radical Q with a 16-dimensional spin module for $L' = D_5$. Take a subgroup $Y = \mathrm{PSL}_2(p) < L'$ containing a regular unipotent element of L' and consider the semidirect product $QY < P$ (note that Y is uniquely determined up to L'-conjugacy). Now one checks that $Q|_Y$ has an 11-dimensional composition factor W with $\dim H^1(Y, W) = 1$, which is a direct summand of Q. It follows that there is a complement $X = \mathrm{PSL}_2(p)$ to Q in QY that is not QY-conjugate to Y. Moreover, one can show that X contains a regular unipotent element of G and there is a unique P-class of such subgroups X (hence X is uniquely determined up to G-conjugacy). We show that the subgroup X constructed in this way is not contained in an A_1-type subgroup of G (this follows from Theorem 2 below). A similar construction can be given in (iii) and again one can show that such a subgroup is both unique up to conjugacy and is not contained in an A_1-type subgroup.

(c) The conclusion of Theorem 1 for $G = G_2$ can be deduced from the proof of [30, Lemma 3.1]. It also follows from Kleidman's classification of the maximal subgroups of $G_2(p)$ in [13]. However, for completeness we provide an alternative proof, following the same approach we use for the other exceptional groups.

(d) Finally, let us comment on the adjoint hypothesis in the statement of the theorem. Let G be a simple exceptional algebraic group and let G_{ad} be the corresponding adjoint group. Suppose $Y = \mathrm{PSL}_2(p)$ or $\mathrm{SL}_2(p)$ is a subgroup of G containing a regular unipotent element y of G. The regularity of y implies that $Z(Y) \leqslant Z(G)$ and thus $YZ(G)/Z(G) = \mathrm{PSL}_2(p)$ is a subgroup of G_{ad} containing a regular unipotent element, so it is determined by Theorem 1.

The next result shows that the subgroups X in part (i) of Theorem 1 are *G-irreducible* in the sense of Serre (that is, X is not contained in a proper parabolic subgroup of G). The proof is given at the end of Section 2. By [36, Theorem 1.2], any connected reductive subgroup of a reductive algebraic group G containing a regular unipotent element is G-irreducible, so we can view Theorem 2 as a partial analogue for subgroups isomorphic to $\mathrm{PSL}_2(p)$ in simple exceptional groups.

THEOREM 2. *Let G be a simple exceptional algebraic group of adjoint type over an algebraically closed field of characteristic $p > 0$ and let $x \in G$ be a regular unipotent element such that*

$$x \in X = \mathrm{PSL}_2(p) < A < G,$$

where A is an A_1-type subgroup of G. Then X is G-irreducible.

REMARK 2. As in Theorem 1, let $X = \mathrm{PSL}_2(p)$ be a subgroup of G containing a regular unipotent element. By combining Theorems 1 and 2, we deduce that X is contained in an A_1-type subgroup of G if and only if X is not contained in a proper parabolic subgroup of G. In particular, the examples arising in parts (ii) and (iii) of Theorem 1 are genuine exceptions to the containment in (i).

The next result follows by combining Theorem 1 with the main results of [29, 30].

COROLLARY 1. *Let G be a simple algebraic group of adjoint type over an algebraically closed field of characteristic $p > 0$ and let $X = \mathrm{PSL}_2(q)$ be a subgroup of G containing a regular unipotent element of G, where $q \geqslant 4$ is a p-power. In addition, if G is classical assume that X is G-irreducible. Then either*

(a) *X is contained in an A_1-type subgroup of G, or*

(b) *$q = p$ and (G, p, X) is one of the cases in parts* (ii) *and* (iii) *in Theorem* 1.

Next we present some further applications of Theorem 1. Let G be a simple algebraic group as in Theorem 1 and recall that a finite subgroup H of G is *Lie primitive* if

(a) H does not contain a subgroup of the form $O^{p'}(G^F)$, where F is a Steinberg endomorphism of G with fixed point subgroup G^F; and

(b) H is not contained in a proper closed subgroup of G of positive dimension.

In [**11**, Section 3], Guralnick and Malle determine the maximal Lie primitive subgroups H of G containing a regular unipotent element (the maximal closed positive-dimensional subgroups of G containing a regular unipotent element were determined in earlier work of Saxl and Seitz [**27**]). More precisely, they give a list of possibilities for H, but they do not claim that all cases actually occur. In particular, their proof relies on [**29**] and thus $H = \mathrm{PSL}_2(p)$ arises as a possibility when $G \in \{F_4, E_6, E_7, E_8\}$ and $h \leqslant p \leqslant N(G)$, where $N(G)$ is the integer in (1). Therefore, by combining [**11**, Theorems 3.3, 3.4] with Theorem 1, we obtain the following refinement.

COROLLARY 2. *Let G be a simple exceptional algebraic group of adjoint type over an algebraically closed field of characteristic $p > 0$. Suppose H is a maximal Lie primitive subgroup of G containing a regular unipotent element. Let H_0 denote the socle of H.*

(i) *If $G = G_2$, then one of the following holds:*

   (a) *$p = 2$ and $H = \mathrm{J}_2$;*
   (b) *$p = 7$ and $H = 2^3.\mathrm{SL}_3(2)$, $G_2(2)$ or $\mathrm{PSL}_2(13)$;*
   (c) *$p = 11$ and $H = \mathrm{J}_1$.*

(ii) *If $G = F_4$, then one of the following holds:*

   (a) *$p = 2$ and $H_0 = \mathrm{PSL}_3(16)$, $\mathrm{PSU}_3(16)$ or $\mathrm{PSL}_2(17)$;*
   (b) *$p = 13$ and $H = 3^3.\mathrm{SL}_3(3)$, or $H_0 = \mathrm{PSL}_2(25)$, $\mathrm{PSL}_2(27)$ or ${}^3D_4(2)$.*

(iii) *If* $G = E_6$, *then one of the following holds:*

(a) $p = 2$ *and* $H_0 = \mathrm{PSL}_3(16)$, $\mathrm{PSU}_3(16)$ *or* Fi_{22}*;*

(b) $p = 13$ *and either* $H = 3^{3+3}.\mathrm{SL}_3(3)$ *or* $H_0 = {}^2F_4(2)'$.

(iv) *If* $G = E_7$, *then* $p = 19$ *and* $H_0 = \mathrm{PSU}_3(8)$ *or* $\mathrm{PSL}_2(37)$.

(v) *If* $G = E_8$, *then one of the following holds:*

(a) $p = 2$ *and* $H_0 = \mathrm{PSL}_2(31)$*;*

(b) $p = 7$ *and* $H_0 = \mathrm{PSp}_8(7)$ *or* $\Omega_9(7)$*;*

(c) $p = 31$ *and* $H = 2^{5+10}.\mathrm{SL}_5(2)$ *or* $5^3.\mathrm{SL}_3(5)$, *or* $H_0 = \mathrm{PSL}_2(32)$, $\mathrm{PSL}_2(61)$ *or* $\mathrm{PSL}_3(5)$.

REMARK 3. By Corollary 2, there are no Lie primitive subgroups containing a regular unipotent element if $p > 31$. This lower bound is best possible: the case $(G, p) = (E_8, 31)$ with $H_0 = \mathrm{PSL}_2(32)$ is a genuine example (this can be deduced from recent work of Litterick [22]). However, we are not claiming that all of the possibilities listed in Corollary 2 are Lie primitive and contain regular unipotent elements (indeed, we expect that this list can be reduced further).

We can also use Theorem 1 to shed new light on the subgroup structure of finite exceptional groups of Lie type. Let G be a simple exceptional algebraic group of adjoint type over $\bar{\mathbb{F}}_p$ with p prime and let $F : G \to G$ be a Steinberg endomorphism of G with fixed point subgroup G^F, an almost simple group over $\mathbb{F}_q$. The maximal subgroups of the Ree groups ${}^2G_2(q)$ and ${}^2F_4(q)$ (and their automorphism groups) have been determined up to conjugacy by Kleidman [13] and Malle [23], respectively, and similarly $G_2(q)$ is handled in [8] for q even and in [13] for q odd. Therefore, we may assume G^F is one of $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ and $E_8(q)$. In these cases, through the work of many authors, the problem of determining the maximal subgroups H of G^F has essentially been reduced to the case where H is an almost simple group of Lie type with socle H_0 over a field $\mathbb{F}_{q_0}$ of characteristic p (see [24, Section 29.1] and the references therein). Here one of the main problems is to determine if such a subgroup is of the form M^F, where M is maximal among positive-dimensional F-stable closed subgroups of G. Significant restrictions on the rank of H_0 and the size of q_0 are established in [16, 18], but the problem of obtaining a complete classification is still open.

The case $H_0 = \mathrm{PSL}_2(q_0)$ is of particular interest. If $q_0 > t(G)\cdot(2, p-1)$, where $t(G)$ is the integer in (2), then the aforementioned work of Liebeck and Seitz [18] shows that $q = q_0$ and $H = M^F$ for some maximal connected subgroup M of G of type A_1. Further results in this direction have recently been obtained by Craven [9] when G^F is one of $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$ or $E_7(q)$. Using the maximality of H, he proves that $H = M^F$ in almost every case, but his approach is unable to eliminate certain values of q_0. In particular, the case where $H = \mathrm{PSL}_2(h+1)$ contains a regular unipotent element of G is problematic (the existence of such subgroups, in a much more general setting, was established by Serre [31], which explains why they are called *Serre embeddings* in [9]). Using Theorem 1, one can show that all maximal Serre embeddings are of the form M^F (we can also handle $G = E_8$, which is excluded in [9]). In particular, it follows that part (1) in [9, Theorem 1.2] is a subcase of part (2), and similarly part (2) in [9, Theorem 1.4] is a subcase of part (3).

To conclude the introduction, let us briefly describe the main steps in the proof of Theorem 1 (we refer the reader to Section 2.5 for more details). Suppose $x \in X = \mathrm{PSL}_2(p) < G$ is a regular unipotent element of G and let $A < G$ be an A_1-type subgroup containing x with maximal torus $T = \{t(c) \mid c \in K^\times\}$. Set $V = \mathrm{Lie}(G)$ and $\mathbb{F}_p^\times = \langle \xi \rangle$. Without loss of generality, replacing X by a suitable G-conjugate, we show that we may assume X contains the toral element $t(\xi) \in T$, which corresponds to a diagonalizable element $s \in \mathrm{SL}_2(p)$ with eigenvalues ξ and ξ^{-1} (see Lemma 2.19). We can use the known action of A on V to determine the eigenvectors and eigenspaces of s on V and this severely restricts the possibilities for $V|_X$. It is possible to obtain further restrictions on the indecomposable summands of $V|_X$ by considering the trace on V of semisimple elements in X of small order (typically, we only need to work with elements of order 2 and 3).

In this way, in almost all cases, we are able to reduce to the situation where $V|_X$ is compatible with the action of a $\mathrm{PSL}_2(p)$ subgroup of A. In this situation, $V|_X$ is given in Table 2 (our notation for indecomposable summands in Table 2 is explained in Section 2.1) and we observe that the socle of $V|_X$ has a 3-dimensional simple summand

$$W = \langle w_2, w_0, w_{-2} \rangle,$$

where w_i is an eigenvector for s with eigenvalue ξ^i. Let E_i be the ξ^i-eigenspace of s on V. Without loss of generality, we may assume that the action of x on W (in terms of this basis) is given by the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

and thus

$$\begin{aligned} w_2 &\in \ker(x-1)\cap E_2, \\ w_0 &\in (\ker((x-1)^2)\setminus \ker(x-1))\cap E_0, \\ w_{-2} &\in (\ker((x-1)^3)\setminus \ker((x-1)^2))\cap E_{-2}. \end{aligned}$$

Our main goal is to show that W is an $\mathfrak{sl}_2$-subalgebra of V.

To do this, we may assume that x is obtained by exponentiating the regular nilpotent element $\sum_{\gamma\in\Pi(G)} e_\gamma \in V$ with respect to a fixed Chevalley basis

$$\mathcal{B} = \{e_\alpha, f_\alpha, h_\gamma \mid \alpha\in\Phi^+(G),\ \gamma\in\Pi(G)\}$$

for V (see Section 2.5 for more details). This allows us to explicitly identify a maximal torus of an A_1-type subgroup of G containing x, which means that we can compute eigenvectors and eigenspaces for s in terms of the Chevalley basis. With the aid of MAGMA [3] to simplify the computations, we can describe the action of x on V in terms of a $\dim G\times\dim G$ matrix with respect to $\mathcal{B}$ and then compute bases for the subspaces $\ker((x-1)^i)$ for $i\geqslant 1$. In this way, we obtain expressions for w_2, w_0 and w_{-2} in terms of $\mathcal{B}$, but with undetermined coefficients. We then derive relations between these coefficients by considering the action of x on W, and further relations can be found by using the fact that $C_V(x)=\ker(x-1)$ is an abelian subalgebra. Apart from a handful of special cases, this allows us to reduce to the case where W is an $\mathfrak{sl}_2$-subalgebra and we complete the argument by showing that the stabilizer of W in G is an A_1-type subgroup.

This process of elimination and extension comprises the bulk of the proof of Theorem 1 (see Sections 3–7). However, there are a handful of possibilities for (G,p) which require further attention; these are the cases arising in part (ii) of Theorem 2.23 and they are handled in Section 8. In each of these cases, the action of X on V is known (up to one of three possibilities if $(G,p)=(E_6,13)$ or $(E_7,19)$) and X stabilizes a nonzero subalgebra of $\langle e_\alpha \mid \alpha\in\Phi^+(G)\rangle$. This allows us to reduce to the case where X is contained in a proper parabolic subgroup $P=QL$ of G. Let $\pi: P\to P/Q$ be the quotient map. Using π, we identify L with P/Q and so we may view $\pi(X)$ as a subgroup of L'. We may as well assume that P is a minimal parabolic (with respect to containing X), so $\pi(X)$ is not contained in a proper parabolic subgroup of L'. Now $\pi(x)\in L'$ is a regular unipotent element which is contained in an A_1-type subgroup H of L' (this follows by combining Theorem 2.23 with the aforementioned earlier work of Seitz and Testerman [29] for classical groups). By inspecting [15], we can determine the action of H on V, which must be compatible with the action of X on V given in Theorem 2.23. In this way we deduce that $(G,p,L')=(E_6,13,D_5)$ and $(E_7,19,E_6)$ are the only possibilities, and this completes the proof of Theorem 1.

Notation

Our notation is fairly standard. For a simple algebraic group G we write $\Phi(G)$, $\Phi^+(G)$ and $\Pi(G) = \{\alpha_1, \dots, \alpha_r\}$ for the set of roots, positive roots and simple roots of G, with respect to a fixed Borel subgroup, and we follow Bourbaki [4] in labelling the simple roots. We often denote a root $\alpha = a_1\alpha_1 + \cdots + a_r\alpha_r$ by writing $\alpha = a_1 \cdots a_r$. If V is a module for a group then $\mathrm{soc}(V)$ and $\mathrm{rad}(V)$ denote the socle and radical of V, respectively, and we write V^m to denote $V \oplus \cdots \oplus V$ (with m summands). It will be convenient to write $[A_1^{n_1}, \dots, A_k^{n_k}]$ for a block-diagonal matrix with a block A_i occurring with multiplicity n_i. In addition, we write J_i for a standard (upper triangular) unipotent Jordan block of size i.

2. Preliminaries

In this section, we record some preliminary results that will be needed in the proof of Theorem 1. We start by recalling some well-known results from the modular representation theory of the simple groups $\mathrm{PSL}_2(p)$. Our main reference is Alperin [1].

2.1. Representation theory. Let K be an algebraically closed field of characteristic $p \geqslant 5$, let $X = \mathrm{PSL}_2(p)$ and let $P = \langle x \rangle \cong Z_p$ be a Sylow p-subgroup of X.

The subgroup P of X has exactly p indecomposable KP-modules, say W_i for $i = 1, \dots, p$, where $\dim W_i = i$ and W_p is the unique projective indecomposable KP-module. The element x has Jordan form $[J_i]$ on W_i. In particular, if W is a projective KP-module, then $\dim W = ap$ for some $a \geqslant 1$, and x has Jordan form $[J_p^a]$ on W.

There are precisely $(p+1)/2$ simple KX-modules, labelled $V_1, V_3, \dots, V_p$ in [1], where $\dim V_i = i$. In particular, every simple KX-module is odd-dimensional. Here V_1 is the trivial module and V_p is the Steinberg module. It is easy to see that x has Jordan form $[J_i]$ on V_i. By a theorem of Steinberg, each V_i is the restriction of a simple module for the corresponding algebraic group of type A_1 (see [33, Section 13]), so we can refer to the highest weight of V_i with respect to a maximal torus of the algebraic A_1. We identify the weights of this 1-dimensional torus with the set of integers, and we often write $V_i = L_X(i-1)$ to highlight the highest weight of V_i.

Similarly, there are precisely $(p+1)/2$ projective indecomposable KX-modules, labelled $P_1, P_3, \dots, P_p$ in [1], where $P_p = V_p$ is simple and the remainder are reducible. Here $\dim P_1 = \dim P_p = p$ and $\dim P_i = 2p$ for $1 < i < p$. The element x has Jordan form $[J_p]$ on P_1 and P_p, and Jordan form $[J_p^2]$ on the remaining P_i. The structure of these modules is described by Alperin [1, pages 48–49]. In terms of composition factors, we have

$$P_1 = V_1|V_{p-2}|V_1$$

and

$$P_i = V_i|(V_{p-i+1} \oplus V_{p-i-1})|V_i$$

where $1 < i < p$ is odd. (Here this notation indicates that $\mathrm{soc}(P_i) \cong P_i/\mathrm{rad}(P_i) \cong V_i$ and $\mathrm{rad}(P_i)/\mathrm{soc}(P_i) \cong V_{p-i+1} \oplus V_{p-i-1}$.) It will be convenient to define

$$U = P_1 = L_X(0)|L_X(p-3)|L_X(0) \tag{3}$$

and

$$W(i) = P_{i+1} = L_X(i)|(L_X(p-i-1) \oplus L_X(p-i-3))|L_X(i) \tag{4}$$

for $i \in \{2, 4, \dots, p-3\}$.

The Green correspondence (see [1, Section 11]) implies that if V is an indecomposable KX-module then $V|_P = W \oplus W'$ where W is projective (or zero) and W' is indecomposable (or zero). In particular, the following lemma holds.

LEMMA 2.1. *Let V be an n-dimensional indecomposable KX-module and write $n = ap + b$, where $a \geqslant 0$ and $0 \leqslant b < p$. Then x has Jordan form $[J_p^a, J_b]$ on V.*

The main result on the structure of indecomposable KX-modules is the following theorem. Here we define a *subtuple* of an n-tuple $[m_1, \dots, m_n]$ to be a tuple of the form $[m_i, m_{i+1}, \dots, m_j]$ for some $1 \leqslant i \leqslant j \leqslant n$. We denote this by writing

$$[m_i, m_{i+1}, \dots, m_j] \subseteq [m_1, \dots, m_n].$$

THEOREM 2.2. *Let V be a reducible indecomposable nonprojective KX-module. Then there exists an integer $\ell \geqslant 2$ and a subtuple*

$$[a_1, \dots, a_\ell] \subseteq [1, p-2, 3, p-4, \dots, p-2, 1]$$

such that

$$\mathrm{soc}(V) = V_{a_1} \oplus V_{a_3} \oplus \cdots \oplus V_{a_{\ell-\epsilon}}, \quad V/\mathrm{soc}(V) = V_{a_2} \oplus V_{a_4} \oplus \cdots \oplus V_{a_{\ell-1+\epsilon}}$$

where $\epsilon = 1$ if ℓ is even, otherwise $\epsilon = 0$.

Proof. This follows from the discussion in [12, Section 3]. Also see [9, Section 7.3]. □

COROLLARY 2.3. *Let V be an indecomposable KX-module with precisely two composition factors. If* $\mathrm{soc}(V) = L_X(i)$ *then*

$$V/\mathrm{soc}(V) \in \{L_X(p-i-3), L_X(p-i-1)\}$$

for some $i \in \{0, 2, \dots, p-3\}$, *hence* $\dim V = p \pm 1$.

COROLLARY 2.4. *Let V be a reducible indecomposable KX-module. Then* $\dim V \geqslant p-1$. *Moreover, if V has at least four composition factors, then* $\dim V \geqslant 2(p-1)$.

2.2. Traces. As in Section 2.1, let K be an algebraically closed field of characteristic $p \geqslant 5$ and set $X = \mathrm{PSL}_2(p)$. Let x_2 and x_3 be representatives of the unique conjugacy classes of elements of order 2 and 3 in X, respectively (note that x_i is semisimple since $p \geqslant 5$). Let V be a KX-module and let $\mathrm{tr}(V, x_i)$ denote the trace of x_i on V.

LEMMA 2.5. *If* $V = L_X(i)$ *then*

$$\mathrm{tr}(V, x_2) = (-1)^{i/2}, \quad \mathrm{tr}(V, x_3) = \begin{cases} 1 & i \equiv 0 \pmod 3 \\ -1 & i \equiv 1 \pmod 3 \\ 0 & i \equiv 2 \pmod 3. \end{cases}$$

Proof. This is a straightforward calculation, using the fact that we can identify $L_X(i)$ with the ith symmetric power $\mathrm{Sym}^i(M)$, where M is the natural module for $\mathrm{SL}_2(K)$. □

If V is a KX-module with composition factors $M_1, \dots, M_k$, then

$$\mathrm{tr}(V, x_i) = \sum_{j=1}^{k} \mathrm{tr}(M_j, x_i)$$

since the action of x_i is diagonalizable. Therefore, the next two results are immediate corollaries of Lemma 2.5 (here we use the notation U and $W(i)$ defined in (3) and (4)).

LEMMA 2.6. *We have*

$$\mathrm{tr}(U, x_2) = \begin{cases} 1 & p \equiv 1 \pmod 4 \\ 3 & p \equiv 3 \pmod 4, \end{cases} \quad \mathrm{tr}(U, x_3) = \begin{cases} 1 & p \equiv 1 \pmod 3 \\ 2 & p \equiv 2 \pmod 3. \end{cases}$$

Table 1. Traces of elements of order 2 and 3 on the adjoint module.

G	$\mathcal{T}_2(G, V)$	$\mathcal{T}_3(G, V)$
G_2	-2	$-1, 5$
F_4	$-4, 20$	$-2, 7$
E_6	$-2, 14$	$-3, 3, 6, 15, 30$
E_7	$-7, 5, 25$	$-2, 7, 34, 52$
E_8	$-8, 24$	$-4, 5, 14, 77$

LEMMA 2.7. *We have*

$$\mathrm{tr}(W(i), x_2) = \begin{cases} 2 & i \equiv 0 \pmod 4 \\ -2 & i \equiv 2 \pmod 4 \end{cases}$$

and

$$\mathrm{tr}(W(i), x_3) = \begin{cases} \begin{cases} 2 & p \equiv 1 \pmod 3 \\ 1 & p \equiv 2 \pmod 3 \end{cases} & i \equiv 0 \pmod 3 \\ \begin{cases} -1 & p \equiv 1 \pmod 3 \\ -2 & p \equiv 2 \pmod 3 \end{cases} & i \equiv 1 \pmod 3 \\ \begin{cases} -1 & p \equiv 1 \pmod 3 \\ 1 & p \equiv 2 \pmod 3 \end{cases} & i \equiv 2 \pmod 3. \end{cases}$$

Let G be a simple algebraic group over K of adjoint type, let V be a KG-module and let m be a positive integer. Define

$$\mathcal{T}_m(G, V) = \{\mathrm{tr}(V, x) \mid x \in G \text{ has order } m\}.$$

Recall that the *adjoint module* for G is the Lie algebra $\mathrm{Lie}(G)$, on which G acts via the adjoint representation.

PROPOSITION 2.8. *Let G be a simple exceptional algebraic group of adjoint type over an algebraically closed field of characteristic $p \geqslant 5$. Let $V = \mathrm{Lie}(G)$ be the adjoint module. Then $\mathcal{T}_m(G, V)$ is recorded in Table 1 for $m \in \{2, 3\}$.*

Proof. This follows by inspecting the dimensions of the centralizers of elements of order m in G (see [10, Tables 4.3.1 and 4.7.1]), using the fact that

$$\dim C_V(g) = \dim C_G(g)$$

for every semisimple element $g \in G$ (see [6, Section 1.14], for example).

Table 2. The action of X on $V = \mathrm{Lie}(G)$ in Proposition 2.12.

G	p	$V\|_X$
G_2	$p \geqslant 11$	$L_X(10) \oplus L_X(2)$
	$p = 7$	$W(2)$
F_4	$p \geqslant 23$	$L_X(22) \oplus L_X(14) \oplus L_X(10) \oplus L_X(2)$
	$p = 19$	$W(14) \oplus L_X(10) \oplus L_X(2)$
	$p = 17$	$W(10) \oplus L_X(14) \oplus L_X(2)$
	$p = 13$	$W(10) \oplus W(2)$
E_6	$p \geqslant 23$	$L_X(22) \oplus L_X(16) \oplus L_X(14) \oplus L_X(10) \oplus L_X(8) \oplus L_X(2)$
	$p = 19$	$W(14) \oplus L_X(16) \oplus L_X(10) \oplus L_X(8) \oplus L_X(2)$
	$p = 17$	$W(10) \oplus L_X(16) \oplus L_X(14) \oplus L_X(8) \oplus L_X(2)$
	$p = 13$	$W(10) \oplus W(8) \oplus W(2)$
E_7	$p \geqslant 37$	$L_X(34) \oplus L_X(26) \oplus L_X(22) \oplus L_X(18) \oplus L_X(14) \oplus L_X(10) \oplus L_X(2)$
	$p = 31$	$W(26) \oplus L_X(22) \oplus L_X(18) \oplus L_X(14) \oplus L_X(10) \oplus L_X(2)$
	$p = 29$	$W(22) \oplus L_X(26) \oplus L_X(18) \oplus L_X(14) \oplus L_X(10) \oplus L_X(2)$
	$p = 23$	$W(18) \oplus W(10) \oplus L_X(22) \oplus L_X(14) \oplus L_X(2)$
	$p = 19$	$W(14) \oplus W(10) \oplus W(2) \oplus L_X(18)$
E_8	$p \geqslant 59$	$L_X(58) \oplus L_X(46) \oplus L_X(38) \oplus L_X(34) \oplus L_X(26) \oplus L_X(22)$ $\oplus\ L_X(14) \oplus L_X(2)$
	$p = 53$	$W(46) \oplus L_X(38) \oplus L_X(34) \oplus L_X(26) \oplus L_X(22) \oplus L_X(14) \oplus L_X(2)$
	$p = 47$	$W(34) \oplus L_X(46) \oplus L_X(38) \oplus L_X(26) \oplus L_X(22) \oplus L_X(14) \oplus L_X(2)$
	$p = 43$	$W(38) \oplus W(26) \oplus L_X(34) \oplus L_X(22) \oplus L_X(14) \oplus L_X(2)$
	$p = 41$	$W(34) \oplus W(22) \oplus L_X(38) \oplus L_X(26) \oplus L_X(14) \oplus L_X(2)$
	$p = 37$	$W(34) \oplus W(26) \oplus W(14) \oplus L_X(22) \oplus L_X(2)$
	$p = 31$	$W(26) \oplus W(22) \oplus W(14) \oplus W(2)$

For instance, if $g \in G = E_8$ has order 3 and $C_G(g) = A_8$, then $\dim C_V(g) = 80$ and the self-duality of V implies that the action of g on V is given by the diagonal matrix $[I_{80}, \omega I_{84}, \omega^2 I_{84}]$, up to conjugacy, where ω is a primitive cube root of unity. Therefore, $\mathrm{tr}(V, g) = -4$. □

REMARK 2.9. Suppose $X = \mathrm{PSL}_2(p)$ is contained in $G = E_6$, where $p \geqslant 5$ and G is adjoint. Write $G = \hat{G}/S$ and $X = \hat{X}/S$, where $\hat{G}$ is the simply connected group of type E_6 and $S = Z_3$ is the centre of $\hat{G}$. Now X has Schur multiplier Z_2, which implies that $\hat{X} = Z_3 \times X$. Therefore, every element $y \in X$ of order 3 lifts to an element in $\hat{G}$ of order 3. In particular, if $y \in X$ has order 3 then $C_G(y)^0 = A_5T_1$, D_4T_2 or A_2^3 (see [10, Table 4.7.1]), whence $\mathrm{tr}(V, y) \in \{-3, 6, 15\}$ with respect to $V = \mathrm{Lie}(G)$.

REMARK 2.10. In a few cases it is helpful to know the eigenvalue multiplicities on V of elements in G of order $m > 3$ for certain values of m; the relevant cases are the following:

$$(G, m) \in \{(F_4, 7), (E_6, 7), (E_7, 5), (E_8, 19)\}.$$

It is straightforward to obtain this information with the aid of MAGMA [3], using an algorithm of Litterick (see [21, Section 3.3.1]), which is heavily based on work of Moody and Patera [25]. We thank Dr. Litterick for his assistance with these computations.

2.3. A_1-type subgroups. Let G be a simple algebraic group and recall that p is a *good* prime for G if $p > 2$ in types B, C and D, $p > 3$ for G_2, F_4, E_6 and E_7, and $p > 5$ when G is of type E_8 (all primes are good in type A).

PROPOSITION 2.11. *Let G be a simple algebraic group of adjoint type over an algebraically closed field of good characteristic $p > 0$. Let $x \in G$ be an element of order p.*

(i) *There is an A_1-type subgroup of G containing x.*

(ii) *If x is regular then the subgroup in* (i) *is unique up to G-conjugacy.*

Proof. Part (i) follows from the main theorem of [35]. Part (ii), for G exceptional, follows from [15, Theorem 4]. Now assume G is classical and let H be an A_1-type subgroup of G containing x. Let V be the natural module for G. By [36, Theorem 1.2], H is not contained in a proper parabolic subgroup of G. In particular, if G is of type A, B or C then H acts irreducibly and tensor indecomposably (see [36, Proposition 2.3]) on V and the conjugacy statement follows from representation theory.

Finally, let us assume $G = D_r$ (with $r \geqslant 4$). We claim that $H < L < G$, where $L = B_{r-1}$ is the stabilizer of a nonsingular 1-space. The result then follows since H is unique in L up to L-conjugacy, and L itself is unique up to G-conjugacy. To justify the claim, first observe that x has Jordan form $[J_{2r-1}, J_1]$ on V, using [27, Lemma 1.2(ii)] and the fact that x has order p, so $p \neq 2$. If H acts irreducibly on V then the Jordan form of x implies that H is tensor decomposable, but this is incompatible with [27, Lemma 1.5]. Therefore, H acts reducibly on V and we complete the argument by applying [20, Lemma 2.2]. □

PROPOSITION 2.12. *Let G be a simple exceptional algebraic group of adjoint type and let $x \in G$ be a regular unipotent element such that*

$$x \in X = \mathrm{PSL}_2(p) < A < G,$$

where A is an A_1-type subgroup. Then the action of X on the adjoint module $V = \mathrm{Lie}(G)$ is given in Table 2.

Proof. A precise description of $V|_A$ as a tilting module is given in [19, Table 10.1] (for $G = E_6$ we may assume that $A < F_4 < G$ so the action of A on V can be deduced from the actions of A on $\mathrm{Lie}(F_4)$ and the minimal module for F_4 (see [19, Table 10.2])). Following [19], we write $T(\lambda; \mu; \ldots)$ for a tilting module having the same composition factors as the direct sum of Weyl modules for A with highest weights $\lambda, \mu, \ldots$ In terms of this notation, we get

$$V|_A = \begin{cases} T(10; 2) & G = G_2 \\ T(22; 14; 10; 2) & G = F_4 \\ T(22; 16; 14; 10; 8; 2) & G = E_6 \\ T(34; 26; 22; 18; 14; 10; 2) & G = E_7 \\ T(58; 46; 38; 34; 26; 22; 14; 2) & G = E_8. \end{cases}$$

As explained at the start of [19, Section 10], we can express $T(\lambda; \mu; \ldots)$ as a direct sum of indecomposable tilting modules of the form $T(c)$, where the highest weight c is at most $2p - 2$. For example, suppose $G = F_4$ and $p = 19$, so $V|_A = T(22; 14; 10; 2)$ as above. The highest weight is 22, so one summand is $T(22)$, which is a uniserial module of shape $14|22|14$ (see [28, Lemma 2.3]). The highest weight not already accounted for is 10, so $T(10) = L_A(10)$ is a summand and we deduce that $V|_A = T(22) \oplus L_A(10) \oplus L_A(2)$ and thus

$$V|_X = T(22)|_X \oplus L_X(10) \oplus L_X(2).$$

By [28, Lemma 2.3], $T(22)|_X$ is a projective indecomposable KX-module of dimension $2p = 38$, so $T(22)|_X = W(i)$ for some i. By comparing socles, it follows that $i = 14$ and thus

$$V|_X = W(14) \oplus L_X(10) \oplus L_X(2)$$

as recorded in Table 2. The other cases are similar and we omit the details. □

For the remainder of this section, let G be a simple exceptional algebraic group of adjoint type over an algebraically closed field K of characteristic $p > 0$, and let r and $h = h(G)$ be the rank and Coxeter number of G, respectively. We assume G contains a regular unipotent element of order p, which means that

$$p \geqslant h. \tag{5}$$

We need to recall the construction of A_1-type subgroups of G containing regular unipotent elements, following the treatment in [31, 34, 35].

First we need some new notation. Let $\mathcal{L}_{\mathbb{C}}$ be a simple Lie algebra over $\mathbb{C}$ of type $\Phi(G)$. Fix a Chevalley basis

$$\mathcal{B} = \{e_\alpha, f_\alpha, h_\gamma \mid \alpha \in \Phi^+(G),\ \gamma \in \Pi(G)\}$$

of $\mathcal{L}_{\mathbb{C}}$ and write $z_i = z_{\alpha_i}$ for $z \in \{e, f, h\}$ and $\Pi(G) = \{\alpha_1, \ldots, \alpha_r\}$. It will be convenient to define $f_\alpha = e_{-\alpha}$ for each $\alpha \in \Phi^+(G)$. Let $\mathcal{L}_{\mathbb{Z}}$ be the $\mathbb{Z}$-span of $\mathcal{B}$ and set $\mathcal{L}_K = \mathcal{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. (By abuse of notation, we also write $e_\alpha, f_\alpha, e_i, f_i, h_i$ for the elements $e_\alpha \otimes 1$, $f_\alpha \otimes 1$, and so forth, in $\mathcal{L}_K$.) Fix a root $\alpha \in \Phi(G)$. As in the familiar Chevalley construction, we have

$$(\mathrm{ad}(e_\alpha)^j/j!)(\mathcal{L}_{\mathbb{Z}}) \subseteq \mathcal{L}_{\mathbb{Z}}$$

for all $j \geqslant 0$, and this allows us to construct the element

$$\exp(\mathrm{ad}(\mathbf{x}e_\alpha)) \in \mathrm{GL}_{\dim G}(\mathbb{Z}[\mathbf{x}]),$$

where $\mathbf{x}$ is an indeterminate. Passing to K, we obtain a 1-dimensional unipotent subgroup

$$U_\alpha = \{\exp(\mathrm{ad}(\gamma e_\alpha)) \mid \gamma \in K\} \leqslant \mathrm{Aut}(\mathcal{L}_K)^0 = G$$

(see [5, Proposition 4.4.2]). Note that $G = \langle U_\alpha \mid \alpha \in \Phi(G)\rangle$.

Given the lower bound on p in (5), we can make a similar construction for more general elements of $\mathcal{L}_{\mathbb{Z}}$. To do this, let $\mathcal{L}_{\mathbb{Z}_{(p)}}$ be the $\mathbb{Z}_{(p)}$-span of $\mathcal{B}$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal $(p) = p\mathbb{Z}$, so that $\mathcal{L}_K = \mathcal{L}_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} K$. By [35, Proposition 1.5] we have

$$(\mathrm{ad}(e)^j/j!)(\mathcal{L}_{\mathbb{Z}_{(p)}}) \subseteq \mathcal{L}_{\mathbb{Z}_{(p)}}$$

for all $e \in \sum_{\alpha\in\Phi^+(G)} \mathbb{Z}e_\alpha$ and all $j \geqslant 0$. Then as in the Chevalley construction, for any nonzero element y in $\sum_{\alpha\in\Phi^+(G)} \mathbb{Z}e_\alpha$ or $\sum_{\alpha\in\Phi^+(G)} \mathbb{Z}f_\alpha$, we can produce

$$x_y(\mathbf{x}) = \exp(\mathrm{ad}(\mathbf{x}y)) \in \mathrm{GL}_{\dim G}(\mathbb{Z}_{(p)}[\mathbf{x}]).$$

In particular, by passing to K, we define

$$U_y = \{x_y(\gamma) = \exp(\mathrm{ad}(\gamma y)) \mid \gamma \in K\} \subseteq \mathrm{Aut}(\mathcal{L}_K)^0 = G. \tag{6}$$

We use this general set-up to construct certain A_1-type subgroups of our group G, following [34, 35]. In order to state the main result (Proposition 2.13 below), recall that an ordered triple of elements (e, h, f) chosen from $\mathcal{L}_K$ (or from $\mathcal{L}_{\mathbb{Z}}$) is an $\mathfrak{sl}_2$-*triple* if the elements satisfy the commutation relations between the standard generators of the Lie algebra $\mathfrak{sl}_2$, namely

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

We have the following result (in part (iii), we use the notation $x_y(\gamma)$ from (6)).

PROPOSITION 2.13. *Suppose $p \geqslant h(G)$ and (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathcal{L}_{\mathbb{Z}}$, with $e = \sum_{i=1}^{r} e_i$ and $f \in \sum_{i=1}^{r} \mathbb{Z} f_i$. Then the following hold:*

(i) *U_e and U_f are 1-dimensional subgroups of G.*

(ii) *$A = \langle U_e, U_f \rangle$ is an A_1-type subgroup of G.*

(iii) *$T = \{t(c) \mid c \in K^{\times}\}$ is a maximal torus of $\langle U_e, U_f \rangle$, where*

$$t(c) = x_e(c)x_f(-c^{-1})x_e(c)x_e(-1)x_f(1)x_e(-1),$$

and the map $t : \mathbb{G}_m \to T$ is a morphism of algebraic groups.

(iv) *The action of T on the basis $\{\bar{v} = v \otimes 1 \mid v \in \mathcal{B}\}$ of $\mathcal{L}_K$ is given by*

$$t(c) \cdot \bar{e}_\alpha = c^{\alpha(h)} \bar{e}_\alpha, \quad t(c) \cdot \bar{h}_i = \bar{h}_i$$

for all $\alpha \in \Phi(G)$, $1 \leqslant i \leqslant r$. Moreover, $\alpha_i(h) = 2$ for all $1 \leqslant i \leqslant r$.

(v) *T normalizes U_e and U_f.*

(vi) *U_e contains a regular unipotent element of G.*

Proof. This follows by combining [34, Lemmas 1 and 2] with [35, Lemma 1.2]. □

The following result will play an important role in the proof of Theorem 1.

PROPOSITION 2.14. *Suppose $p \geqslant h(G)$ and (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathcal{L}_{\mathbb{Z}}$, with $e = \sum_{i=1}^{r} e_i$ and $f \in \sum_{i=1}^{r} \mathbb{Z} f_i$. Let W be the 3-dimensional subalgebra of $\mathcal{L}_K$ generated by $\{e, f\}$ and let H be the stabilizer of W in G. Then H is an A_1-type subgroup of G.*

Proof. Let A be the A_1-type subgroup of G constructed in Proposition 2.13(ii). Note that A contains a regular unipotent element and it clearly stabilizes W by construction, so $A \leqslant H$. Let M_1 be a maximal closed positive-dimensional subgroup of G with $H \leqslant M_1$. By the main theorem of [36], A is not contained in a proper parabolic subgroup of G, so Borel–Tits [2, Corollary 3.9] (also see Weisfeiler [37]) implies that M_1 is reductive. By [27, Theorem A], either M_1 is an A_1-type subgroup (and thus $A = H = M_1$), or $G = E_6$ and $M_1 = F_4$. In the latter case, $H \neq M_1$ since M_1 does not stabilize a 3-dimensional subspace of $\mathcal{L}_K$, so let M_2 be a maximal closed positive-dimensional subgroup of M_1 with $H \leqslant M_2$. As above, M_2 is reductive and by applying [27, Theorem A] once again, we conclude that $A = H = M_2$. □

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We would like to be able to use Proposition 2.14 to identify the stabilizers of other $\mathfrak{sl}_2$-subalgebras of $\mathcal{L}_K$. With this aim in mind, we present Proposition 2.15 below. In order to state this result, we need some additional notation.

Suppose we have an $\mathfrak{sl}_2$-triple (e, h, f) as in Proposition 2.13. Let T be the 1-dimensional torus constructed in part (iii) of the proposition. Let $\alpha_0 \in \Phi(G)$ be the highest root and recall that $h(G) = \mathrm{ht}(\alpha_0) + 1$, where $\mathrm{ht} : \Phi(G) \to \mathbb{N}$ is the familiar height function (that is, if $\alpha = \sum_i a_i\alpha_i$ then $\mathrm{ht}(\alpha) = \sum_i a_i$). Then

$$\{2i \mid -\mathrm{ht}(\alpha_0) \leqslant i \leqslant \mathrm{ht}(\alpha_0)\}$$

is the set of weights of T on both $\mathcal{L}_{\mathbb{Z}}$ and $\mathcal{L}_K$. For each T-weight m, write $(\mathcal{L}_{\mathbb{Z}})_m$ for the corresponding T-weight space and similarly for $\mathcal{L}_K$. In both the statement and proof of the following result, we use the notation $\bar{a} = a \otimes 1 \in \mathcal{L}_K$ for $a \in \mathcal{L}_{\mathbb{Z}}$.

PROPOSITION 2.15. *Suppose $p \geqslant h(G)$ and (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathcal{L}_{\mathbb{Z}}$, with $e = \sum_{i=1}^r e_i$ and $f \in \sum_{i=1}^r \mathbb{Z}f_i$. Suppose $y \in (\mathcal{L}_{\mathbb{Z}})_{p-1} \cap C_{\mathcal{L}_{\mathbb{Z}}}(e)$ and $z \in (\mathcal{L}_{\mathbb{Z}})_{p-3}$ are chosen so that*

(i) $[y, z] = 0$ *in $\mathcal{L}_{\mathbb{Z}}$; and*

(ii) $(\bar{e}, \bar{h} + \gamma\bar{y}, \bar{f} + \gamma\bar{z})$ *is an $\mathfrak{sl}_2$-triple in $\mathcal{L}_K$ for some $\gamma \in K$.*

Then there exists $g \in C_G(\bar{e})$ such that $g \cdot \bar{h} = \bar{h} + \gamma\bar{y}$ and $g \cdot \bar{f} = \bar{f} + \gamma\bar{z}$ in $\mathcal{L}_K$. Moreover, the stabilizer in G of the subalgebra W of $\mathcal{L}_K$ generated by $\{\bar{e}, \bar{f} + \gamma\bar{z}\}$ is an A_1-type subgroup.

Proof. First observe that $y \in \sum_{\alpha \in \Phi^+(G)} \mathbb{Z}e_\alpha$ since $y \in (\mathcal{L}_{\mathbb{Z}})_{p-1}$, so we can take $g = x_y(\gamma) \in G$ as in (6). Note that $g \in C_G(\bar{e})$ since $y \in C_{\mathcal{L}_{\mathbb{Z}}}(e)$. Now y is an eigenvector for $\mathrm{ad}(h)$ (since y is a T-weight vector), so $[y, [y, h]] = 0$ and thus

$$g \cdot \bar{h} = \bar{h} + \gamma[\bar{y}, \bar{h}] = \bar{h} - \gamma[\bar{h}, \bar{y}] = \bar{h} - \gamma(p-1)\bar{y} = \bar{h} + \gamma\bar{y}.$$

The maximum T-weight in $\mathcal{L}_{\mathbb{Z}}$ is $2\mathrm{ht}(\alpha_0)$, which is at most $2(p-1)$ since $p \geqslant h(G) > \mathrm{ht}(\alpha_0)$, so $\mathrm{ad}(y)^i(f) \in (\mathcal{L}_{\mathbb{Z}})_{i(p-1)-2} = 0$ for all $i \geqslant 3$ and thus

$$g \cdot \bar{f} = \bar{f} + \gamma[\bar{y}, \bar{f}] + \tfrac{1}{2}\gamma^2[\bar{y}, [\bar{y}, \bar{f}]].$$

In addition, since $[y, z] = 0$ and $z \in (\mathcal{L}_{\mathbb{Z}})_{p-3}$, we have

$$[h + y, f + z] = -2f + (p-3)z + [y, f]. \tag{7}$$

The $\mathfrak{sl}_2$ commutation relations imply that $[\bar{h} + \bar{y}, \bar{f} + \bar{z}] = -2(\bar{f} + \bar{z})$, which is equal to $-2\bar{f} - 3\bar{z} + [\bar{y}, \bar{f}]$ by (7). Therefore, $[\bar{y}, \bar{f}] = \bar{z}$ and it follows that $[\bar{y}, [\bar{y}, \bar{f}]] = 0$. We conclude that $g \cdot \bar{h} = \bar{h} + \gamma\bar{y}$ and $g \cdot \bar{f} = \bar{f} + \gamma\bar{z}$ in $\mathcal{L}_K$, as required. The final statement concerning the stabilizer of W follows immediately from Proposition 2.14. □

2.4. Exponentiation. In this section, we turn to a different notion of 'exponentiation', following Seitz [28]. As before, let G be a simple exceptional algebraic group of adjoint type over an algebraically closed field K of characteristic $p > 0$ and let r and h denote the rank and Coxeter number of G, respectively. Let $U = \langle U_\alpha \mid \alpha \in \Phi^+(G)\rangle$ be the unipotent radical of a fixed Borel subgroup B of G corresponding to our choice of base $\Pi(G) = \{\alpha_1, \ldots, \alpha_r\}$, where the root subgroup U_α is defined as in (6). As explained in [28, Section 5], we may view $\mathrm{Lie}(U)$ as an algebraic group via the Hausdorff formula. Set $V = \mathrm{Lie}(G)$.

We start by recalling [28, Proposition 5.3].

PROPOSITION 2.16. *Suppose $p \geqslant h$. Then there exists a unique isomorphism of algebraic groups*

$$\theta : \mathrm{Lie}(U) \to U \tag{8}$$

whose tangent map is the identity and which is B-equivariant; that is, $\theta(b \cdot n) = b\theta(n)b^{-1}$ for all $n \in \mathrm{Lie}(U)$, $b \in B$.

Suppose G contains a regular unipotent element x of order p, so $p \geqslant h$ and we are in a position to use Proposition 2.16 to study the structure of $C_G(x)$. Replacing x by a suitable conjugate, we may assume that

$$x = x_e(1) = \exp(\mathrm{ad}(e)),$$

where $e = \sum_{i=1}^r e_i$. As in Proposition 2.13, let A be an A_1-type subgroup of G containing x, and let $T = \{t(c) \mid c \in K^\times\}$ be the given maximal torus of A. Without loss of generality, we may assume that T is contained in the Borel subgroup B defined above. From the description of the action of A on $V = \mathrm{Lie}(G)$ in the proof of Proposition 2.12, it follows that $t(c)$ acts on the 1-eigenspace $C_V(x) = \mathrm{Lie}(C_G(x))$ as

$$\mathrm{diag}(c^{d_1}, \ldots, c^{d_r}), \tag{9}$$

where the d_i are recorded in Table 3 (we label the d_i so that they form a decreasing sequence).

PROPOSITION 2.17. *Let $x = x_e(1) \in G$ be a regular unipotent element of order p, where $e = \sum_{i=1}^r e_i$, and let $T = \{t(c) \mid c \in K^\times\}$ be the torus constructed in Proposition 2.13. Then there exist connected 1-dimensional unipotent subgroups $X_i = \{x_i(\gamma) \mid \gamma \in K\}$ such that the following hold:*

(i) *$C_G(x) = \langle X_i \mid 1 \leqslant i \leqslant r\rangle$. In particular, each $z \in C_G(x)$ can be written as a commuting product of the form $z = \prod_{i=1}^r x_i(\gamma_i)$ for some $\gamma_i \in K$.*

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Table 3. The integers $d_1, \dots, d_r$ in (9).

G	d_i
G_2	10, 2
F_4	22, 14, 10, 2
E_6	22, 16, 14, 10, 8, 2
E_7	34, 26, 22, 18, 14, 10, 2
E_8	58, 46, 38, 34, 26, 22, 14, 2

(ii) *We have*

$$t(c)x_i(\gamma)t(c)^{-1} = x_i(c^{d_i}\gamma) \tag{10}$$

for all $c \in K^\times$, $\gamma \in K$, $1 \leqslant i \leqslant r$.

Proof. First note that $p \geqslant h$ since x has order p. As above, let

$$U = \langle U_\alpha \mid \alpha \in \Phi^+(G)\rangle$$

be the unipotent radical of a Borel subgroup B of G and note that $x \in U$ and $T \leqslant B$. Moreover, we have $C_G(x) \leqslant U$ and thus $C_V(x) = \mathrm{Lie}(C_G(x)) \subseteq \mathrm{Lie}(U)$. Choose $v_r \in \mathrm{Lie}(U)$ such that $\theta(v_r) = x$, where θ is the map in Proposition 2.16. Extend to a basis $\{v_1, \dots, v_r\}$ of the 1-eigenspace $C_V(x)$, where $t(c) \cdot v_i = c^{d_i} v_i$ for each i, and construct the corresponding connected 1-dimensional unipotent subgroups

$$X_i = \{x_i(\gamma) = \theta(\gamma v_i) \mid \gamma \in K\} \leqslant G.$$

Recall that $C_G(x)$ is abelian, so $C_V(x) = \mathrm{Lie}(C_G(x))$ is an abelian subalgebra and the proof of [**28**, Proposition 5.4] implies that each X_i is contained in $C_G(x)$. Therefore, $H = \langle X_i \mid 1 \leqslant i \leqslant r\rangle$ is a closed connected unipotent subgroup of $C_G(x)$. Moreover, $v_i \in \mathrm{Lie}(X_i)$ for each i, so $\dim H \geqslant r$ and thus $H = C_G(x)$ (note that $C_G(x)$ is connected since G is adjoint). Part (i) now follows since $C_G(x)$ is abelian. Finally, part (ii) follows from the B-equivariance of θ (see Proposition 2.16). □

PROPOSITION 2.18. *Let U be the unipotent radical of a Borel subgroup B of G, let W be a proper nonzero subalgebra of* $\mathrm{Lie}(U)$ *and let H be the stabilizer of W in G. Assume H contains a regular unipotent element of G of order p. Then either*

(i) *H is contained in a proper parabolic subgroup of G; or*

(ii) *H is contained in an A_1-type subgroup of G.*

Proof. Since $p \geqslant h$, we can consider the isomorphism $\theta : \mathrm{Lie}(U) \to U$ in (8). Let $Z = Z(W)$ be the centre of W, which is a nonzero abelian subalgebra of W stabilized by H. We claim that $\theta(Z) \leqslant H$. To see this, let $z \in Z$, $w \in W$ and note that $\theta(z)$ and $\theta(w)$ commute since $[z, w] = 0$ in $\mathrm{Lie}(U)$ (see the proof of [28, Proposition 5.4]). The B-equivariance of θ implies that

$$\theta(\theta(z) \cdot w) = \theta(z)\theta(w)\theta(z)^{-1} = \theta(w),$$

so $\theta(z) \cdot w = w$ and the claim follows. Therefore, H is a positive-dimensional subgroup of G containing a regular unipotent element.

To complete the argument, we proceed as in the proof of Proposition 2.14, using Borel–Tits [2, Corollary 3.9]. Let us assume H is not contained in a proper parabolic subgroup of G. Then $H \leqslant M_1$, where M_1 is a maximal closed reductive positive-dimensional subgroup of G. By the main theorem of [27], either M_1 is an A_1-type subgroup, or $G = E_6$ and $M_1 = F_4$, so we may assume that we are in the latter situation. Suppose $H = M_1$. Since

$$V|_{M_1} = \mathrm{Lie}(M_1) \oplus V_{26},$$

where V_{26} is the minimal module for M_1, it follows that $W = V_{26}$ is the only possibility. But V_{26} must contain nonzero elements in the Lie algebra of a maximal torus of G (just by comparing dimensions) and this is a contradiction. Therefore H is a proper subgroup of M_1 and thus $H \leqslant M_2$ for some maximal closed reductive subgroup M_2 of M_1. By a further application of [27] we conclude that H is contained in an A_1-type subgroup of G. □

2.5. Methods. In this section, we discuss the proof of Theorem 1, highlighting the main steps and ideas.

Let G be a simple exceptional algebraic group of adjoint type defined over an algebraically closed field K of characteristic $p > 0$. Let r be the rank of G and let $V = \mathrm{Lie}(G)$ be the adjoint module. Suppose $x \in X = \mathrm{PSL}_2(p) < G$ is a regular unipotent element of G, so $p \geqslant h$ where h is the Coxeter number of G. The embedding of X in G corresponds to an abstract homomorphism $\varphi : \mathrm{SL}_2(p) \to G$ with kernel $Z = Z(\mathrm{SL}_2(p))$ and image X.

As before, let $\mathcal{L}_{\mathbb{C}}$ be a simple Lie algebra over $\mathbb{C}$ of type $\Phi(G)$ and fix a Chevalley basis

$$\mathcal{B} = \{e_\alpha, f_\alpha, h_\gamma \mid \alpha \in \Phi^+(G),\ \gamma \in \Pi(G)\}. \tag{11}$$

Since $p \geqslant h$, we can view $\mathcal{B}$ as a basis for V, where e_α, f_α are in the appropriate root spaces with respect to the Cartan subalgebra spanned by the h_γ. It will be convenient to write $z_i = z_{\alpha_i}$ for $z \in \{e, f, h\}$ and $\Pi(G) = \{\alpha_1, \ldots, \alpha_r\}$.

Set $e = \sum_{i=1}^{r} e_i$ and let (e, h, f) be an $\mathfrak{sl}_2$-triple as in Proposition 2.13. Let A be the corresponding A_1-type subgroup of G constructed in Proposition 2.13, with maximal torus $T = \{t(c) \mid c \in K^\times\}$ and associated morphism $t : \mathbb{G}_m \to T$. By replacing X by a suitable G-conjugate, we may assume that $x = \exp(\mathrm{ad}(e)) \in A$. Let $v : \mathbb{G}_a \to A$ be a morphism of algebraic groups such that

$$t(c)v(\gamma)t(c)^{-1} = v(c^2\gamma)$$

for all $c \in K^\times$, $\gamma \in K$. We may assume $v : \mathbb{G}_a \to \mathrm{im}(v)$ is an isomorphism of algebraic groups.

Consider the elements

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \tag{12}$$

in $\mathrm{SL}_2(p)$, where $\mathbb{F}_p^\times = \langle \xi \rangle = \{1, \ldots, p-1\}$. Without loss of generality, we may assume that $x = \varphi(u) = v(1)$ so $\varphi(sus^{-1}) = \varphi(u^m) = v(m)$ with $m = \xi^2$. Then $t(\xi)xt(\xi)^{-1} = \varphi(s)x\varphi(s)^{-1}$ and thus $\varphi(s) = t(\xi)z$ for some $z \in C_G(x)$. Set $\bar{s} = \varphi(s) \in X$.

LEMMA 2.19. *There exists a $C_G(x)$-conjugate of X containing x and $t(\xi)$.*

Proof. As noted above, we have $\bar{s} = t(\xi)z$ for some $z \in C_G(x)$. By Proposition 2.17 there are scalars $\gamma_i \in K$ such that $z = \prod_{i=1}^{r} x_i(\gamma_i)$. Let us consider a general element $y = \prod_{i=1}^{r} x_i(\delta_i) \in C_G(x)$. In view of (10), we get

$$\begin{aligned} y\bar{s}y^{-1} &= \prod_i x_i(\delta_i)t(\xi)\prod_i x_i(\gamma_i - \delta_i)t(\xi)^{-1}t(\xi) \\ &= \prod_i x_i(\delta_i)\prod_i x_i(\xi^{d_i}(\gamma_i - \delta_i))t(\xi) \\ &= \prod_i x_i(\delta_i + \xi^{d_i}(\gamma_i - \delta_i))t(\xi) \end{aligned}$$

where the d_i are the integers appearing in Table 3.

Since $p \geqslant h$ and $\mathbb{F}_p^\times = \langle \xi \rangle$, it is easy to see that there is at most one i such that $\xi^{d_i} = 1$. If there is no such i then we can set $\delta_i = \xi^{d_i}\gamma_i/(\xi^{d_i} - 1)$ for all i, so $y\bar{s}y^{-1} = t(\xi)$ and X^y is the desired conjugate of X. Finally, suppose $\xi^{d_j} = 1$ and $\gamma_j \neq 0$ for some j. By defining δ_i as above for all $i \neq j$, we get $y\bar{s}y^{-1} = t(\xi)x_j(\gamma_j)$ with $[t(\xi), x_j(\gamma_j)] = 1$. But this implies that $y\bar{s}y^{-1}$ is a nonsemisimple element, which contradicts the semisimplicity of $\bar{s}$. □

In view of the lemma, we may assume that X contains $t(\xi)$, which corresponds to a diagonalizable element $s \in \mathrm{SL}_2(p)$ with eigenvalues ξ and ξ^{-1}. Since

$t(\xi) \in T$, we can use the known action of A on V (see the proof of Proposition 2.12) to determine the eigenvectors and eigenspaces of s on V. For example,

$$\{\xi^{d_1}, \xi^{d_2}, \ldots, \xi^{d_r}\} \tag{13}$$

is the collection of eigenvalues of s on $C_V(x)$, where the d_i are given in Table 3. We set $\bar{s} = t(\xi) = sZ \in X$, where Z is the centre of $\mathrm{SL}_2(p)$. Note that A contains the Borel subgroup $\langle \bar{s}, x\rangle$ of X.

The proof of Theorem 1 has three main steps, which we now describe.

Step 1: Elimination. Our initial aim is to reduce to the situation where the action of X on V is compatible with the decomposition of V as an A-module given in Table 2. In almost all cases, we are able to achieve this goal. To do this, we consider the possible decompositions of $V|_X$ as a direct sum of indecomposable KX-modules, using the description of these modules given in Section 2.1, with the aim of eliminating all but one possibility.

First we use the fact that the decomposition of $V|_X$ has to be compatible with the Jordan form of x on V (this can be read off from the relevant tables in [**14**]). In addition, it must be compatible with the known eigenvalues of s on V (as noted above, these are just the eigenvalues of $t(\xi)$ on V, which we can compute from the known action of A on V). Note that if M is an indecomposable summand of $V|_X$ then the restriction of M to $\langle s\rangle$ is completely reducible, so we just need to identify the KX-composition factors of M in order to compute the eigenvalues of s on this summand. Often it is sufficient to compare the eigenvalues of s on $C_V(x)$ with the expected eigenvalues in (13), and we can also use our earlier calculations on the traces of elements of order 2 and 3 to obtain further restrictions on $V|_X$ (see Section 2.2). With this approach in mind, the following lemma will be useful.

LEMMA 2.20. *Let M be an indecomposable KX-module of the form $L_X(i)$, U or $W(j)$, where $i \in \{0, 2, \ldots, p-1\}$ and $j \in \{2, 4, \ldots, p-3\}$. Then the eigenvalues of s on $C_M(x)$ are ξ^i, 1 and $\{\xi^j, \xi^{-j}\}$, respectively.*

Proof. First recall that x has Jordan form $[J_{i+1}]$, $[J_p]$ and $[J_p^2]$ on $L_X(i)$, U and $W(j)$, respectively. The fixed point of x on the simple module $L_X(i)$ has highest weight i, so the result is clear in this case. Similarly, $\mathrm{soc}(U) = L_X(0)$ so s has eigenvalue ξ^0 on $C_U(x)$. Finally, suppose $M = W(j)$. The highest weight of $\mathrm{soc}(M) = L_X(j)$ is j, so ξ^j is one of the eigenvalues of s on $C_M(x)$. To determine the second eigenvalue, it is helpful to view $W(j)$ as the restriction to X of the tilting module $T(2p-2-j)$ for the ambient algebraic group of type A_1 (see [**28**, Lemma 2.3]). On the latter module, x has a fixed point of weight $2p-2-j$ (the high weight), so the eigenvalue of s is $\xi^{2p-2-j} = \xi^{-j}$ as required. □

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Let us illustrate how Step 1 is carried out in the specific case $(G, p) = (E_8, 31)$.

EXAMPLE 2.21. Suppose $G = E_8$ and $p = 31$, so x has Jordan form $[J_{31}^8]$ on V (see [14, Table 9]). In particular, $V|_X$ is projective and thus every indecomposable summand of $V|_X$ is also projective. In terms of the notation introduced in Section 2.1, the possibilities for $V|_X$ are as follows

$$\begin{cases} M_1 \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_5 \oplus M_6 \oplus M_7 \oplus M_8 \\ W(a_1) \oplus M_1 \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_5 \oplus M_6 \\ W(a_1) \oplus W(a_2) \oplus M_1 \oplus M_2 \oplus M_3 \oplus M_4 \\ W(a_1) \oplus W(a_2) \oplus W(a_3) \oplus M_1 \oplus M_2 \\ W(a_1) \oplus W(a_2) \oplus W(a_3) \oplus W(a_4) \end{cases}$$

where $M_i \in \{L_X(30), U\}$ and $a_i \in \{2, 4, \dots, 28\}$. If $V|_X$ has an M_i summand, then s has an eigenvalue $\xi^{30} = \xi^0$ on $C_V(x)$, which contradicts (13), so we must have

$$V|_X = W(a_1) \oplus W(a_2) \oplus W(a_3) \oplus W(a_4).$$

Since $\xi^{30} = 1$ and s has eigenvalues ξ^i, ξ^{-i} on $C_{W(i)}(x)$ (see Lemma 2.20), it follows that

$$\{\xi^{a_1}, \xi^{-a_1}, \xi^{a_2}, \xi^{-a_2}, \xi^{a_3}, \xi^{-a_3}, \xi^{a_4}, \xi^{-a_4}\} = \{\xi^{28}, \xi^{16}, \xi^{8}, \xi^{4}, \xi^{26}, \xi^{22}, \xi^{14}, \xi^{2}\}$$

(see Table 3). Up to a reordering of summands, this immediately implies that

$$a_1 \in \{2, 28\}, \quad a_2 \in \{4, 26\}, \quad a_3 \in \{8, 22\}, \quad a_4 \in \{14, 16\}.$$

Let $y \in X$ be an involution. Since $\mathrm{tr}(W(i), y) = \pm 2$ and $\mathrm{tr}(V, y) \in \{-8, 24\}$ (see Lemma 2.7 and Proposition 2.8), it follows that $\mathrm{tr}(W(a_i), y) = -2$ for all i, whence $a_i \equiv 2 \pmod 4$ and thus $(a_1, a_2, a_3, a_4) = (2, 26, 22, 14)$ is the only possibility. We have now reduced to the case where the decomposition of $V|_X$ is compatible with $V|_A$ (see Table 2).

Step 2: Extension. Next observe that if $V|_X$ has the decomposition given in Table 2 then the socle of $V|_X$ has a simple summand $W = L_X(2)$. To complete the argument, we aim to show that W is an $\mathfrak{sl}_2$-subalgebra of V and its stabilizer in G is an A_1-type subgroup. We can do this in almost every case; the exceptions are the two special cases appearing in the statement of Theorem 1.

Let $\{w_2, w_0, w_{-2}\}$ be a basis for W, where w_i is an eigenvector for s with eigenvalue ξ^i. We may assume that the action of x on W is given by the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

with respect to this basis (that is, $x(w_0) = w_0 + w_2$, and so forth). If we define $\bar{s} = sZ \in X$ as above then $\langle \bar{s}, x\rangle$ is a Borel subgroup of X and we can consider the opposite Borel subgroup $\langle \bar{s}, x'\rangle$ of X, where $x' \in X$ is also a regular unipotent element of order p. With respect to the above basis, we may assume that x' acts on W via the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \tag{15}$$

If all these conditions are satisfied, then we say that $\{w_2, w_0, w_{-2}\}$ is a *standard basis* for W.

With the aid of MAGMA [3] we can construct a $\dim G \times \dim G$ matrix to represent the action of x on V with respect to our Chevalley basis $\mathcal{B}$. Let us illustrate this with an example.

EXAMPLE 2.22. For $(G, p) = (G_2, 7)$ we proceed as follows in MAGMA:

```
G:=GroupOfLieType("G2",Rationals());
L:=LieAlgebra(G);
e,f,h:=ChevalleyBasis(L);
I1:=[1..6]; I2:=[1..2];

B:=[f[7-i] : i in I1] cat [e[i]*f[i] : i in I2] cat [e[i] : i in I1];
L:=ChangeBasis(L,B);
B:=Basis(L);
e:=[B[8+i] : i in I1]; f:=[B[7-i] : i in I1]; h:=[B[6+i] : i in I2];

ad:=AdjointRepresentation(L);
y:=ad(e[1]+e[2]);
A:=MatrixAlgebra(Rationals(),14);
x:=Identity(A);  y:=A!y;
for i in [1..10] do x:=x+(1/Factorial(i))*y^i; end for;
B:=MatrixAlgebra(GF(7),14);
x:=B!x;
```

In this example, we are working with a Chevalley basis

$$\mathcal{B} = \{e[i], f[i], h[j] : i \in \{1, \dots, 6\}, j \in \{1, 2\}\}$$

where $e[i]$ spans the root space of the ith positive root, $f[i]$ is in the root space of the corresponding negative root, and $h[j] = [e[j], f[j]]$ for $j = 1, 2$, with respect to the following ordering

$$\alpha_1, \quad \alpha_2, \quad \alpha_1 + \alpha_2, \quad 2\alpha_1 + \alpha_2, \quad 3\alpha_1 + \alpha_2, \quad 3\alpha_1 + 2\alpha_2$$

of positive roots (note that this agrees with the ordering given by the MAGMA command PositiveRoots(G)). We adopt an analogous set-up in all cases.

Moreover, we can use Proposition 2.13(iv) to compute the eigenvalues and eigenvectors of $t(\xi)$ (and thus s) on V in terms of $\mathcal{B}$. For $i \in \mathbb{Z}$, it will be convenient to write E_i for the ξ^i-eigenspace of s on V (so $w_i \in E_i$ for the elements in a standard basis of W).

Next we identify a basis $\{v_1, \ldots, v_r\}$ of the 1-eigenspace $C_V(x) = \ker(x - 1)$ in terms of $\mathcal{B}$, where $v_i \in E_{d_i}$ (see Table 3). Since $w_2 \in C_V(x) \cap E_2$ we can write

$$w_2 = \sum_{i=1}^{r} a_i v_i$$

for some $a_i \in K$, where $a_i \neq 0$ only if $\xi^{d_i} = \xi^2$. Similarly,

$$w_0 \in (\ker((x-1)^2) \setminus \ker(x-1)) \cap E_0,$$
$$w_{-2} \in (\ker((x-1)^3) \setminus \ker((x-1)^2)) \cap E_{-2}.$$

Using MAGMA it is straightforward to compute bases for the relevant kernels; these computations can be done by hand, but it is much quicker and more efficient to use a machine.

Given these bases, say $\mathcal{B}_2$, $\mathcal{B}_0$ and $\mathcal{B}_{-2}$, we can write

$$w_2 = \sum_{v \in \mathcal{B}_2} a_v v, \quad w_0 = \sum_{v \in \mathcal{B}_0} b_v v, \quad w_{-2} = \sum_{v \in \mathcal{B}_{-2}} c_v v$$

for $a_v, b_v, c_v \in K$ and our goal is to determine these scalars. To do this, we can use the specified actions of x and x' on W to derive relations between the coefficients. Further relations can be determined by exploiting the fact that x and x' are regular unipotent elements. For example, we observe that $x' \cdot w_{-2} = w_{-2}$ and

$$x' \cdot [w_{-2}, w_0] = [w_{-2}, w_0 + w_{-2}] = [w_{-2}, w_0],$$

where $[\,,\,]$ is the Lie bracket on V, so $w_{-2}, [w_{-2}, w_0] \in C_V(x')$. Since the regularity of x' implies that $C_G(x')$ is abelian, it follows that $C_V(x') = \mathrm{Lie}(C_G(x'))$ is an abelian subalgebra of V (for the latter equality, recall that $p \geqslant h$) and thus

$$[w_{-2}, [w_{-2}, w_0]] = 0. \tag{16}$$

Proceeding in this way, our goal is to reduce to the case where $W = \langle w_2, w_0, w_{-2} \rangle$ is an $\mathfrak{sl}_2$-subalgebra, with $w_2 = \sum_{i=1}^{r} e_i$ and $w_{-2} \in \sum_{i=1}^{r} \mathbb{Z} f_i$. Moreover, we want to find integers λ, μ such that $(w_2, \lambda w_0, \mu w_{-2})$ is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ (that is, an $\mathfrak{sl}_2$-triple of $\mathcal{L}_{\mathbb{Z}}$ in the notation of Section 2.3). Indeed, if we can do this, then Proposition 2.14 implies that the stabilizer of W in G is an A_1-type subgroup and so we are in the generic situation described in part (i) of Theorem 1. In a few cases,

Table 4. The exceptional cases $(G, p, V|_X)$ in Theorem 2.23.

G	p	$V\|_X$
F_4	13	$W(10) \oplus W(2)$
E_6	13	$W(10) \oplus W(8) \oplus W(2)$
		$W(10) \oplus W(4) \oplus W(2)$
		$W(10)^2 \oplus W(4)$
E_7	19	$W(8) \oplus W(4) \oplus W(2) \oplus U$
		$W(16) \oplus W(10) \oplus W(4) \oplus U$
		$W(16) \oplus W(14) \oplus W(8) \oplus U$
E_8	37	$W(34) \oplus W(26) \oplus W(14) \oplus L_X(22) \oplus L_X(2)$

we are unable to force $w_{-2} \in \sum_{i=1}^{r} \mathbb{Z} f_i$, but by appealing to Proposition 2.15 we can still show that the same conclusion holds.

In the remaining cases where W is not an $\mathfrak{sl}_2$-subalgebra, or the action of X on V is incompatible with $V|_A$, we show that X stabilizes a nonzero subalgebra of $\langle e_\alpha \mid \alpha \in \Phi^+(G)\rangle$. More precisely, we establish the following result, which reduces the proof of Theorem 1 to the handful of cases appearing in Table 4 (see Remark 1(a) for the conjugacy statement in part (i)).

THEOREM 2.23 (Reduction Theorem). *Let G be a simple exceptional algebraic group of adjoint type over an algebraically closed field of characteristic $p > 0$. Let $X = \mathrm{PSL}_2(p)$ be a subgroup of G containing a regular unipotent element of G and set $V = \mathrm{Lie}(G)$ with Chevalley basis as in (11). Then one of the following holds:*

(i) *X is contained in an A_1-type subgroup of G and X is uniquely determined up to G-conjugacy;*

(ii) *X stabilizes a nonzero subalgebra of $\langle e_\alpha \mid \alpha \in \Phi^+(G)\rangle$ and $(G, p, V|_X)$ is one of the cases in Table 4.*

We prove the Reduction Theorem in Sections 3–7, considering each possibility for G in turn.

Step 3: Parabolic analysis. The final step in our proof of Theorem 1 concerns the cases arising in Theorem 2.23(ii), given in Table 4. In view of Proposition 2.18, we may assume that X is contained in a proper parabolic subgroup $P = QL$ of G and we proceed by studying the possible embeddings of X in such a subgroup. Take P to be a minimal such parabolic and let $\pi : P \to P/Q$ be the quotient map. By identifying L with P/Q, we may view $\pi(X)$ as a subgroup of L'. Now we can show that $\pi(X) < H$, where H is an A_1-type subgroup of L' containing a regular unipotent element of L' (namely, $\pi(x)$), so we can use [15, Tables 1–5] to study the composition factors of $V|_H$ for each (G, L'). In turn, this imposes restrictions on the decomposition of $V|_X$. But the possibilities for $V|_X$ are listed in Table 4 and in this way we arrive at the two special cases in the statement of Theorem 1. See Section 8 for the details. (Notice that we adopt a similar approach in the proof of Theorem 2 below.)

EXAMPLE 2.24. To illustrate some of the above ideas, let us explain how we handle the case $(G, p) = (E_8, 31)$. Recall that in Example 2.21 we reduced to the situation where

$$V|_X = W(2) \oplus W(26) \oplus W(22) \oplus W(14),$$

which is compatible with the decomposition of $V|_A$. By following the approach in Example 2.22, we use MAGMA to determine the action of x on V in terms of a Chevalley basis $\mathcal{B}$.

Let $W = \mathrm{soc}(W(2)) = L_X(2)$ and let $\{w_2, w_0, w_{-2}\}$ be a standard basis of W as above. First consider $w_2 \in C_V(x)$. Now $C_V(x) \cap E_2$ is 1-dimensional (indeed, by inspecting Table 3 we see that there is a unique d_i which is congruent to 2 modulo 30), spanned by the sum of the simple root vectors, so we must have

$$w_2 = a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8)$$

for some nonzero scalar $a_1 \in K$. Similarly, w_0 is contained in the 1-dimensional space $\ker((x-1)^2) \cap E_0$ and by considering the relation $x(w_0) = w_0 + w_2$ we take

$$w_0 = a_2(h_1 + 19h_2 + 4h_3 + 9h_4 + 28h_5 + 18h_6 + 10h_7 + 4h_8).$$

Finally, w_{-2} is in the 2-dimensional space $\ker((x-1)^3) \cap E_{-2}$ (note that $\xi^{-2} = \xi^{58}$ since $p = 31$) and it follows that

$$w_{-2} = a_3(8f_1 + 28f_2 + f_3 + 10f_4 + 7f_5 + 20f_6 + 18f_7 + f_8) + a_4 e_{\alpha_0}$$

where $\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$ is the highest root. Note that $a_3 \neq 0$ since $w_{-2} \in \ker((x-1)^3) \setminus \ker((x-1)^2)$ and $e_{\alpha_0} \in C_V(x)$.

By considering the action of x on W (see (14)) we quickly deduce that $a_2 = 16a_1$ and $a_3 = 4a_1$. Finally, one checks that the condition in (16) yields $a_4 = 0$, so setting $a_1 = 1$ we have

$$w_2 = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8$$

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$$w_0 = 16(h_1 + 19h_2 + 4h_3 + 9h_4 + 28h_5 + 18h_6 + 10h_7 + 4h_8)$$
$$w_{-2} = 4(8f_1 + 28f_2 + f_3 + 10f_4 + 7f_5 + 20f_6 + 18f_7 + f_8)$$

and it is easy to see that w_2, w_0 and w_{-2} satisfy the relations

$$[w_2, w_{-2}] = 2w_0, \quad [w_2, w_0] = w_2, \quad [w_0, w_{-2}] = w_{-2}, \tag{17}$$

and thus $W = \langle w_2, w_0, w_{-2}\rangle$ is an $\mathfrak{sl}_2$-subalgebra. If we set

$$w_2' = w_2, \quad w_0' = -2w_0, \quad w_{-2}' = -w_{-2}, \tag{18}$$

then (w_2', w_0', w_{-2}') is an $\mathfrak{sl}_2$-triple. Moreover, working mod p, we have

$$w_{-2}' = 92f_1 + 136f_2 + 182f_3 + 270f_4 + 220f_5 + 168f_6 + 114f_7 + 58f_8$$

and thus (w_2', w_0', w_{-2}') is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ (see the proof of [**35**, Proposition 2.4]). Since X stabilizes W, it is contained in an A_1-type subgroup of G by Proposition 2.14. This completes the proof of Theorem 1 for $G = E_8$ with $p = 31$.

We close this section by presenting a proof of Theorem 2.

Proof of Theorem 2. Let $V = \mathrm{Lie}(G)$ be the adjoint module for G. Seeking a contradiction, suppose $X < P$, where $P = QL$ is a proper parabolic subgroup of G with unipotent radical Q and Levi factor L. We may as well assume that P is minimal with respect to the containment of X. In particular, if $\pi : P \to P/Q$ is the quotient map and we identify L with P/Q, then $\pi(X)$ is not contained in a proper parabolic subgroup of L'. Now $\pi(x)$ is a regular unipotent element of L' (see [**36**, Lemma 2.6]). Writing $L' = L_1 \cdots L_t$, where each L_i is a simple factor, let $\pi_i : L' \to L_i$ be the naturally defined projection map. Then $\pi_i(\pi(X)) < L_i$ contains a regular unipotent element of L_i and does not lie in a proper parabolic subgroup of L_i.

If L_i is of classical type, we apply the main theorem of [**29**] to see that $\pi_i(\pi(X))$ is contained in an A_1-type subgroup of L_i. On the other hand, if L_i is of exceptional type, then G is of type E_n and L_i is of type E_m for $m < n$. In this case, we apply Theorem 2.23 to conclude once again that $\pi_i(\pi(X))$ is contained in an A_1-type subgroup of L_i for all relevant values of p. In particular, in all cases we deduce that $\pi(X)$ lies in an A_1-type subgroup H of L'.

Now the KH-composition factors of $V|_H$ can be read off from the information in [**15**, Tables 1–5] and we can use this to determine the KX-composition factors of $V|_X$ (to do this, note that we may set all $q_i = 1$ in terms of the notation in [**15**, Tables 1–5]). Indeed, each composition factor of $V|_P$ is an irreducible KL'-module (the unipotent radical Q acts trivially on the KP-composition factors of $V|_P$), so the decompositions of $V|_X$ and $V|_H$ have to be compatible. But the decomposition of $V|_X$ is given in Table 2 and in this way we reach a contradiction.

To see this, first observe that X has at least one trivial composition factor on V, coming from $Z(L)$. By inspecting Table 2, this immediately implies that

$$(G, p) \in \{(F_4, 13), (E_6, 13), (E_8, 37)\}.$$

Suppose $(G, p) = (E_8, 37)$. From Table 2, the KX-composition factors of $V|_X$ are as follows:

$$\begin{gathered} L_X(34)^2, \quad L_X(26)^2, \quad L_X(22)^2, \quad L_X(20), \quad L_X(14)^2, \\ L_X(10), \quad L_X(8), \quad L_X(2)^2, \quad L_X(0). \end{gathered} \tag{19}$$

By inspecting [15, Table 5], using the fact that $V|_X$ has a unique trivial composition factor, we deduce that $L' = A_4A_2A_1$, A_4A_3 or D_5A_2. However, in each of these cases we see that $V|_X$ has an $L_X(6)$ composition factor, which is incompatible with (19). The other two possibilities for (G, p) can be eliminated in a similar fashion. For example, if $(G, p) = (E_6, 13)$ then the composition factors of $V|_X$ are

$$L_X(10)^3, \quad L_X(8)^3, \quad L_X(4), \quad L_X(2)^4, \quad L_X(0).$$

By inspecting [15, Table 3], just considering trivial composition factors, we deduce that $L' = A_2^2A_1$, A_4A_1 or D_5, but in each case we find that $V|_X$ has two or more $L_X(4)$ factors. This is a contradiction. □

As mentioned above, the proof of Theorem 2.23 will be given in Sections 3–7, where we carry out Steps 1 and 2 (elimination and extension) for each group in turn. We handle Step 3 in Section 8, thus completing the proof of Theorem 1.

3. The case $G = G_2$

We begin the proof of Theorem 1 by handling the case $G = G_2$. As noted in Remark 1(c), the result in this case can be deduced from the proof of [30, Lemma 3.1] (it also follows from Kleidman's classification of the maximal subgroups of $G_2(p)$ in [13]).

THEOREM 3.1. *Let G be a simple algebraic group of type G_2 over an algebraically closed field of characteristic $p > 0$. Let $X = \mathrm{PSL}_2(p)$ be a subgroup of G containing a regular unipotent element x of G. Then X is contained in an A_1-type subgroup of G.*

Proof. The Coxeter number of G is 7, so we have $p \geqslant 7$. Let $V = \mathrm{Lie}(G)$ be the adjoint module for G and fix a Chevalley basis for V as in (11). We use the notation introduced in Section 2.5. In particular, $\langle \bar{s}, x\rangle$ is a Borel subgroup of X, where $\bar{s} = t(\xi) = sZ$ and

$$\{\xi^{10}, \xi^2\} \tag{20}$$

are the eigenvalues of $s \in \mathrm{SL}_2(p)$ on $C_V(x)$, where $\mathbb{F}_p^\times = \langle \xi \rangle$. Let E_i be the ξ^i-eigenspace of s on V and recall from Section 2.3 that we may assume x is obtained by exponentiating the regular nilpotent element $e = e_1 + e_2 \in V$ (that is, we assume $x = \exp(\mathrm{ad}(e))$). According to [14, Table 2], the Jordan form of x on V is as follows:

$$\begin{cases} [J_{11}, J_3] & p \geqslant 11 \\ [J_7^2] & p = 7. \end{cases} \tag{21}$$

We use the notation U and $W(i)$ for the projective indecomposable KX-modules defined in (3) and (4), respectively.

Case 1. $V|_X$ is semisimple.

First assume $V|_X$ is semisimple and recall that x has Jordan form $[J_{m+1}]$ on $L_X(m)$ (for $0 \leqslant m < p$). In view of (21), it follows that

$$V|_X = \begin{cases} L_X(10) \oplus L_X(2) & p \geqslant 11 \\ L_X(6)^2 & p = 7. \end{cases}$$

If $p = 7$ then the above decomposition implies that ξ^6 is an eigenvalue of s on $C_V(x)$, but this is not compatible with (20).

Now assume $p \geqslant 11$, so $V|_X = L_X(10) \oplus L_X(2)$. As in Section 2.5, let $\{w_2, w_0, w_{-2}\}$ be a standard basis for the summand $W = L_X(2)$, so $w_i \in E_i$ and the action of x on W is given by the matrix in (14). Our goal is to show that W is an $\mathfrak{sl}_2$-subalgebra with $w_2 = e$ and $w_{-2} \in \sum_{i=1}^2 \mathbb{Z} f_i$. Furthermore, we seek integers λ, μ so that $(w_2, \lambda w_0, \mu w_{-2})$ is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$, which will allow us to apply Proposition 2.14.

For $p \geqslant 17$ we find that each space

$$\ker(x-1) \cap E_2, \quad \ker((x-1)^2) \cap E_0, \quad \ker((x-1)^3) \cap E_{-2} \tag{22}$$

is 1-dimensional, which gives us

$$w_2 = a_1(e_1 + e_2), \quad w_0 = a_2(h_1 + 13h_2), \quad w_{-2} = a_3(4f_1 + f_2)$$

for some nonzero scalars $a_i \in K$ (in the expressions for w_0 and w_{-2}, the specific coefficients of the h_i and f_i will depend on the characteristic p; the coefficients presented here are for $p = 17$). If we set $a_1 = 1$ then by considering the action of x on V we deduce that $a_2 = 14$ and $a_3 = 7$. Now w_2, w_0 and w_{-2} satisfy the relations in (17), so we get an $\mathfrak{sl}_2$-triple (w_2', w_0', w_{-2}') as in (18). Here $w_{-2}' = -w_{-2} = 6f_1 + 10f_2$ (for $p = 17$) and thus (w_2', w_0', w_{-2}') is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ (see the proof of [35, Proposition 2.4]). Finally, by applying Proposition 2.14, we conclude that X is contained in an A_1-type subgroup of G.

Next assume $p = 13$. Once again $\ker(x-1) \cap E_2$ and $\ker((x-1)^2) \cap E_0$ are 1-dimensional, but now $\ker((x-1)^3) \cap E_{-2}$ is 2-dimensional, spanned by the vectors $11f_1 + f_2$ and e_{32} (here we use the notation e_{32} for e_γ with $\gamma = 3\alpha_1 + 2\alpha_2$). Therefore

$$w_2 = a_1(e_1 + e_2), \quad w_0 = a_2(h_1 + 6h_2), \quad w_{-2} = a_3(11f_1 + f_2) + a_4 e_{32}$$

for some $a_i \in K$. By considering the action of x on W we deduce that $a_2 = 10a_1$ and $a_3 = 3a_1$. Moreover, (16) implies that $a_4 = 0$ and by arguing as above, setting $a_1 = 1$ and using Proposition 2.14, we deduce that X is contained in an A_1-type subgroup of G.

Now suppose $p = 11$. Here we have

$$w_2 = a_1(e_1 + e_2), \quad w_0 = a_2(h_1 + 9h_2) + a_3 e_{32}, \quad w_{-2} = a_4(5f_1 + f_2) + a_5 e_{31}$$

and by considering the action of x on W we deduce that $a_2 = 8a_1$, $a_4 = a_1$ and $a_5 = 2a_3$. We may as well set $a_1 = 1$, so

$$w_2 = e_1 + e_2, \quad w_0 = 8(h_1 + 9h_2) + \gamma e_{32}, \quad w_{-2} = 5f_1 + f_2 + 2\gamma e_{31}$$

for some $\gamma \in K$. One now checks that the relations in (17) are satisfied (for all γ), so $W = \langle w_2, w_0, w_{-2} \rangle$ is an $\mathfrak{sl}_2$-subalgebra. Moreover, if we take

$$e = w_2, \quad h = -2(8(h_1 + 9h_2)) = 6h_1 + 4h_2, \quad f = -(5f_1 + f_2) = 6f_1 + 10f_2,$$

then (e, h, f) is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ and we can apply Proposition 2.15 (with $y = e_{32}$ and $z = e_{31}$). It follows that the stabilizer of W in G is an A_1-type subgroup.

Case 2. $\mathrm{rad}(V|_X) \neq 0$, $p \geqslant 11$.

To complete the proof of the theorem we may assume that $\mathrm{rad}(V|_X) \neq 0$. Suppose $p \geqslant 11$ and W is a reducible indecomposable summand of $V|_X$. If $p \geqslant 13$ then $\dim W \geqslant 12$ (see Corollary 2.4) and thus Lemma 2.1 implies that x has a Jordan block of size $n \geqslant 12$ on W, but this is incompatible with (21). Now assume $p = 11$. Here (21) implies that W has at least three composition factors (if there were only two, then Lemma 2.1 and Corollary 2.3 would imply that x has Jordan form $[J_{10}]$ or $[J_{11}, J_1]$ on W, which contradicts (21)). By Lemma 2.1, it

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follows that x has Jordan form $[J_{11}, J_i]$ on W with $i \in \{0, 3\}$, so $\dim W \in \{11, 14\}$. By considering Theorem 2.2, it is easy to see that $i = 0$ is the only possibility, so $W = U$ is projective and thus

$$V|_X = U \oplus L_X(2).$$

However, this implies that an involution $x_2 \in X$ has trace 2 on V (see Section 2.2), which is incompatible with Proposition 2.8. This is a contradiction.

Case 3. $\mathrm{rad}(V|_X) \neq 0$, $p = 7$.

Finally, let us assume $p = 7$. Let $P = \langle x \rangle$ be a Sylow p-subgroup of X and observe that $V|_P$ is projective. Then [**1**, Corollary 3, Section 9] implies that $V|_X$ is projective and thus each indecomposable summand is also projective. Since the eigenvalues of s on $C_V(x)$ are $\{\xi^4, \xi^2\}$, we deduce that $V|_X = W(2)$ or $W(4)$. In fact, by considering the trace of x_2, we see that $V|_X = W(2)$ is the only option. This is compatible with the decomposition of V with respect to an A_1-type subgroup of G containing a regular unipotent element (see Table 2).

Let W be the $L_X(2)$ summand in the socle of $V|_X$ and let $\{w_2, w_0, w_{-2}\}$ be a standard basis. The spaces $\ker(x-1) \cap E_2$ and $\ker((x-1)^2) \cap E_0$ are 1-dimensional, whereas $\ker((x-1)^3) \cap E_{-2}$ is 2-dimensional and we get

$$w_2 = a_1(e_1 + e_2), \quad w_0 = a_2(h_1 + 4h_2), \quad w_{-2} = a_3(2f_1 + f_2) + a_4 e_{32}$$

for some $a_i \in K$. Set $a_1 = 1$, so $w_2 = e$. By considering the action of x on W we deduce that $a_2 = a_3 = 4$. Moreover, (16) implies that $a_4 = 0$ and we deduce that $W = \langle w_2, w_0, w_{-2} \rangle$ is an $\mathfrak{sl}_2$-subalgebra and the relations in (17) are satisfied. As before, the desired result now follows by applying Proposition 2.14.

This completes the proof of Theorem 3.1. □

4. A reduction for $G = F_4$

In this section, our goal is to establish Theorem 2.23 when $G = F_4$. The proof of Theorem 1 in this case will be completed in Section 8. Our main result is the following.

THEOREM 4.1. *Let G be a simple algebraic group of type F_4 over an algebraically closed field of characteristic $p > 0$. Let $X = \mathrm{PSL}_2(p)$ be a subgroup of G containing a regular unipotent element x of G and set $V = \mathrm{Lie}(G)$. Then one of the following holds:*

(i) *X is contained in an A_1-type subgroup of G;*

(ii) $p=13$, $V|_X = W(10) \oplus W(2)$ *and* X *stabilizes a nonzero subalgebra of* $\langle e_\alpha \mid \alpha \in \Phi^+(G)\rangle$.

Proof. Here $p \geqslant 13$ and we set up the standard notation as before. In particular,

$$\{\xi^{22}, \xi^{14}, \xi^{10}, \xi^{2}\} \tag{23}$$

are the eigenvalues of s on $C_V(x)$, where $\mathbb{F}_p^\times = \langle \xi \rangle$, and

$$\begin{cases} [J_{23}, J_{15}, J_{11}, J_3] & p \geqslant 23 \\ [J_{19}^2, J_{11}, J_3] & p = 19 \\ [J_{17}^2, J_{15}, J_3] & p = 17 \\ [J_{13}^4] & p = 13 \end{cases} \tag{24}$$

is the Jordan form of x on V (see [**14**, Table 4]). We may assume that x is obtained by exponentiating the regular nilpotent element $e = e_1 + e_2 + e_3 + e_4$ in V, with respect to a Chevalley basis for V as in (11). It will also be useful to note that $V|_X$ is self-dual.

Case 1. $V|_X$ *is semisimple*

If $p \in \{13, 17, 19\}$ then (24) implies that

$$V|_X = \begin{cases} L_X(18)^2 \oplus L_X(10) \oplus L_X(2) & p = 19 \\ L_X(16)^2 \oplus L_X(14) \oplus L_X(2) & p = 17 \\ L_X(12)^4 & p = 13 \end{cases}$$

but none of these decompositions are compatible with the eigenvalues of s on $C_V(x)$ given in (23). For example, if $p = 19$ then the given decomposition implies that the relevant eigenvalues are $\{\xi^0, \xi^0, \xi^{10}, \xi^2\}$, but this contradicts (23).

Now assume $p \geqslant 23$, so

$$V|_X = L_X(22) \oplus L_X(14) \oplus L_X(10) \oplus L_X(2).$$

Let W be the $L_X(2)$ summand and let $\{w_2, w_0, w_{-2}\}$ be a standard basis for W as in Section 2.5, so $w_i \in E_i$ (the ξ^i-eigenspace of s on V) and the action of x and x' on W is given by the matrices in (14) and (15), respectively, where $\langle \bar{s}, x' \rangle$ is the opposite Borel subgroup of X. If $p \geqslant 29$ then the spaces in (22) are 1-dimensional and we get

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4) \\ w_0 &= a_2(h_1 + 23h_2 + 4h_3 + 6h_4) \end{aligned}$$

$$w_{-2} = a_3(5f_1 + 28f_2 + 20f_3 + f_4)$$

for some $a_i \in K$ (in the expressions for w_0 and w_{-2}, the specific coefficients depend on the characteristic p; the ones given here are for $p = 29$). If we set $a_1 = 1$ then we can use the action of x on V to deduce that $a_2 = 18$ and $a_3 = 13$. Moreover, the relations in (17) are satisfied and it follows that (w'_2, w'_0, w'_{-2}) is an $\mathfrak{sl}_2$-triple, where these elements are defined in (18). Now

$$w'_{-2} = -w_{-2} = -13(5f_1 + 28f_2 + 20f_3 + f_4) = 22f_1 + 42f_2 + 30f_3 + 16f_4$$

working mod p (for $p = 29$), so (w'_2, w'_0, w'_{-2}) is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ (see the proof of [**35**, Proposition 2.4]). By applying Proposition 2.14, we conclude that X is contained in an A_1-type subgroup of G.

Now suppose $p = 23$. Here

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4) \\ w_0 &= a_2(h_1 + 4h_2 + 16h_3 + 7h_4) + a_3 e_{2342} \\ w_{-2} &= a_4(10f_1 + 17f_2 + 22f_3 + f_4) + a_5 e_{1342} \end{aligned}$$

for some $a_i \in K$ (we use the notation e_{2342} for e_γ with $\gamma = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, and similarly for e_{1342}). By considering the action of x on W we deduce that $a_2 = 12a_1$, $a_4 = 7a_1$ and $a_5 = 2a_3$. Setting $a_1 = 1$ we get

$$\begin{aligned} w_2 &= e_1 + e_2 + e_3 + e_4 \\ w_0 &= 12(h_1 + 4h_2 + 16h_3 + 7h_4) + \gamma e_{2342} \\ w_{-2} &= 7(10f_1 + 17f_2 + 22f_3 + f_4) + 2\gamma e_{1342} \end{aligned}$$

for some $\gamma \in K$, and one can check that the relations in (17) are satisfied. In particular, W is an $\mathfrak{sl}_2$-subalgebra of V. Set

$$e = w_2, \quad h = -2(12(h_1 + 4h_2 + 16h_3 + 7h_4)) = 22h_1 + 15h_2 + 14h_3 + 9h_4$$

and

$$f = -(7(10f_1 + 17f_2 + 22f_3 + f_4)) = 22f_1 + 19f_2 + 7f_3 + 16f_4.$$

Then (e, h, f) is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ and by applying Proposition 2.15 (with $y = e_{2342}$ and $z = e_{1342}$) we deduce that the stabilizer of W in G is an A_1-type subgroup.

Case 2. $\mathrm{rad}(V|_X) \neq 0$, $p \geqslant 19$.

For the remainder we may assume that $\mathrm{rad}(V|_X) \neq 0$. First assume $p \geqslant 19$. By arguing as in Case 2 in the proof of Theorem 3.1, it is straightforward to reduce to the case $p = 19$. For example, suppose $p = 23$ and W is a reducible indecomposable summand of $V|_X$. The Jordan form of x on V (see (24)) implies that W has at least three composition factors and we can use Lemma 2.1 to see that x has Jordan form $[J_{23}, J_i]$ on W with $i \in \{0, 3, 11, 15\}$, so $\dim W \in \{23, 26, 34, 38\}$. Using Theorem 2.2, we deduce that $i = 0$ is the only option, so

$$V|_X = U \oplus L_X(14) \oplus L_X(10) \oplus L_X(2).$$

But this implies that an involution $x_2 \in X$ has trace 0 on V, which is incompatible with Proposition 2.8.

Now assume $p = 19$. Suppose W is a reducible nonprojective indecomposable summand of $V|_X$. By combining Lemma 2.1 and Theorem 2.2 we deduce that x has Jordan form $[J_{19}^2, J_{11}]$ or $[J_{19}^2, J_3]$ on W, so there is a unique such summand (and the other summand is simple). However, this is incompatible with the self-duality of $V|_X$. For example, if x has Jordan form $[J_{19}^2, J_3]$ on W, then $V|_X = W \oplus L_X(10)$ and Theorem 2.2 implies that

$$\mathrm{soc}(W) = L_X(0) \oplus L_X(2) \oplus L_X(4), \quad W/\mathrm{soc}(W) = L_X(16) \oplus L_X(14)$$

(up to duality) so $V|_X$ is not self-dual.

Therefore, we may assume that each indecomposable summand is either simple or projective, so the possibilities for $V|_X$ are as follows:

$$\begin{cases} U \oplus L_X(18) \oplus L_X(10) \oplus L_X(2) \\ U^2 \oplus L_X(10) \oplus L_X(2) \\ W(i) \oplus L_X(10) \oplus L_X(2) \end{cases}$$

with $i \in \{2, 4, \dots, 16\}$. As in (23), the eigenvalues of s on $C_V(x)$ are

$$\{\xi^4, \xi^{14}, \xi^{10}, \xi^2\}.$$

Since s has eigenvalues 1 and $\{\xi^i, \xi^{-i}\}$ on $C_U(x)$ and $C_{W(i)}(x)$, respectively (see Lemma 2.20), it follows that

$$V|_X = W(i) \oplus L_X(10) \oplus L_X(2)$$

with $i \in \{4, 14\}$. The case $i = 4$ can be ruled out by considering the trace of x_2; hence $i = 14$ and $V|_X$ is compatible with the containment of X in an A_1-type subgroup of G (see Table 2). We need to show that X is contained in such a subgroup. To do this we can repeat the argument in Case 1 for $p \geqslant 29$ (the details are entirely similar).

Case 3. $\mathrm{rad}(V|_X) \neq 0$, $p = 17$.

Now assume $p = 17$. Suppose W is a reducible nonprojective indecomposable summand of $V|_X$. It is easy to check that the Jordan form of x on W is either $[J_{17}^2, J_3]$ or $[J_{17}, J_{15}]$, so there is a unique such summand. If x has Jordan form $[J_{17}^2, J_3]$ on W then Theorem 2.2 implies that $V|_X = W \oplus L_X(14)$ (up to duality) where

$$\mathrm{soc}(W) = L_X(0) \oplus L_X(2) \oplus L_X(4), \quad W/\mathrm{soc}(W) = L_X(16) \oplus L_X(14),$$

but this is incompatible with the self-duality of $V|_X$. Similarly, in the other case we have $V|_X = W \oplus V_1 \oplus L_X(2)$ and

$$\mathrm{soc}(W) = L_X(i) \oplus L_X(i+2), \quad W/\mathrm{soc}(W) = L_X(14-i) \oplus L_X(12-i)$$

with $i \in \{0, 2, 4, \dots, 12\}$ and $V_1 \in \{L_X(16), U\}$. By self-duality, $i = 6$ is the only option. But this implies that x_2 has trace 0 on V, which contradicts Proposition 2.8.

It follows that each indecomposable summand of $V|_X$ is either simple or projective. By arguing as above (the case $p = 19$), using the fact that s has eigenvalues $\{\xi^6, \xi^{14}, \xi^{10}, \xi^2\}$ on $C_V(x)$, we deduce that

$$V|_X = W(i) \oplus L_X(14) \oplus L_X(2)$$

with $i \in \{6, 10\}$. If $i = 6$ then one can check that an element $x_3 \in X$ of order 3 has trace 1 on V, so Proposition 2.8 implies that $i = 10$. Therefore, the action of X is compatible with an A_1-type subgroup of G (see Table 2) and it remains to establish the desired containment.

As before, let $\{w_2, w_0, w_{-2}\}$ be a standard basis of the $L_X(2)$ summand W in the decomposition of $V|_X$. In the usual manner we deduce that

$$w_2 = a_1(e_1 + e_2 + e_3 + e_4)$$
$$w_0 = a_2(h_1 + 5h_2 + 6h_3 + 10h_4)$$

for some nonzero scalars $a_1, a_2 \in K$. We may assume $a_1 = 1$. Now w_{-2} is contained in $\ker((x-1)^3) \cap E_{-2}$, which is 2-dimensional, and we get

$$w_{-2} = a_3(12f_1 + 9f_2 + 4f_3 + f_4) + a_4(e_{1231} - e_{1222}).$$

Since the action of x on W is given by the matrix in (14) we deduce that $a_2 = 6$ and $a_3 = 1$. Finally, the condition in (16), which is obtained by considering the action of x' on W, implies that $a_4 = 0$. It is easy to see that the relations in (17) are satisfied and we complete the argument in the usual manner, via Proposition 2.14.

Case 4. $\mathrm{rad}(V|_X) \neq 0$, $p = 13$.

Finally, let us assume that $p = 13$. Here $V|_X$ is projective and thus each indecomposable summand is also projective. Since the eigenvalues of s on $C_V(x)$ are $\{\xi^{10}, \xi^2, \xi^{10}, \xi^2\}$, we quickly deduce that $V|_X$ is one of the following:

$$W(2) \oplus W(10), \quad W(2)^2, \quad W(10)^2.$$

Let $y = \hat{y}Z \in X$ be an element of order 7, where $\hat{y} \in \mathrm{SL}_2(13)$ is $\mathrm{SL}_2(K)$-conjugate to a diagonal matrix $\mathrm{diag}(\omega, \omega^{-1})$ and $\omega \in K$ is a nontrivial 7th root of unity. For each decomposition we can compute the eigenvalues of y on V and then compare the results with the list of eigenvalue multiplicities of all elements in G of order 7 (as noted in Remark 2.10, the latter can be computed using Litterick's algorithm in [21]). For example, if $V|_X = W(2)^2$ then $y \in \mathrm{GL}_{52}(K)$ is conjugate to the diagonal matrix

$$[I_8, \omega I_8, \omega^2 I_8, \omega^3 I_6, \omega^4 I_6, \omega^5 I_8, \omega^6 I_8],$$

but one checks that no element in G of order 7 acts on V with these eigenvalues. In this way, we deduce that $V|_X = W(2) \oplus W(10)$ is the only possibility.

Let W be the $L_X(2)$ summand in the socle of $V|_X$ and let $\{w_2, w_0, w_{-2}\}$ be a standard basis. The spaces $\ker(x-1) \cap E_2$ and $\ker((x-1)^2) \cap E_0$ are 2-dimensional and we get

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4) + a_2(e_{1231} - e_{1222}) \\ w_0 &= a_3(h_1 + 9h_2 + 12h_3 + 9h_4) + a_4(e_{1221} + 10e_{1122}) \end{aligned}$$

for some $a_i \in K$. Finally, one checks that $\ker((x-1)^3) \cap E_{-2}$ is 4-dimensional and we take

$$w_{-2} = a_5(3f_1 + f_2 + 10f_3 + f_4) + a_6(e_{1220} + 3e_{0122}) + a_7(e_{1121} + 8e_{0122}) + a_8 e_{2342}.$$

In the usual manner, by considering the action of x on W, we get $a_3 = 2a_1$, $a_4 = a_2$, $a_5 = 10a_1$ and $a_7 = 2a_2 + a_6$. In addition, the condition in (16) yields the following system of equations:

$$\begin{aligned} a_1^2a_2 + 8a_1^2a_6 &= 0 \\ 7a_1^2a_2 + 11a_1^2a_6 &= 0 \\ 11a_1^2a_2 + 5a_1^2a_6 &= 0 \\ 4a_1^2a_8 + 8a_1a_2^2 + 7a_1a_2a_6 &= 0. \end{aligned}$$

If $a_1 \neq 0$ then these equations imply that $a_2 = a_6 = a_8 = 0$, so

$$w_2 = a_1(e_1 + e_2 + e_3 + e_4)$$

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$$w_0 = -a_1(h_1 + 9h_2 + 12h_3 + 9h_4)$$
$$w_{-2} = 10a_1(3f_1 + f_2 + 10f_3 + f_4)$$

and by setting $a_1 = 1$ we can use Proposition 2.14 to show that X is contained in an A_1-type subgroup. On the other hand, if $a_1 = 0$ then we can set $a_2 = 1$, so

$$\begin{aligned} w_2 &= e_{1231} - e_{1222} \\ w_0 &= e_{1221} + 10e_{1122} \\ w_{-2} &= a_6(e_{1220} + 3e_{0122}) + (2 + a_6)(e_{1121} + 8e_{0122}) + a_8e_{2342}. \end{aligned}$$

One checks that $[w_2, w_0] = 0$, so

$$0 = x' \cdot [w_2, w_0] = [w_2 + 2w_0 + w_{-2}, w_0 + w_{-2}]$$

and we deduce that $a_6 = 7$, hence

$$w_{-2} = 7e_{1220} + 9e_{1121} + 2e_{0122} + a_8e_{2342}.$$

It is now easy to check that $W \subseteq \langle e_\alpha \mid \alpha \in \Phi^+(G)\rangle$ is a subalgebra, which gives case (ii) in the statement of the theorem.

This completes the proof of Theorem 4.1. □

5. A reduction for $G = E_6$

The following result, which we prove in this section, establishes Theorem 2.23 for groups of type E_6.

THEOREM 5.1. *Let G be a simple adjoint algebraic group of type E_6 over an algebraically closed field of characteristic $p > 0$. Let $X = \mathrm{PSL}_2(p)$ be a subgroup of G containing a regular unipotent element x of G and set $V = \mathrm{Lie}(G)$. Then one of the following holds:*

(i) *X is contained in an A_1-type subgroup of G;*

(ii) *$p = 13$, $V|_X$ is one of*

$$W(10) \oplus W(8) \oplus W(2), \quad W(10) \oplus W(4) \oplus W(2), \quad W(10)^2 \oplus W(4)$$

and X stabilizes a nonzero subalgebra of $\langle e_\alpha \mid \alpha \in \Phi^+(G)\rangle$.

Proof. Here $p \geqslant 13$, $V|_X$ is self-dual and

$$\{\xi^{22}, \xi^{16}, \xi^{14}, \xi^{10}, \xi^{8}, \xi^{2}\} \tag{25}$$

are the eigenvalues of s on $C_V(x)$, where $\mathbb{F}_p^\times = \langle \xi \rangle$ (see Section 2.5). By inspecting [14, Table 6], we see that

$$\begin{cases} [J_{23}, J_{17}, J_{15}, J_{11}, J_9, J_3] & p \geqslant 23 \\ [J_{19}^2, J_{17}, J_{11}, J_9, J_3] & p = 19 \\ [J_{17}^3, J_{15}, J_9, J_3] & p = 17 \\ [J_{13}^6] & p = 13 \end{cases} \tag{26}$$

is the Jordan form of x on V. We adopt the notation introduced in Section 2.

Case 1. $V|_X$ is semisimple.

If $p \in \{13, 17, 19\}$ then

$$V|_X = \begin{cases} L_X(18)^2 \oplus L_X(16) \oplus L_X(10) \oplus L_X(8) \oplus L_X(2) & p = 19 \\ L_X(16)^3 \oplus L_X(14) \oplus L_X(8) \oplus L_X(2) & p = 17 \\ L_X(12)^6 & p = 13 \end{cases}$$

but not one of these decompositions is compatible with the eigenvalues of s on $C_V(x)$ (see (25)) so we may assume $p \geqslant 23$ and

$$V_X = L_X(22) \oplus L_X(16) \oplus L_X(14) \oplus L_X(10) \oplus L_X(8) \oplus L_X(2).$$

Let W be the $L_X(2)$ summand and let $\{w_2, w_0, w_{-2}\}$ be a standard basis for W. If $p \geqslant 29$ then one checks that each of the spaces in (22) are 1-dimensional and the result quickly follows via Proposition 2.14. For example, if $p = 29$ then

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) \\ w_0 &= a_2(h_1 + 5h_2 + 20h_3 + 28h_4 + 20h_5 + h_6) \\ w_{-2} &= a_3(f_1 + 5f_2 + 20f_3 + 28f_4 + 20f_5 + f_6) \end{aligned}$$

and by setting $a_1 = 1$ and considering the action of x on W (see (14)), we deduce that $a_2 = 21$ and $a_3 = 13$. One now checks that $(w_2, -2w_0, -w_{-2})$ is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ (see the proof of [35, Proposition 2.4]) and by applying Proposition 2.14 we deduce that X is contained in an A_1-type subgroup of G.

Now assume $p = 23$. Here $\ker((x-1)^2) \cap E_0$ and $\ker((x-1)^3) \cap E_{-2}$ are 2-dimensional and we get

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) \\ w_0 &= a_2(h_1 + 10h_2 + 22h_3 + 17h_4 + 22h_5 + h_6) + a_3 e_{122321} \\ w_{-2} &= a_4(f_1 + 10f_2 + 22f_3 + 17f_4 + 22f_5 + f_6) + a_5 e_{112321} \end{aligned}$$

where $a_1a_2a_4 \neq 0$. Set $a_1 = 1$. From the action of x on W we deduce that $a_2 = 15$, $a_4 = 7$ and $a_5 = 2a_3$, so

$$\begin{aligned} w_2 &= e_1 + e_2 + e_3 + e_4 + e_5 + e_6 \\ w_0 &= 15(h_1 + 10h_2 + 22h_3 + 17h_4 + 22h_5 + h_6) + \gamma e_{122321} \\ w_{-2} &= 7(f_1 + 10f_2 + 22f_3 + 17f_4 + 22f_5 + f_6) + 2\gamma e_{112321} \end{aligned}$$

for some $\gamma \in K$. If we take

$$e = w_2, \quad h = -2(15(h_1 + 10h_2 + 22h_3 + 17h_4 + 22h_5 + h_6))$$

and

$$\begin{aligned} f &= -7(f_1 + 10f_2 + 22f_3 + 17f_4 + 22f_5 + f_6) \\ &= 16f_1 + 22f_2 + 7f_3 + 19f_4 + 7f_5 + 16f_6 \end{aligned}$$

then (e, h, f) is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ and using Proposition 2.15 we conclude that X is contained in an A_1-type subgroup of G.

Case 2. $\mathrm{rad}(V|_X) \neq 0$, $p \geqslant 19$.

If $p \geqslant 23$ then we can essentially repeat the argument in the proof of Theorem 4.1 (see the first paragraph in Case 2). Indeed, it is easy to reduce to the case where $p = 23$ and

$$V|_X = U \oplus L_X(16) \oplus L_X(14) \oplus L_X(10) \oplus L_X(8) \oplus L_X(2),$$

but this is not compatible with (25).

Now assume $p = 19$. First suppose $V|_X$ has a reducible nonprojective indecomposable summand W. By applying Lemma 2.1 and Theorem 2.2, we deduce that the Jordan form of x on W is one of the following:

$$[J_{19}^2, J_{11}], \quad [J_{19}^2, J_9], \quad [J_{19}^2, J_3], \quad [J_{19}, J_{17}].$$

In particular, $V|_X$ has a unique such summand. The structure of W is described in Theorem 2.2 and it is easy to see that the existence of such a summand contradicts the self-duality of $V|_X$. For instance, suppose x has Jordan form $[J_{19}^2, J_9]$ on W. Then up to duality we have

$$\mathrm{soc}(W) = L_X(6) \oplus L_X(8) \oplus L_X(10), \quad W/\mathrm{soc}(W) = L_X(10) \oplus L_X(8)$$

and thus $V|_X = W \oplus L_X(16) \oplus L_X(10) \oplus L_X(2)$ is not self-dual. The other cases are very similar.

Therefore, we may assume that each indecomposable summand of $V|_X$ is either simple or projective. By considering the eigenvalues of s in (25), we deduce that

$$V|_X = W(i) \oplus L_X(16) \oplus L_X(10) \oplus L_X(8) \oplus L_X(2)$$

with $i \in \{4, 14\}$. If $i = 4$ then we find that x_2 has trace 2 on V, which contradicts Proposition 2.8, hence $i = 14$ is the only possibility. In the usual manner, we now construct a basis

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) \\ w_0 &= a_2(h_1 + 18h_2 + 9h_3 + 5h_4 + 9h_5 + h_6) \\ w_{-2} &= a_3(f_1 + 18f_2 + 9f_3 + 5f_4 + 9f_5 + f_6) + a_4(e_{112211} - e_{111221}) \end{aligned}$$

(with $a_1a_2a_3 \neq 0$) of the summand $W = L_X(2)$ of $V|_X$. If we set $a_1 = 1$ and consider the action of x on W (see (14)) we deduce that $a_2 = 11$ and $a_3 = 3$, and one checks that the condition in (16) gives $a_4 = 0$. The result now follows in the usual manner via Proposition 2.14.

Case 4. $\mathrm{rad}(V|_X) \neq 0$, $p = 17$.

First assume that $V|_X$ has a reducible indecomposable nonprojective summand W. In the usual way, by combining Lemma 2.1 and Theorem 2.2, we deduce that the Jordan form of x on W is one of the following:

$$[J_{17}^3, J_3], \quad [J_{17}^2, J_9], \quad [J_{17}^2, J_3], \quad [J_{17}, J_{15}].$$

Suppose that x has Jordan form $[J_{17}^3, J_3]$ on W, so $V|_X = W \oplus L_X(14) \oplus L_X(8)$. By applying Theorem 2.2, using the self-duality of $V|_X$, we deduce that

$$\mathrm{soc}(W) = W/\mathrm{soc}(W) = L_X(10) \oplus L_X(8) \oplus L_X(6)$$

is the only possibility, but this is incompatible with the eigenvalues of s on $C_V(x)$. We can rule out $[J_{17}^2, J_9]$ and $[J_{17}^2, J_3]$ by the self-duality of $V|_X$, so let us assume x has Jordan form $[J_{17}, J_{15}]$ on W. By self-duality it follows that

$$\mathrm{soc}(W) = W/\mathrm{soc}(W) = L_X(8) \oplus L_X(6)$$

and thus $V|_X$ is one of the following:

$$\begin{cases} W \oplus M_1 \oplus M_2 \oplus L_X(8) \oplus L_X(2) \\ W \oplus W(i) \oplus L_X(8) \oplus L_X(2) \end{cases}$$

where $M_j \in \{L_X(16), U\}$ and $i \in \{2, 4, \ldots, 14\}$. However, it is clear that none of these decompositions are compatible with (25).

For the remainder, we may assume that each indecomposable summand of $V|_X$ is either simple or projective. By considering the eigenvalues of s, we deduce that

$$V|_X = W(i) \oplus M_1 \oplus L_X(14) \oplus L_X(8) \oplus L_X(2)$$

with $i \in \{6, 10\}$ and $M_1 \in \{L_X(16), U\}$. By computing the trace of x_3 and appealing to Proposition 2.8 (and also Remark 2.9), it follows that $i = 10$ and $M_1 = L_X(16)$ is the only possibility. In particular, we have now reduced to the case where the decomposition of $V|_X$ is compatible with containment in an A_1-type subgroup of G (see Table 2).

As before, let W be the $L_X(2)$ summand of $V|_X$ and let $\{w_2, w_0, w_{-2}\}$ be a standard basis of W. The reader can check that

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) \\ w_0 &= a_2(h_1 + 12h_2 + 4h_3 + 9h_4 + 4h_5 + h_6) + a_3(e_{112211} - e_{111221}) \\ w_{-2} &= a_4(f_1 + 12f_2 + 4f_3 + 9f_4 + 4f_5 + f_6) + a_5(e_{112210} + e_{011221}) \\ &\quad + a_6(e_{111211} + 15e_{011221}) \end{aligned}$$

with $a_1a_2a_4 \neq 0$. Set $a_1 = 1$. By considering the action of x on W we deduce that $a_2 = 9$, $a_4 = 1$ and $a_6 = 2a_3 + a_5$. The condition in (16) yields $a_5 = 15a_3$, so $a_6 = 0$ and thus

$$\begin{aligned} w_2 &= e_1 + e_2 + e_3 + e_4 + e_5 + e_6 \\ w_0 &= 9(h_1 + 12h_2 + 4h_3 + 9h_4 + 4h_5 + h_6) + \gamma(e_{112211} - e_{111221}) \\ w_{-2} &= f_1 + 12f_2 + 4f_3 + 9f_4 + 4f_5 + f_6 + 15\gamma(e_{112210} + e_{011221}) \end{aligned}$$

for some $\gamma \in K$. In addition, the relations in (17) are satisfied and W is an $\mathfrak{sl}_2$-subalgebra of V. Set

$$e = w_2, \quad h = -2(9(h_1 + 12h_2 + 4h_3 + 9h_4 + 4h_5 + h_6))$$

and

$$\begin{aligned} f &= -(f_1 + 12f_2 + 4f_3 + 9f_4 + 4f_5 + f_6) \\ &= 16f_1 + 5f_2 + 13f_3 + 8f_4 + 13f_5 + 16f_6. \end{aligned}$$

Then (e, h, f) is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ and by applying Proposition 2.15, where we set $y = e_{112211} - e_{111221}$ and $z = -(e_{112210} + e_{011221})$, we conclude that X is contained in an A_1-type subgroup of G.

Case 5. $\mathrm{rad}(V|_X) \neq 0$, $p = 13$.

Here (26) implies that $V|_X$ is projective, so each indecomposable summand is also projective. In view of (25), we must have $V|_X = W(i) \oplus W(j) \oplus W(k)$ with $i, j \in \{2, 10\}$ and $k \in \{4, 8\}$. In each case, the traces of x_2 and x_3 are -2 and -3, respectively, so we need to work harder to eliminate some of these decompositions. Let $y = \hat{y}Z \in X$ be an element of order 7, where $\hat{y} \in \mathrm{SL}_2(13)$ is $\mathrm{SL}_2(K)$-conjugate to a diagonal matrix $\mathrm{diag}(\omega, \omega^{-1})$ and $\omega \in K$ is a nontrivial 7th root of unity. We can compute the eigenvalues of y on V and then compare with the eigenvalue multiplicities of all elements in G of order 7, which we obtain using the algorithm in [21]. In this way, we deduce that $V|_X$ is one of the following:

$$W(10) \oplus W(8) \oplus W(2), \quad W(10) \oplus W(4) \oplus W(2), \quad W(10)^2 \oplus W(4).$$

Case 5(a). $p = 13$, $V|_X = W(10) \oplus W(8) \oplus W(2)$.

Here $V|_X$ is compatible with the containment of X in an A_1-type subgroup of G (see Table 2). Let W be the $L_X(2)$ summand in the socle of $V|_X$ and let $\{w_2, w_0, w_{-2}\}$ be a standard basis. In the usual manner, we deduce that

$$\begin{aligned}
w_2 &= a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) + a_2(e_{112210} + e_{111211} - e_{011221}) \\
w_0 &= a_3(h_1 + 3h_2 + 10h_3 + h_4 + 10h_5 + h_6) + a_4(e_{111210} + 3e_{111111} - e_{011211}) \\
w_{-2} &= a_5(f_1 + 3f_2 + 10f_3 + f_4 + 10f_5 + f_6) + a_6(e_{111110} + 7e_{011210} - e_{011111}) \\
&\quad + a_7(e_{101111} + 9e_{011210}) + a_8(e_{122321})
\end{aligned}$$

for some scalars $a_i \in K$. By considering the action of x on W, together with the condition in (16), we see that

$$a_3 = 5a_1, \quad a_4 = a_2, \quad a_5 = 10a_1, \quad a_7 = 6a_2 + 2a_6$$

and either $a_1 = 0$ or $a_2 = a_6 = a_8 = 0$. In the latter situation, we set $a_1 = 1$ and then check that the relations in (17) are satisfied – this allows us to apply Proposition 2.14 to conclude that X is contained in an A_1-type subgroup of G. Now assume $a_1 = 0$ and set $a_2 = 1$. Here one checks that $[w_{-2}, w_2] = 0$ and thus $[w_{-2}, w_2 + 2w_0 + w_{-2}] = 0$ since x' preserves the Lie bracket on V. This yields $a_6 = 9$, so

$$\begin{aligned}
w_2 &= e_{112210} + e_{111211} - e_{011221} \\
w_0 &= e_{111210} + 3e_{111111} - e_{011211} \\
w_{-2} &= 9e_{111110} + 11e_{101111} + 6e_{011210} + 4e_{011111} + a_8(e_{122321}).
\end{aligned} \tag{27}$$

We conclude that W is an X-invariant subalgebra of $\langle e_\alpha \mid \alpha \in \Phi^+(G)\rangle$, as in part (ii) of the theorem.

Case 5(b). $p = 13$, $V|_X = W(10) \oplus W(4) \oplus W(2)$ or $W(10)^2 \oplus W(4)$.

Let W be the $L_X(4)$ summand in the socle of $V|_X$ and let $\{w_4, w_2, w_0, w_{-2}, w_{-4}\}$ be a basis of W with $w_i \in E_i$. We may assume that the actions of x and x' on W are given by the matrices

$$\begin{pmatrix} 1&1&1&1&1\\ 0&1&2&3&4\\ 0&0&1&3&6\\ 0&0&0&1&4\\ 0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0\\ 4&1&0&0&0\\ 6&3&1&0&0\\ 4&3&2&1&0\\ 1&1&1&1&1 \end{pmatrix} \tag{28}$$

respectively (in terms of this basis). One checks that $\ker(x-1) \cap E_4$ is 1-dimensional, whereas the spaces

$$\ker((x-1)^2) \cap E_2, \quad \ker((x-1)^3) \cap E_0$$

are 3-dimensional, and

$$\ker((x-1)^4) \cap E_{-2}, \quad \ker((x-1)^5) \cap E_{-4}$$

have dimension 5 and 6, respectively, and we get

$$\begin{aligned}
w_4 &= a_1(e_{112211} - e_{111221})\\
w_2 &= a_2(e_{112210} + e_{011221}) + a_3(e_{111211} + 11e_{011221})\\
&\quad + a_4(e_1 + e_2 + e_3 + e_4 + e_5 + e_6)\\
w_0 &= a_5(e_{111210} + e_{011211}) + a_6(e_{111111} + 8e_{011211})\\
&\quad + a_7(h_1 + 3h_2 + 10h_3 + h_4 + 10h_5 + h_6)\\
w_{-2} &= a_8(e_{111110} + e_{011111}) + a_9(e_{101111} + 10e_{011111}) + a_{10}(e_{011210} + 9e_{011111})\\
&\quad + a_{11}(f_1 + 3f_2 + 10f_3 + f_4 + 10f_5 + f_6) + a_{12}(e_{122321})\\
w_{-4} &= a_{13}(e_{111100} + 3e_{001111}) + a_{14}(e_{101110} + e_{001111}) + a_{15}(e_{011110} + 6e_{001111})\\
&\quad + a_{16}(e_{010111} + 3e_{001111}) + a_{17}(12f_{101000} + 10f_{010100} + 2f_{001100}\\
&\quad + 11f_{000110} + f_{000011}) + a_{18}(e_{112321})
\end{aligned}$$

for some $a_i \in K$.

Set $a_1 = 1$ and consider the relations among the a_i obtained from the action of x on this basis. It is also helpful to note that x' is a regular unipotent element, so $C_V(x')$ is abelian and we see that $[w_{-4}, [w_{-4}, w_{-2}]] = 0$ since $[w_{-4}, w_{-2}] \in C_V(x')$. In this way, we deduce that

$$w_4 = e_{112211} - e_{111221}$$

$$w_2 = a_2(e_{112210} + e_{011221}) + (1 + a_2)(e_{111211} + 11e_{011221})$$
$$w_0 = 2a_2(e_{111210} + e_{011211}) + (6 + 6a_2)(e_{111111} + 8e_{011211})$$
$$w_{-2} = a_8(e_{111110} + e_{011111}) + (4 + 5a_2 + 2a_8)(e_{101111} + 10e_{011111}) + (6a_2 + 12a_8)(e_{011210} + 9e_{011111}) + a_{12}(e_{122321})$$
$$w_{-4} = a_{13}(e_{111100} + 3e_{001111}) + (2a_2 + 8a_8 + a_{13})(e_{101110} + e_{001111}) + (11a_2 + 9a_8)(e_{011110} + 6e_{001111}) + (1 + 2a_2 + 2a_8 + 12a_{13})(e_{010111} + 3e_{001111}) + 4a_{12}(e_{112321}).$$

Next one checks that $[w_2, w_{-2}] = 0$, so $x' \cdot [w_2, w_{-2}] = 0$ and thus

$$[w_2 + 3w_0 + 3w_{-2} + w_{-4}, w_{-2} + w_{-4}] = 0$$

since x' preserves the Lie bracket. This yields $a_{13} = 12a_2 + 2a_2^2 + 12a_2a_8$. Similarly, $[w_4, w_2] = 0$ and thus

$$[w_4 + 4w_2 + 6w_0 + 4w_{-2} + w_{-4}, w_2 + 3w_0 + 3w_{-2} + w_{-4}] = 0.$$

This relation implies that $a_2^2 + 12a_2a_8 + 9a_2 + 12a_8 = 0$ and it is now straightforward to check that $W \subseteq \langle e_\alpha \mid \alpha \in \Phi^+(G)\rangle$ is a subalgebra.

This completes the proof of Theorem 5.1. □

6. A reduction for $G = E_7$

In this section, we establish the following result, which proves Theorem 2.23 for groups of type E_7.

THEOREM 6.1. *Let G be a simple adjoint algebraic group of type E_7 over an algebraically closed field of characteristic $p > 0$. Let $X = \mathrm{PSL}_2(p)$ be a subgroup of G containing a regular unipotent element x of G and set $V = \mathrm{Lie}(G)$. Then one of the following holds:*

(i) *X is contained in an A_1-type subgroup of G;*

(ii) *$p = 19$, $V|_X$ is one of*

$$W(8) \oplus W(4) \oplus W(2) \oplus U, \quad W(16) \oplus W(10) \oplus W(4) \oplus U,$$

$$W(16) \oplus W(14) \oplus W(8) \oplus U$$

and X stabilizes a nonzero subalgebra of $\langle e_\alpha \mid \alpha \in \Phi^+(G)\rangle$.

Proof. Here we have $p \geqslant 19$ and

$$\{\xi^{34}, \xi^{26}, \xi^{22}, \xi^{18}, \xi^{14}, \xi^{10}, \xi^{2}\} \tag{29}$$

is the collection of eigenvalues of s on $C_V(x)$, where $\mathbb{F}_p^\times = \langle \xi \rangle$. By [14, Table 8], the Jordan form of x on V is as follows:

$$\begin{cases} [J_{35}, J_{27}, J_{23}, J_{19}, J_{15}, J_{11}, J_3] & p \geqslant 37 \\ [J_{31}^2, J_{23}, J_{19}, J_{15}, J_{11}, J_3] & p = 31 \\ [J_{29}^2, J_{27}, J_{19}, J_{15}, J_{11}, J_3] & p = 29 \\ [J_{23}^5, J_{15}, J_3] & p = 23 \\ [J_{19}^7] & p = 19. \end{cases} \tag{30}$$

Note that $V|_X$ is self-dual.

Case 1. $V|_X$ is semisimple.

If $p < 37$ then the eigenvalues of s on $C_V(x)$ are incompatible with (29), so we may assume $p \geqslant 37$ and thus

$$V|_X = L_X(34) \oplus L_X(26) \oplus L_X(22) \oplus L_X(18) \oplus L_X(14) \oplus L_X(10) \oplus L_X(2)$$

in view of (30). Let W be the $L_X(2)$ summand and let $\{w_2, w_0, w_{-2}\}$ be a standard basis for W. In the usual manner, it is straightforward to show that W is an appropriate $\mathfrak{sl}_2$-subalgebra and we can use Proposition 2.14 to show that (i) holds in the statement of the theorem. For example, if $p = 37$ we get

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7) \\ w_0 &= a_2(h_1 + 33h_2 + 15h_3 + 5h_4 + 12h_5 + 32h_6 + 28h_7) \\ w_{-2} &= a_3(4f_1 + 21f_2 + 23f_3 + 20f_4 + 11f_5 + 17f_6 + f_7) + a_4 e_{2234321}. \end{aligned}$$

If we set $a_1 = 1$, then by considering the action of x on W, we deduce that $a_2 = 20$ and $a_3 = 10$. Furthermore, the relation in (16) implies that $a_4 = 0$ and we deduce that $(w_2, -2w_0, -w_{-2})$ is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ (see the proof of [35, Proposition 2.4]). Now apply Proposition 2.14.

Case 2. $\mathrm{rad}(V|_X) \neq 0$, $p \geqslant 29$.

If $p \geqslant 37$ then a combination of Lemma 2.1 and Corollary 2.4 implies that x has a Jordan block of size $n \geqslant 36$ on V, but this contradicts (30).

Next assume $p = 31$. In the usual way, by applying Lemma 2.1 and Theorem 2.2, and by appealing to the self-duality of $V|_X$, we can reduce to the case where each indecomposable summand of $V|_X$ is either simple or projective. By considering the eigenvalues in (29), it follows that

$$V|_X = W(i) \oplus L_X(22) \oplus L_X(18) \oplus L_X(14) \oplus L_X(10) \oplus L_X(2)$$

with $i \in \{4, 26\}$. If $i = 4$ then an involution $x_2 \in X$ has trace -3 on V, which contradicts Proposition 2.8. Therefore $i = 26$ and it is entirely straightforward to show that the $L_X(2)$ summand of $V|_X$ is an appropriate $\mathfrak{sl}_2$-subalgebra. The result follows via Proposition 2.14 in the usual fashion.

A similar argument applies when $p = 29$. If $V|_X$ has a reducible nonprojective summand then the self-duality of $V|_X$ implies that

$$V|_X = M_1 \oplus M_2 \oplus L_X(18) \oplus L_X(14) \oplus L_X(10) \oplus L_X(2)$$

is the only possibility, where

$$\mathrm{soc}(M_1) \cong M_1/\mathrm{soc}(M_1) = L_X(12) \oplus L_X(14)$$

and $M_2 \in \{L_X(28), U\}$. However, this implies that x_2 has trace -3 on V, which is a contradiction. Therefore, the indecomposable summands of $V|_X$ are simple or projective, and by considering the eigenvalues in (29) we deduce that

$$V|_X = W(i) \oplus L_X(26) \oplus L_X(18) \oplus L_X(14) \oplus L_X(10) \oplus L_X(2)$$

with $i \in \{6, 22\}$. We can rule out $i = 6$ by computing the trace of x_3, so $i = 22$ and we complete the argument as in the previous case.

Case 3. $\mathrm{rad}(V|_X) \neq 0,\ p = 23.$

As before, it is not difficult to reduce to the case where each indecomposable summand of $V|_X$ is either simple or projective. By considering the eigenvalues in (29) we deduce that

$$V|_X = W(i) \oplus W(j) \oplus M_1 \oplus L_X(14) \oplus L_X(2)$$

where $i \in \{4, 18\}$, $j \in \{10, 12\}$ and $M_1 \in \{L_X(22), U\}$. By computing the trace of x_2 we see that $(i, j) = (4, 12)$ or $(18, 10)$, and we can rule out the first possibility by considering the trace of x_3. This calculation with x_3 also implies that $M_1 = L_X(22)$, so

$$V|_X = W(18) \oplus W(10) \oplus L_X(22) \oplus L_X(14) \oplus L_X(2).$$

Let W be the $L_X(2)$ summand and fix a standard basis $\{w_2, w_0, w_{-2}\}$. By considering the spaces

$$\ker(x-1) \cap E_2, \quad \ker((x-1)^2) \cap E_0, \quad \ker((x-1)^3) \cap E_{-2},$$

we deduce that

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7) \\ w_0 &= a_2(h_1 + 17h_2 + 6h_3 + 15h_4 + 11h_5 + 11h_6 + 15h_7) \\ &\quad + a_3(e_{1223210} + e_{1123211} - e_{1122221}) \\ w_{-2} &= a_4(20f_1 + 18f_2 + 5f_3 + f_4 + 13f_5 + 13f_6 + f_7) \\ &\quad + a_5(e_{1123210} + 2e_{1122211} + 20e_{1112221}). \end{aligned}$$

Setting $a_1 = 1$ and using the action of x on W, we deduce that $a_2 = 6$, $a_4 = 19$ and $a_5 = 2a_3$, so we have

$$\begin{aligned} w_2 &= e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 \\ w_0 &= 6(h_1 + 17h_2 + 6h_3 + 15h_4 + 11h_5 + 11h_6 + 15h_7) \\ &\quad + \gamma(e_{1223210} + e_{1123211} - e_{1122221}) \\ w_{-2} &= 19(20f_1 + 18f_2 + 5f_3 + f_4 + 13f_5 + 13f_6 + f_7) \\ &\quad + 2\gamma(e_{1123210} + 2e_{1122211} + 20e_{1112221}) \end{aligned}$$

for some $\gamma \in K$. One can check that the relations in (17) are satisfied, so W is an $\mathfrak{sl}_2$-subalgebra of V. Set

$$e = w_2, \quad h = -2(6(h_1 + 17h_2 + 6h_3 + 15h_4 + 11h_5 + 11h_6 + 15h_7)),$$

$$\begin{aligned} f &= -19(20f_1 + 18f_2 + 5f_3 + f_4 + 13f_5 + 13f_6 + f_7) \\ &= 11f_1 + 3f_2 + 20f_3 + 4f_4 + 6f_5 + 6f_6 + 4f_7 \end{aligned}$$

and

$$y = e_{1223210} + e_{1123211} - e_{1122221}, \qquad z = e_{1123210} + 2e_{1122211} + 20e_{1112221}.$$

Then (e, h, f) is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ and we can use Proposition 2.15 to deduce that X is contained in an A_1-type subgroup of G.

Case 4. $\operatorname{rad}(V|_X) \neq 0$, $p = 19$.

Finally, let us assume $p = 19$ so $V|_X$ is projective and each indecomposable summand is also projective. By considering the eigenvalues in (29), it follows that

$$V|_X = W(i) \oplus W(j) \oplus W(k) \oplus M_1$$

where $i \in \{2, 16\}$, $j \in \{4, 14\}$, $k \in \{8, 10\}$ and $M_1 \in \{L_X(18), U\}$. By computing the trace of x_2 we see that (i, j, k, M_1) is one of the following:

$$(2, 14, 10, L_X(18)), \quad (16, 4, 8, L_X(18)), \quad (16, 4, 10, U),$$
$$(16, 14, 8, U), \quad (2, 4, 8, U).$$

In all of these cases, x_3 has trace 2 on V, which is compatible with Proposition 2.8. If $V|_X = W(16) \oplus W(4) \oplus W(8) \oplus L_X(18)$ then there is an element $y \in X$ of order 5 with eigenvalues $[I_{25}, \omega I_{27}, \omega^2 I_{27}, \omega^3 I_{27}, \omega^4 I_{27}]$ on V, but one checks that there are no elements in G that act on V in this way (for example, see [7, Table 6]), so this possibility is ruled out.

If $V|_X$ is one of

$$W(8) \oplus W(4) \oplus W(2) \oplus U, \quad W(16) \oplus W(10) \oplus W(4) \oplus U,$$

$$W(16) \oplus W(14) \oplus W(8) \oplus U,$$

then X stabilizes the 1-dimensional subalgebra of V spanned by the vector

$$w = e_{1122111} - e_{1112211} + e_{0112221}.$$

Indeed, X stabilizes $\mathrm{soc}(U) = L_X(0)$, which is spanned by a vector in $C_V(x) \cap E_0$. But one checks that $C_V(x) \cap E_0 = \langle w \rangle$ so we are in case (ii) of the theorem.

Finally, suppose $V|_X = W(2) \oplus W(14) \oplus W(10) \oplus L_X(18)$, which is compatible with the containment of X in an A_1-type subgroup of G (see Table 2). Let W be the $L_X(2)$ summand in the socle of $V|_X$ and let $\{w_2, w_0, w_{-2}\}$ be a standard basis. In the usual way we obtain

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7) \\ w_0 &= a_2(h_1 + 2h_2 + 12h_3 + 14h_4 + 5h_5 + 6h_6 + 17h_7) \\ &\quad + a_3(e_{1122111} - e_{1112211} + e_{0112221}) \\ w_{-2} &= a_4(9f_1 + 18f_2 + 13f_3 + 12f_4 + 7f_5 + 16f_6 + f_7) \\ &\quad + a_5(e_{1122110} - e_{1112210} + 13e_{1112111} + 12e_{0112211}) + a_6(e_{2234321}) \end{aligned}$$

and we may assume $a_1 = 1$. By considering the action of x on W, together with the condition in (16), we deduce that $a_2 = 2$, $a_4 = 11$, $a_5 = 16a_3$ and $a_6 = 13a_3^2$, so we have

$$\begin{aligned} w_2 &= e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 \\ w_0 &= 2(h_1 + 2h_2 + 12h_3 + 14h_4 + 5h_5 + 6h_6 + 17h_7) \\ &\quad + \gamma(e_{1122111} - e_{1112211} + e_{0112221}) \\ w_{-2} &= 11(9f_1 + 18f_2 + 13f_3 + 12f_4 + 7f_5 + 16f_6 + f_7) \\ &\quad + 16\gamma(e_{1122110} - e_{1112210} + 13e_{1112111} + 12e_{0112211}) + 13\gamma^2(e_{2234321}) \end{aligned}$$

for some $\gamma \in K$. Set

$$e = w_2, \quad h = -2(2(h_1 + 2h_2 + 12h_3 + 14h_4 + 5h_5 + 6h_6 + 17h_7)),$$

$$\begin{aligned} f &= -11(9f_1 + 18f_2 + 13f_3 + 12f_4 + 7f_5 + 16f_6 + f_7) \\ &= 15f_1 + 11f_2 + 9f_3 + f_4 + 18f_5 + 14f_6 + 8f_7 \end{aligned}$$

and

$$\begin{aligned} y &= e_{1122111} - e_{1112211} + e_{0112221} \\ z_1 &= 8(e_{1122110} - e_{1112210} + 13e_{1112111} + 12e_{0112211}) \\ z_2 &= 11e_{2234321}. \end{aligned}$$

Then (e, h, f) is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$, but we cannot directly apply Proposition 2.15. However, a minor modification of the argument in the proof of that proposition will work.

First observe that $y \in (\mathcal{L}_{\mathbb{Z}})_{p-1} \cap C_{\mathcal{L}_{\mathbb{Z}}}(e)$, $z_1 \in (\mathcal{L}_{\mathbb{Z}})_{p-3}$ and $z_2 \in (\mathcal{L}_{\mathbb{Z}})_{2p-4}$ (in terms of the notation used in the proof of Proposition 2.15). Setting $\delta = -2\gamma$, we see that

$$(w_2, -2w_0, -w_{-2}) = (\bar{e}, \bar{h} + \delta\bar{y}, \bar{f} + \delta\bar{z}_1 + \delta^2\bar{z}_2)$$

is an $\mathfrak{sl}_2$-triple in $\mathcal{L}_K$ for all choices of $\gamma \in K$. Put $g = \exp(\mathrm{ad}(\delta y)) \in G$ and note that

$$g \cdot \bar{e} = \bar{e}, \quad g \cdot \bar{h} = \bar{h} + \delta[\bar{y}, \bar{h}] = \bar{h} + \delta\bar{y}, \quad g \cdot \bar{f} = \bar{f} + \delta[\bar{y}, \bar{f}] + \tfrac{1}{2}\delta^2[\bar{y}, [\bar{y}, \bar{f}]]$$

(for the final equality, note that all higher degree terms are zero since the maximum T-weight on $\mathcal{L}_{\mathbb{Z}}$ is $2\mathrm{ht}(\alpha_0) \leqslant 2(p-1)$). Now calculating (in $\mathcal{L}_{\mathbb{Z}}$), we have

$$[h + y, f + z_1 + z_2] = -2f + (p-3)z_1 + (2p-4)z_2 + [y, f] + [y, z_1]$$

and passing to $\mathcal{L}_K$, setting $\gamma = -\frac{1}{2}$, we deduce that

$$-2\bar{f} - 3\bar{z}_1 - 4\bar{z}_2 + [\bar{y}, \bar{f}] + [\bar{y}, \bar{z}_1] = -2(\bar{f} + \bar{z}_1 + \bar{z}_2).$$

Therefore $[\bar{y}, \bar{f}] + [\bar{y}, \bar{z}_1] = \bar{z}_1 + 2\bar{z}_2$ and by comparing T-weights we deduce that $[\bar{y}, \bar{f}] = \bar{z}_1$ and $[\bar{y}, \bar{z}_1] = 2\bar{z}_2$. Finally, this implies that

$$g \cdot \bar{f} = \bar{f} + \delta\bar{z}_1 + \tfrac{1}{2}\delta^2[\bar{y}, \bar{z}_1] = \bar{f} + \delta\bar{z}_1 + \delta^2\bar{z}_2$$

and we can now conclude as in the proof of Proposition 2.15. In particular, X is contained in an A_1-type subgroup of G.

This completes the proof of Theorem 6.1. □

7. A reduction for $G = E_8$

In this section, we complete the proof of the Reduction Theorem (see Theorem 2.23). Our main result is the following:

THEOREM 7.1. *Let G be a simple algebraic group of type E_8 over an algebraically closed field of characteristic $p > 0$. Let $X = \mathrm{PSL}_2(p)$ be a subgroup of G containing a regular unipotent element x of G and set $V = \mathrm{Lie}(G)$. Then one of the following holds:*

(i) *X is contained in an A_1-type subgroup of G;*

(ii) *$p = 37$, $V|_X = W(34) \oplus W(26) \oplus W(14) \oplus L_X(22) \oplus L_X(2)$ and X stabilizes a nonzero subalgebra of $\langle e_\alpha \mid \alpha \in \Phi^+(G)\rangle$.*

Proof. First note that $p \geqslant 31$. In fact, we may assume $p \geqslant 37$ since the case $p = 31$ was handled in Section 2 (see Examples 2.21 and 2.24). Recall that

$$\{\xi^{58}, \xi^{46}, \xi^{38}, \xi^{34}, \xi^{26}, \xi^{22}, \xi^{14}, \xi^{2}\} \tag{31}$$

is the collection of eigenvalues of s on $C_V(x)$ and note that $V|_X$ is self-dual. The Jordan form of x on V is as follows:

$$\begin{cases} [J_{59}, J_{47}, J_{39}, J_{35}, J_{27}, J_{23}, J_{15}, J_3] & p \geqslant 59 \\ [J_{53}^2, J_{39}, J_{35}, J_{27}, J_{23}, J_{15}, J_3] & p = 53 \\ [J_{47}^3, J_{39}, J_{27}, J_{23}, J_{15}, J_3] & p = 47 \\ [J_{43}^4, J_{35}, J_{23}, J_{15}, J_3] & p = 43 \\ [J_{41}^4, J_{39}, J_{27}, J_{15}, J_3] & p = 41 \\ [J_{37}^6, J_{23}, J_3] & p = 37 \\ [J_{31}^8] & p = 31 \end{cases} \tag{32}$$

(see [14, Table 9]).

Case 1. $V|_X$ is semisimple.

By considering the eigenvalues in (31) we deduce that $p \geqslant 59$ and

$$\begin{aligned} V|_X = L_X(58) \oplus L_X(46) \oplus L_X(38) \oplus L_X(34) \oplus L_X(26) \\ \oplus L_X(22) \oplus L_X(14) \oplus L_X(2). \end{aligned}$$

Let W be the $L_X(2)$ summand and let $\{w_2, w_0, w_{-2}\}$ be a standard basis. If $p \geqslant 61$ then it is straightforward to show that W is an appropriate $\mathfrak{sl}_2$-subalgebra and the result follows by applying Proposition 2.14 (note that for $p = 61$ we find that $\ker((x-1)^3) \cap E_{-2}$ is 2-dimensional, but this does not cause any special difficulties). Now assume $p = 59$. Here $\ker((x-1)^2) \cap E_0$ and $\ker((x-1)^3) \cap E_{-2}$ are both 2-dimensional and we get

$$w_2 = a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8)$$

$$w_0 = a_2(h_1 + 22h_2 + 52h_3 + 35h_4 + 46h_5 + 48h_6 + 41h_7 + 25h_8) \\ + a_3(e_{23465432})$$

$$w_{-2} = a_4(26f_1 + 41f_2 + 54f_3 + 25f_4 + 16f_5 + 9f_6 + 4f_7 + f_8) + a_5(e_{23465431}).$$

Set $a_1 = 1$ and consider the action of x on W (see (14)). We deduce that $a_2 = 13$, $a_4 = 1$ and $a_5 = 2a_3$, so

$$w_2 = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8$$

$$w_0 = 13(h_1 + 22h_2 + 52h_3 + 35h_4 + 46h_5 + 48h_6 + 41h_7 + 25h_8) \\ + \gamma(e_{23465432})$$

$$w_{-2} = 26f_1 + 41f_2 + 54f_3 + 25f_4 + 16f_5 + 9f_6 + 4f_7 + f_8 + 2\gamma(e_{23465431})$$

for some $\gamma \in K$. Set

$$e = w_2, \quad h = -2(13(h_1 + 22h_2 + 52h_3 + 35h_4 + 46h_5 + 48h_6 + 41h_7 + 25h_8))$$

and

$$f = -(26f_1 + 41f_2 + 54f_3 + 25f_4 + 16f_5 + 9f_6 + 4f_7 + f_8) \\ = 33f_1 + 18f_2 + 5f_3 + 34f_4 + 43f_5 + 50f_6 + 55f_7 + 58f_8.$$

Then (e, h, f) is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ (see the proof of [35, Proposition 2.4]) and by applying Proposition 2.15 (with $y = e_{23465432}$ and $z = e_{23465431}$) we deduce that X is contained in an A_1-type subgroup of G.

Case 2. $\mathrm{rad}(V|_X) \neq 0$, $p \geqslant 53$.

If $p \geqslant 61$ then the dimension of each indecomposable summand of $V|_X$ is at least 60, which implies that the Jordan form of x has a block of size $n \geqslant 60$. This is a contradiction.

Next assume $p = 59$. Suppose W is a reducible indecomposable nonprojective summand of $V|_X$, so $\dim W \geqslant 58$ (see Corollary 2.4). In view of (32) and Lemma 2.1, we deduce that x has Jordan form $[J_{59}, J_i]$ on W for some odd integer i between 3 and 47. But this implies that $\dim W$ is even, so Corollary 2.3 implies that W has at least four composition factors and thus $i \geqslant 57$ (again, by Corollary 2.4). This is a contradiction. Therefore, we may assume that each indecomposable summand of $V|_X$ is either simple or projective. Clearly,

$$V|_X = U \oplus L_X(46) \oplus L_X(38) \oplus L_X(34) \oplus L_X(26) \oplus L_X(22) \oplus L_X(14) \oplus L_X(2)$$

is the only possibility. However, this implies that x_2 has trace -4 on V, which contradicts Proposition 2.8.

Now assume $p = 53$. As in the previous case, by applying Lemma 2.1 and Theorem 2.2, and by appealing to the self-duality of $V|_X$, it is straightforward to reduce to the case where the indecomposable summands of $V|_X$ are either simple or projective. Moreover, by considering the eigenvalues in (31), we deduce that

$$V|_X = W(i) \oplus L_X(38) \oplus L_X(34) \oplus L_X(26) \oplus L_X(22) \oplus L_X(14) \oplus L_X(2)$$

with $i \in \{6, 46\}$. By computing the trace of $x_3 \in X$, it follows that $i = 46$. It is now entirely straightforward to show that the $L_X(2)$ summand of $V|_X$ is an appropriate $\mathfrak{sl}_2$-subalgebra and the result follows via Proposition 2.14.

Case 3. $\mathrm{rad}(V|_X) \neq 0,\ p = 47.$

As in the previous case, we can quickly reduce to the situation where each indecomposable summand of $V|_X$ is simple or projective, in which case

$$V|_X = W(i) \oplus M_2 \oplus L_X(38) \oplus L_X(26) \oplus L_X(22) \oplus L_X(14) \oplus L_X(2)$$

with $i \in \{12, 34\}$ and $M_1 \in \{L_X(46), U\}$. By computing the trace of x_2 we deduce that $i = 34$ and $M_1 = L_X(46)$, in which case $V|_X$ is compatible with the containment of X in an A_1-type subgroup of G (see Table 2).

As usual, let W be the $L_X(2)$ summand of $V|_X$ and let $\{w_2, w_0, w_{-2}\}$ be a standard basis for W. We get

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8) \\ w_0 &= a_2(h_1 + 26h_2 + 3h_3 + 6h_4 + 31h_5 + 10h_6 + 37h_7 + 18h_8) \\ &\quad + a_3(e_{23354321} - e_{22454321}) \\ w_{-2} &= a_4(34f_1 + 38f_2 + 8f_3 + 16f_4 + 20f_5 + 11f_6 + 36f_7 + f_8) \\ &\quad + a_5(e_{22354321} - 2e_{13354321}). \end{aligned}$$

We may set $a_1 = 1$. By considering the action of x on this basis we deduce that $a_2 = 1$, $a_4 = 36$ and $a_5 = 45a_3$, so

$$\begin{aligned} w_2 &= e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 \\ w_0 &= h_1 + 26h_2 + 3h_3 + 6h_4 + 31h_5 + 10h_6 + 37h_7 + 18h_8 \\ &\quad + \gamma(e_{23354321} - e_{22454321}) \\ w_{-2} &= 36(34f_1 + 38f_2 + 8f_3 + 16f_4 + 20f_5 + 11f_6 + 36f_7 + f_8) \\ &\quad + 45\gamma(e_{22354321} - 2e_{13354321}) \end{aligned}$$

for some $\gamma \in K$. One now checks that the relations in (17) are satisfied and thus W is an $\mathfrak{sl}_2$-subalgebra of V. Set

$$e = w_2, \quad h = -2(h_1 + 26h_2 + 3h_3 + 6h_4 + 31h_5 + 10h_6 + 37h_7 + 18h_8)$$

and

$$\begin{aligned} f &= -36(34f_1 + 38f_2 + 8f_3 + 16f_4 + 20f_5 + 11f_6 + 36f_7 + f_8) \\ &= 45f_1 + 42f_2 + 41f_3 + 35f_4 + 32f_5 + 27f_6 + 20f_7 + 11f_8. \end{aligned}$$

Then (e, h, f) is an $\mathfrak{sl}_2$-triple over $\mathbb{Z}$ (see the proof of [35, Proposition 2.4]) and we can use Proposition 2.15 to conclude that X is contained in an A_1-type subgroup of G.

Case 4. $\mathrm{rad}(V|_X) \neq 0$, $p = 43$.

By arguing in the usual manner, it is straightforward to reduce to the case where each indecomposable summand of $V|_X$ is either simple or projective. By considering the eigenvalues in (31), we deduce that

$$V|_X = W(i) \oplus W(j) \oplus L_X(34) \oplus L_X(22) \oplus L_X(14) \oplus L_X(2)$$

with $i \in \{4, 38\}$ and $j \in \{16, 26\}$. By computing the trace of x_2, we see that $(i, j) = (38, 26)$ is the only option, in which case $V|_X$ is compatible with the desired containment of X in an A_1-type subgroup of G. As usual, we now construct the summand $W = L_X(2)$ of $V|_X$ in terms of a standard basis $\{w_2, w_0, w_{-2}\}$; it is easy to show that W is an appropriate $\mathfrak{sl}_2$-subalgebra and we can conclude by applying Proposition 2.14.

Case 5. $\mathrm{rad}(V|_X) \neq 0$, $p = 41$.

First assume that $V|_X$ has a reducible indecomposable nonprojective summand W. In the usual way, we deduce that the Jordan form of x on W is one of the following:

$$\begin{cases} [J_{41}^4, J_{27}], [J_{41}^4, J_{15}] \\ [J_{41}^3, J_3] \\ [J_{41}^2, J_{27}], [J_{41}^2, J_{15}], [J_{41}^2, J_3] \\ [J_{41}, J_{39}]. \end{cases}$$

If the Jordan form is either $[J_{41}^4, J_{27}]$ or $[J_{41}^4, J_{15}]$ then there is a unique such summand. Moreover, W has an odd number of composition factors and it is easy to see that this is incompatible with the self-duality of $V|_X$. Similar reasoning rules out the cases where x has Jordan form $[J_{41}^2, J_i]$. Finally, suppose x has Jordan form $[J_{41}^3, J_3]$ or $[J_{41}, J_{39}]$. Here the self-duality of $V|_X$ implies that

$$\mathrm{soc}(W) \cong W/\mathrm{soc}(W) = L_X(22) \oplus L_X(20) \oplus L_X(18)$$

or

$$\mathrm{soc}(W) \cong W/\mathrm{soc}(W) = L_X(20) \oplus L_X(18),$$

respectively. However, the existence of such a summand would mean that ξ^{20} is an eigenvalue of s on $C_V(x)$, which is not the case (see (31)). Therefore, we conclude that every indecomposable summand of $V|_X$ is either simple or projective. More precisely, in view of (31), it follows that

$$V|_X = W(i) \oplus W(j) \oplus L_X(38) \oplus L_X(26) \oplus L_X(14) \oplus L_X(2)$$

with $i \in \{6, 34\}$ and $j \in \{18, 22\}$. By computing the trace of x_3 we deduce that $(i, j) = (34, 22)$, in which case $V|_X$ is compatible with the containment of X in an A_1-type subgroup.

Let W be the $L_X(2)$ summand of $V|_X$ and let $\{w_2, w_0, w_{-2}\}$ be a standard basis. In the usual manner we deduce that

$$\begin{aligned}
w_2 &= a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8) \\
w_0 &= a_2(h_1 + 30h_2 + 10h_3 + 27h_4 + 22h_5 + 25h_6 + 36h_7 + 14h_8) \\
w_{-2} &= a_3(3f_1 + 8f_2 + 30f_3 + 40f_4 + 25f_5 + 34f_6 + 26f_7 + f_8) \\
&\quad + a_4(e_{22343221} - e_{12343321} + e_{12244321}).
\end{aligned}$$

Set $a_1 = 1$. By considering the action of x on this basis we get $a_2 = 36$ and $a_3 = 24$. Finally, one can check that the condition in (16) implies that $a_4 = 0$ and now the desired result follows from Proposition 2.14.

To complete the proof of the theorem, we may assume that $p = 37$ (recall that the case $p = 31$ was handled earlier in Examples 2.21 and 2.24).

Case 6. $\mathrm{rad}(V|_X) \neq 0$, $p = 37$.

As usual, let us first assume that $V|_X$ has a reducible indecomposable nonprojective summand W. By applying Lemma 2.1 and Theorem 2.2, we deduce that the Jordan form of x on W is one of the following:

$$[J_{37}^6, J_{23}], \quad [J_{37}^4, J_{23}], \quad [J_{37}^3, J_3], \quad [J_{37}^2, J_{23}], \quad [J_{37}^2, J_3].$$

In fact, the self-duality of $V|_X$ implies that $[J_{37}^3, J_3]$ is the only possibility, with

$$\mathrm{soc}(W) \cong W/\mathrm{soc}(W) = L_X(16) \oplus L_X(18) \oplus L_X(20).$$

But if this is a summand of $V|_X$ then ξ^{20} is an eigenvalue of s on $C_V(x)$, contradicting (31). Therefore, we have reduced to the case where each indecomposable summand of $V|_X$ is simple or projective. Again, by considering (31) we deduce that

$$V|_X = W(i) \oplus W(j) \oplus W(k) \oplus L_X(22) \oplus L_X(2) \qquad (33)$$

with $i \in \{2, 34\}$, $j \in \{10, 26\}$ and $k \in \{14, 22\}$.

We claim that $(i, j, k) = (34, 26, 14)$, in which case the decomposition of $V|_X$ is compatible with the containment of X in an A_1 subgroup of G. One can check that all of the eight decompositions above are compatible with the trace of x_2 and x_3, so we consider the traces of elements of larger order. Let $y \in X$ be an element of order 19. In each case it is straightforward to compute the eigenvalues of y on V. Using Litterick's algorithm in [21], we can compute the eigenvalues on V of every element in G of order 19 and in this way we deduce that $(i, j, k) = (34, 26, 14)$ as claimed.

Let W be the $L_X(2)$ summand of $V|_X$ with standard basis $\{w_2, w_0, w_{-2}\}$. The spaces

$$\ker(x-1) \cap E_2, \quad \ker((x-1)^2) \cap E_0, \quad \ker((x-1)^3) \cap E_{-2}$$

have respective dimensions 2, 2 and 3, which gives

$$\begin{aligned} w_2 &= a_1(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8) \\ &\quad + a_2(e_{22343221} - e_{12343321} + e_{12244321}) \\ w_0 &= a_3(h_1 + 24h_2 + 6h_3 + 15h_4 + 4h_5 + 34h_6 + 31h_7 + 32h_8) \\ &\quad + a_4(e_{22343211} + 24e_{12343221} + 25e_{12243321}) \\ w_{-2} &= a_5(22f_1 + 10f_2 + 21f_3 + 34f_4 + 14f_5 + 8f_6 + 16f_7 + f_8) \\ &\quad + a_6(e_{22343210} + 13e_{12243221} - e_{12233321}) \\ &\quad + a_7(e_{12343211} + 23e_{12243221} + 2e_{12233321}). \end{aligned}$$

By considering the action of x on this basis, we deduce that $a_3 = 28a_1$, $a_4 = 3a_2$, $a_5 = 16a_1$ and $a_7 = 6a_2 + a_6$. The condition in (16) yields the equations

$$\begin{aligned} 16a_1^2a_2 + 3a_1^2a_6 &= 0 \\ 19a_1^2a_2 + 6a_1^2a_6 &= 0. \end{aligned}$$

If $a_1 \neq 0$ then these equations imply that $a_2 = a_6 = 0$, so we can set $a_1 = 1$ and then apply Proposition 2.14 to show that X is contained in an A_1-type subgroup. On the other hand, if $a_1 = 0$ then we may assume $a_2 = 1$, so

$$\begin{aligned} w_2 &= e_{22343221} - e_{12343321} + e_{12244321} \\ w_0 &= 3(e_{22343211} + 24e_{12343221} + 25e_{12243321}) \\ w_{-2} &= a_6(e_{22343210} + 13e_{12243221} - e_{12233321}) \\ &\quad + (6 + a_6)(e_{12343211} + 23e_{12243221} + 2e_{12233321}). \end{aligned}$$

It is straightforward to check that $W \subseteq \langle e_\alpha \mid \alpha \in \Phi^+(G) \rangle$ is a subalgebra and this puts us in case (ii) of the theorem.

This completes the proof of Theorem 7.1. □

8. Proof of Theorem 1

In this final section, we complete the proof of Theorem 1. In view of Theorem 3.1, we may assume that G is of type F_4, E_6, E_7 or E_8. Moreover, by our work in Sections 4–7, it remains to handle the cases appearing in Table 4. In each of these cases, X stabilizes a nonzero subalgebra $W \subseteq \langle e_\alpha \mid \alpha \in \Phi^+(G) \rangle$ of $V = \mathrm{Lie}(G)$ and by applying Proposition 2.18 we can assume that X is contained in a proper parabolic subgroup $P = QL$ of G with unipotent radical Q and Levi factor L. The following result, when combined with Theorem 2, completes the proof of Theorem 1. (Recall that Craven [9] has constructed a subgroup X satisfying the conditions in parts (ii) and (iii) of Theorem 1, and he has established its uniqueness up to conjugacy; see Remark 1(b).)

THEOREM 8.1. *Let G be a simple exceptional algebraic group of adjoint type over an algebraically closed field of characteristic $p > 0$. Let $X = \mathrm{PSL}_2(p)$ be a subgroup of G containing a regular unipotent element of G and let $V = \mathrm{Lie}(G)$ be the adjoint module. If X is contained in a proper parabolic subgroup $P = QL$ of G, then either*

(i) *$G = E_6$, $p = 13$, $L' = D_5$ and $V|_X = W(10)^2 \oplus W(4)$; or*

(ii) *$G = E_7$, $p = 19$, $L' = E_6$ and $V|_X = W(16) \oplus W(14) \oplus W(8) \oplus U$.*

Proof. We may assume that P is minimal with respect to containing X. Let us write $\pi : P \to P/Q$ for the quotient map and identify L with P/Q. By arguing as in the first paragraph in the proof of Theorem 2 (see the end of Section 2), we deduce that $\pi(X)$ is contained in an A_1-type subgroup H of L'. In addition, Theorem 2 implies that X is not contained in an A_1-type subgroup of G, so $(G, p, V|_X)$ must be one of the cases in Table 4. As noted in the proof of Theorem 2, the composition factors of $V|_H$ can be read off from [15, Tables 1–5] and this imposes restrictions on the composition factors of $V|_X$. By considering each possibility for (G, L') in turn, comparing composition factors with Table 4, we show that the cases labelled (i) and (ii) in the statement of the theorem are the only compatible options.

First assume $(G, p) = (F_4, 13)$, so the composition factors of $V|_X$ are $L_X(10)^3$, $L_X(8)$, $L_X(2)^3$ and $L_X(0)$. By inspecting [15, Table 2] it is easy to see that there is no compatible Levi subgroup L. Similarly, if $(G, p) = (E_8, 37)$ then the composition factors of $V|_X$ are given in (19) and thus we can eliminate this case by repeating the argument in the proof of Theorem 2.

Next suppose $(G, p) = (E_6, 13)$. The three possibilities for $V|_X$ (and their composition factors) are as follows:

$$\begin{aligned} W(10) \oplus W(8) \oplus W(2) &: L_X(10)^3, L_X(8)^3, L_X(4), L_X(2)^4, L_X(0) \\ W(10) \oplus W(4) \oplus W(2) &: L_X(10)^3, L_X(8)^2, L_X(6), L_X(4)^2, L_X(2)^3, L_X(0) \\ W(10) \oplus W(10) \oplus W(4) &: L_X(10)^4, L_X(8), L_X(6), L_X(4)^2, L_X(2)^2, L_X(0)^2. \end{aligned}$$

In all three cases, we see that $V|_X$ has at most two trivial composition factors, so [**15**, Table 3] implies that

$$L' = A_2A_1^2,\ A_2^2A_1,\ A_4A_1 \text{ or } D_5.$$

If $L' = A_2A_1^2$ then $V|_X$ has five or more $L_X(2)$ factors, which is incompatible with all three possibilities for $V|_X$. Similarly, if $L' = A_4A_1$ then there are too many $L_X(4)$ factors, and we can rule out $L' = A_2^2A_1$ because we would get $L_X(1)$ composition factors, which is absurd. Finally, suppose $L' = D_5$. Since the Weyl module $W_X(14)$ has an $L_X(10)$ composition factor, we see that $V|_X$ has four such factors and thus

$$V|_X = W(10) \oplus W(10) \oplus W(4)$$

is the only option.

Finally, let us assume $(G, p) = (E_7, 19)$. The three possibilities for $V|_X$ are as follows:

$$\begin{aligned} W(8) \oplus W(4) \oplus W(2) \oplus U &: L_X(16)^2, L_X(14)^2, L_X(12), L_X(10), L_X(8)^3, \\ &\quad\ L_X(4)^2, L_X(2)^2, L_X(0)^2 \\ W(16) \oplus W(10) \oplus W(4) \oplus U &: L_X(16)^3, L_X(14), L_X(12), L_X(10)^2, L_X(8), \\ &\quad\ L_X(6), L_X(4)^2, L_X(2), L_X(0)^3 \\ W(16) \oplus W(14) \oplus W(8) \oplus U &: L_X(16)^3, L_X(14)^2, L_X(10), L_X(8)^3, L_X(4), \\ &\quad\ L_X(2)^2, L_X(0)^3. \end{aligned}$$

By inspecting [**15**, Table 4], counting the number of trivial composition factors, we quickly reduce to a small number of possibilities for L'. By considering nontrivial composition factors, it is straightforward to reduce further to the case $L' = E_6$. For example, we can rule out $L' = A_6$ because there would be too many $L_X(4)$ factors. Similarly, $L' = D_5A_1$ is out because we would have an $L_X(6)$ and at least three $L_X(8)$ factors, which is not compatible with any of the three possibilities above. We can rule out $L' = D_6$ because it would imply that $V|_X$ has an $L_X(5)$ factor. Finally, suppose $L' = E_6$. Here $V|_X$ has at least three $L_X(16)$ and $L_X(8)$ composition factors, so

$$V|_X = W(16) \oplus W(14) \oplus W(8) \oplus U$$

is the only possibility. □

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References

[1] J. L. Alperin, *Local Representation Theory*, Cambridge Studies in Advanced Mathematics, 11 (Cambridge University Press, Cambridge, 1986).

[2] A. Borel and J. Tits, 'Éléments unipotents et sous-groupes paraboliques de groupes réductifs. I', *Invent. Math.* **12** (1971), 95–104.

[3] W. Bosma, J. Cannon and C. Playoust, 'The Magma algebra system I: the user language', *J. Symbolic Comput.* **24** (1997), 235–265.

[4] N. Bourbaki, *Groupes et algebrès de Lie (Chapitres 4, 5 et 6)* (Hermann, Paris, 1968).

[5] R. W. Carter, *Simple Groups of Lie Type* (Wiley, London, 1972).

[6] R. W. Carter, *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters* (Wiley, London, 1985).

[7] A. M. Cohen and R. L. Griess, 'On finite subgroups of the complex Lie group of type E_8', *Proc. Sympos. Pure Math.* **47** (1987), 367–405.

[8] B. N. Cooperstein, 'Maximal subgroups of $G_2(2^n)$', *J. Algebra* **70** (1981), 23–36.

[9] D. A. Craven, 'Maximal PSL_2 subgroups of exceptional groups of Lie type', *Mem. Amer. Math. Soc.* to appear.

[10] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups, Number 3*, Mathematical Surveys and Monographs, 40 (Amererican Mathematical Society, Providence, RI, 1998).

[11] R. Guralnick and G. Malle, 'Rational rigidity for $E_8(p)$', *Compos. Math.* **150** (2014), 1679–1702.

[12] G. J. Janusz, 'Indecomposable representations of groups with a cyclic Sylow subgroup', *Trans. Amer. Math. Soc.* **125** (1966), 288–295.

[13] P. B. Kleidman, 'The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups', *J. Algebra* **117** (1988), 30–71.

[14] R. Lawther, 'Jordan block sizes of unipotent elements in exceptional algebraic groups', *Comm. Algebra* **23** (1995), 4125–4156.
[15] R. Lawther and D. M. Testerman, 'A_1 subgroups of exceptional algebraic groups', *Mem. Amer. Math. Soc.* **674** (1999), viii+131 pp.
[16] M. W. Liebeck, J. Saxl and D. M. Testerman, 'Simple subgroups of large rank in groups of Lie type', *Proc. Lond. Math. Soc.* **72** (1996), 425–457.
[17] M. W. Liebeck and G. M. Seitz, 'Subgroups generated by root elements in groups of Lie type', *Ann. Math.* **139** (1994), 293–361.
[18] M. W. Liebeck and G. M. Seitz, 'On the subgroup structure of exceptional groups of Lie type', *Trans. Amer. Math. Soc.* **350** (1998), 3409–3482.
[19] M. W. Liebeck and G. M. Seitz, 'The maximal subgroups of positive dimension in exceptional algebraic groups', *Mem. Amer. Math. Soc.* **802** (2004), vi+227 pp.
[20] M. W. Liebeck and D. M. Testerman, 'Irreducible subgroups of algebraic groups', *Q. J. Math.* **55** (2004), 47–55.
[21] A. J. Litterick, 'Finite simple subgroups of exceptional algebraic groups', PhD Thesis, Imperial College London, 2013.
[22] A. J. Litterick, 'On non-generic finite subgroups of exceptional algebraic groups', *Mem. Amer. Math. Soc.* **1207** (2018), v+156 pp.
[23] G. Malle, 'The maximal subgroups of ${}^2F_4(q^2)$', *J. Algebra* **139** (1991), 52–69.
[24] G. Malle and D. Testerman, *Linear Algebraic Groups and Finite Groups of Lie Type*, Cambridge Advanced Studies in Mathematics, 133 (Cambridge University Press, Cambridge, 2011).
[25] R. V. Moody and J. Patera, 'Characters of elements of finite order in Lie groups', *SIAM J. Algebr. Discrete Methods* **5** (1984), 359–383.
[26] R. Proud, J. Saxl and D. M. Testerman, 'Subgroups of type A_1 containing a fixed unipotent element in an algebraic group', *J. Algebra* **231** (2000), 53–66.
[27] J. Saxl and G. M. Seitz, 'Subgroups of algebraic groups containing regular unipotent elements', *J. Lond. Math. Soc.* **55** (1997), 370–386.
[28] G. M. Seitz, 'Unipotent elements, tilting modules, and saturation', *Invent. Math.* **141** (2000), 467–502.
[29] G. M. Seitz and D. M. Testerman, 'Extending morphisms from finite to algebraic groups', *J. Algebra* **131** (1990), 559–574.
[30] G. M. Seitz and D. M. Testerman, 'Subgroups of type A_1 containing semiregular unipotent elements', *J. Algebra* **196** (1997), 595–619.
[31] J.-P. Serre, 'Exemples de plongements des groupes $\mathrm{PSL}_2(\mathbb{F}_p)$ dans des groupes de Lie simples', *Invent. Math.* **124** (1996), 525–562.
[32] R. Steinberg, 'Representations of algebraic groups', *Nagoya Math. J.* **22** (1963), 33–56.
[33] R. Steinberg, 'Endomorphisms of linear algebraic groups', *Mem. Amer. Math. Soc.* **80** (1968), 108 pp.
[34] D. M. Testerman, 'The construction of the maximal A_1's in the exceptional algebraic groups', *Proc. Amer. Math. Soc.* **116** (1992), 635–644.
[35] D. M. Testerman, 'A_1-type overgroups of elements of order p in semisimple algebraic groups and the associated finite groups', *J. Algebra* **177** (1995), 34–76.
[36] D. M. Testerman and A. Zalesski, 'Irreducibility in algebraic groups and regular unipotent elements', *Proc. Amer. Math. Soc.* **141** (2013), 13–28.
[37] B. Weisfeiler, 'On one class of unipotent subgroups of semisimple algebraic groups', Preprint, 2000, arXiv:0005149.