

DUALITY FOR CONVEX POLYTOPES

A. B. ROMANOWSKA , P. ŚLUSARSKI and J. D. H. SMITH

(Received 21 May 2007; accepted 1 October 2007)

Communicated by M. G. Jackson

Abstract

This paper establishes a duality between the category of polytopes (finitely generated real convex sets considered as barycentric algebras) and a certain category of intersections of hypercubes, considered as barycentric algebras with additional constant operations.

2000 *Mathematics subject classification*: primary 18C05, 52A01; secondary 08A35, 08A05, 18A35.

Keywords and phrases: duality, modes, affine spaces, barycentric algebras, convex sets, simplex, hypercube.

1. Introduction

This paper represents a continuation of the work of Pszczoła and the latter two authors [10–12]. The main motivation for this work is the search for a duality theory of barycentric algebras. The aforementioned papers contain some partial results in this direction. In particular, from [11] one obtains a duality for finite-dimensional real simplices, as finitely generated free barycentric algebras. Another partial case was considered in [12], where a duality for the class of quadrilaterals was established. Pontryagin duality for semilattices [6] may also be considered as a duality for a limited class of barycentric algebras.

In this paper, we offer a duality for real convex polytopes, considered as (cancellative) barycentric algebras. As in the earlier cases of simplices and quadrilaterals, this duality is again of ‘classical’ type, and is given by an infinite schizophrenic object, the unit real interval. In the new duality, the class of representation spaces is also a class of certain polytopes, namely a class of certain intersections of hypercubes. However, as representation spaces these polytopes are considered as barycentric algebras with additional constant operations (and with the corresponding homomorphisms preserving these constants). Nevertheless our duality

This paper was written within the framework of INTAS project no. 03-51-4110.

© 2009 Australian Mathematical Society 1446-7887/2009 \$16.00

is not a direct generalization of the duality for quadrilaterals. The description of the representation spaces dual to quadrilaterals given in [12] is too complicated to use in the general case of all polytopes. We propose a somewhat different approach, closer in spirit to the description of dual spaces for simplices given in [11], using a duality for real affine spaces described in this paper, and the fact that convex polytopes embed into affine spaces [14, Ch. 7]. In a sense, three dualities are carried out in parallel fashion: for simplices, for polytopes, and for barycentric algebra reducts of finitely generated real affine spaces. These dualities do not involve any topology or additional relations. In subsequent work, we hope to show that the current approach can be extended to obtain dualities for some more general classes of convex sets.

Section 2 gives a brief introduction to real affine spaces, convex sets and barycentric algebras. A special description of convex polytopes is given in Section 3. Section 4 provides the background necessary for the dualities considered in this paper. The next two sections deal with the first and second duals. The main result (Theorem 6.5) establishes the full duality between the category of all convex polytopes and the category of corresponding representation spaces.

We use notation and conventions similar to those of [15] and the earlier papers. For details and further information on affine spaces and barycentric algebras, we refer the reader to those papers, and to the monographs [13, 14]. For convex polytopes, see [1, 5].

2. Affine spaces, convex sets and barycentric algebras

Real *affine spaces* may be defined as the subreducts (subalgebras of reducts) $(A, \underline{\mathbb{R}})$ of modules $(A, +, \mathbb{R})$, where $\underline{\mathbb{R}}$ consists of the set of binary operations

$$\underline{r}: A^2 \rightarrow A; (x, y) \mapsto xy\underline{r} = x(1 - r) + yr$$

for each $r \in \mathbb{R}$. The class of all real affine spaces forms a variety, denoted by $\underline{\mathbb{R}}$. Equivalently, the variety $\underline{\mathbb{R}}$ is defined as the variety of Mal'tsev modes with binary operations \underline{r} for $r \in \mathbb{R}$ satisfying certain identities that can easily be deduced from more general theorems [13, Corollary 255] and [14, Theorem 6.3.4]). Each real affine space $(A, \underline{\mathbb{R}})$ generated by a finite set, say $X = \{x_0, x_1, \dots, x_k\}$, is in fact the free real affine space $X\underline{\mathbb{R}}$ (or $(k + 1)\underline{\mathbb{R}}$) on X . Its underlying set is then described as

$$\left\{ x_0r_0 + \dots + x_kr_k \mid r_i \in \mathbb{R}, \sum_{i=0}^k r_i = 1 \right\}.$$

The affine space $(k + 1)\underline{\mathbb{R}}$ is isomorphic to the affine space \mathbb{R}^k freely generated by the standard set of generators

$$e_0 = (0, \dots, 0), \quad e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1).$$

Let I denote the closed unit interval $[0, 1] \subset \mathbb{R}$, and let I° denote the open unit interval $]0, 1[\subset \mathbb{R}$. Convex subsets of real affine spaces can be defined as subreducts (C, \underline{I}°) of affine spaces $(A, \underline{\mathbb{R}})$. The variety \underline{B} generated by the class of real convex

sets (C, \underline{I}^o) is called the variety of *barycentric algebras*. The class of convex sets forms the subquasivariety \underline{C} of cancellative barycentric algebras, and cancellative barycentric algebras are known to be precisely those barycentric algebras that embed into real affine spaces as \underline{I}^o -subreducts. (Romanowska and Smith [13, Section 2.6] discuss the universal affine space extension of a cancellative barycentric algebra.) The variety \underline{B} is also considered as a category, having barycentric algebras as objects and barycentric algebra homomorphisms as morphisms. The (full) subcategory \underline{S} of finite-dimensional simplices and the (full) subcategory \underline{P} of convex polytopes (finitely generated cancellative barycentric algebras) play an important role in this paper. We are also interested in category \underline{B} of cancellative barycentric algebras with two constant operations denoted by $\bar{0}$ and $\bar{1}$, and with barycentric algebra homomorphisms preserving these constants as morphisms. In particular, we will consider the class consisting of subalgebras of hypercubes $\hat{I}^{n+1} = (I^{n+1}, \underline{I}^o, \bar{0}, \bar{1})$, equipped with constants $\bar{0} = (0, \dots, 0)$ and $\bar{1} = (1, \dots, 1)$.

Finitely generated free barycentric algebras may be characterized by the following theorem (see [9, 13, Section 2.1, 14, Section 5.8]).

THEOREM 2.1. *Let $X = \{x_0, x_1, \dots, x_k\}$ be a finite set. Then the following objects coincide:*

- (1) *the free barycentric algebra XB over X ;*
- (2) *the simplex Δ_k spanned by X ;*
- (3) *the \underline{I}^o -subreduct generated by X of the free affine space $X\mathbb{R}$ generated by X .*

The elements of the \underline{I}^o -subreduct are described as follows:

$$\left\{ x_0 r_0 + \dots + x_k r_k \mid r_i \in I, \sum_{i=0}^k r_i = 1 \right\}.$$

We say that a convex set A is *k-dimensional*, and write $\dim A = k$, if k is the smallest positive integer such that A embeds as a subreduct into the affine space $(k+1)\mathbb{R}$. In such a situation, the convex set A generates the affine space $(k+1)\mathbb{R}$. Each convex polytope A is finite-dimensional and, considered as a subset of the topological space \mathbb{R}^k with the usual topology, it is closed. The minimal set of generators of A is uniquely determined: it is the set of vertices of A . If A has n vertices, then we say that A is *n-generated*. If $\dim A = k$, then the number of vertices is at least $k+1$.

3. Convex polytopes

First, we will provide a description of convex polytopes of dimension at least 1. We will use the concept of a wall of a polytope, however, in both geometric and algebraic senses [13, 14]. Recall that a *wall* of a polytope A is a subalgebra, say B , of A such that if $b_1 b_2 \underline{r} \in B$ for some $b_1, b_2 \in A$ and $r \in I^o$, then $b_1, b_2 \in B$. In geometry, walls are also called ‘faces’ and maximal faces are called ‘facets’.

The following theorem (whose proof we sketch) seems to be folklore in the theory of convex polytopes¹. Let $G(A)$ denote the set of vertices (the minimal set of generators) of a given convex polytope A .

THEOREM 3.1. *Let A be a k -dimensional convex polytope with the set $G(A)$ of $n + 1$ vertices, where $n \geq k$. Let g be an element of $G(A)$. Then A is the union of k -dimensional simplices, each generated by a $(k + 1)$ -element subset of $G(A)$ containing g . Moreover, any two of these simplices have a common wall that is a simplex containing g .*

PROOF. The theorem is obvious for $k = 2$. Assume now that $k > 2$, and that the theorem holds for $(k - 1)$ -dimensional convex polytopes. In particular, the theorem holds for the facets of A that do not contain g . Consider the set S of $(k - 1)$ -dimensional simplices subdividing these facets. The required division of A is obtained by taking the simplices generated by g and the vertices of each of the $(k - 1)$ -dimensional simplices from the set S . □

For a fixed g , each simplex of the division described in Theorem 3.1 has dimension k , and any two of them are isomorphic. The set of vertices of each such simplex will be called a *frame of A centered in g* or a *g -frame*. The element g will be called its *center*.

COROLLARY 3.2. *Suppose that $n \geq k$. Then each vertex of a k -dimensional and $(n + 1)$ -generated convex polytope A is the center of a frame of A .*

Each frame F of A generates a subalgebra of A that is isomorphic to a free barycentric algebra (a k -dimensional simplex). At the same time, it (freely) generates the k -dimensional real affine space $\mathbb{R}(A) = F\mathbb{R}$. Obviously, A is an \underline{I}^o -subreduct of the affine space $\mathbb{R}(A)$. Then $\mathbb{R}(A)$ is called the *canonical extension* of A . Choose a generator $g_0 \in G(A)$ and a g_0 -frame $\{g_0, g_1, \dots, g_k\}$, and let $G(A) = \{g_0, g_1, \dots, g_k, \dots, g_n\}$. Corollary 3.2 allows us to consider the generators g_0, g_1, \dots, g_k as the (free) generators x_0, \dots, x_k of the affine space $\mathbb{R}(A)$. Then each of the generators g_i , for $i = k + 1, \dots, n$, can be described in $\mathbb{R}(A)$ as

$$g_i = x_0r_{i0} + \dots + x_kr_{ik}, \tag{3.1}$$

where $\sum_{j=0}^k r_{ij} = 1$. (Other elements of A have a similar description in $\mathbb{R}(A)$.) Note that passage from one frame of A to another will induce an automorphism of the canonical affine space extension $\mathbb{R}(A)$, determined by a bijective mapping between the first and second frames, assigning the center of the second frame to the center of the first.

THEOREM 3.3. *Suppose that A and B are convex polytopes. Then each barycentric algebra homomorphism $h : A \rightarrow B$ extends uniquely (to within isomorphism) to an affine space homomorphism $\bar{h} : \mathbb{R}(A) \rightarrow \mathbb{R}(B)$ between the corresponding canonical affine space extensions.*

¹ We are grateful to G. Bergman and B. Grünbaum for discussion concerning this result.

PROOF. Let F be a frame of A . Then the restriction

$$h|_F : F \rightarrow B \hookrightarrow \mathbb{R}(B)$$

extends uniquely (up to isomorphism) to $\bar{h} : \mathbb{R}(A) \rightarrow \mathbb{R}(B)$. Moreover, $\bar{h}|_A = h$. \square

4. Duality

Let \mathcal{A} and \mathcal{X} be categories. We say that there is a *full dual equivalence*, or simply *full duality*, between \mathcal{A} and \mathcal{X} if there are contravariant functors

$$D : \mathcal{A} \longrightarrow \mathcal{X} \quad \text{and} \quad E : \mathcal{X} \longrightarrow \mathcal{A}$$

such that both $DE = E \circ D$ and $ED = D \circ E$ are naturally isomorphic with the corresponding identity functors on \mathcal{A} and \mathcal{X} , respectively (compare [7] and [8, p. 91]). In the cases of current interest, \mathcal{A} is a category of convex sets with the corresponding homomorphisms as morphisms, while the category \mathcal{X} of representation spaces is a certain category of subalgebras of hypercubes \hat{I}^{k+1} with constants, in which the morphisms are barycentric algebra homomorphisms preserving these constants. The functors D and E are defined on objects and morphisms by

$$\begin{aligned} D : (f : A \rightarrow B) &\mapsto (f^D : \mathcal{A}(B, I) \rightarrow \mathcal{A}(A, I); x \mapsto fx), \\ E : (\varphi : \hat{X} \rightarrow \hat{Y}) &\mapsto (\varphi^E : \mathcal{X}(\hat{Y}, \hat{I}) \rightarrow \mathcal{X}(\hat{X}, \hat{I}); \alpha \mapsto \varphi\alpha). \end{aligned}$$

In left-handed notation,

$$D(A) = A^D = \mathcal{A}(A, I) \quad \text{and} \quad D(f) = f^D.$$

The natural isomorphisms e of DE with the identity functor and ε of ED with the identity functor are given by the evaluations

$$\begin{aligned} e_A : A &\rightarrow A^{DE}; & a &\mapsto (ae_A : x \mapsto ax), \\ \varepsilon_{\hat{X}} : \hat{X} &\rightarrow \hat{X}^{ED}; & x &\mapsto (x\varepsilon_{\hat{X}} : \alpha \mapsto x\alpha). \end{aligned}$$

In left-handed notation,

$$e_A(a)(x) = x(a) \quad \text{and} \quad \varepsilon_{\hat{X}}(x)(\alpha) = \alpha(x).$$

Observe that such dualities are given by the schizophrenic object I . Note that the so-called ‘natural dualities’ considered in [2–4] are of similar type. However, they require finite schizophrenic objects and satisfaction of certain additional conditions that do not necessarily obtain for the cases considered in this paper.

Two known dualities for convex sets concern the classes $\mathcal{A} = \underline{\underline{S}}$ of finite-dimensional simplices and $\mathcal{A} = \underline{\underline{Q}}$ of quadrilaterals, while the duality under consideration in this paper deals with the class $\mathcal{A} = \underline{\underline{P}}$ of convex polytopes. All three dualities are based on the following theorem, an obvious corollary of [13, Proposition 159]).

THEOREM 4.1. *For each pair A and B of cancellative barycentric algebras, the set $\underline{\underline{B}}(B, A)$ is a cancellative barycentric algebra, a subalgebra of A^B .*

Note also the following.

LEMMA 4.2. *Let A and B be convex polytopes. Let $h \in \underline{\underline{B}}(B, A)$. Then h^D is a morphism of $\check{\underline{\underline{B}}}$. Similarly, if \hat{A} and \hat{B} are convex polytopes with constants and $\hat{\varphi} \in \check{\underline{\underline{B}}}(\hat{B}, \hat{A})$, then $\hat{\varphi}^E$ is a $\underline{\underline{B}}$ -morphism.*

PROOF. We omit the routine details of the proof based on the following two facts. The barycentric algebra structure on the set I^A is defined by

$$r : (I^A)^2 \rightarrow I^A; \quad (f_1, f_2) \mapsto f_1 f_2 r =: f,$$

where

$$f : A \rightarrow I; \quad x \mapsto x f = x(f_1 f_2 r) = x^{f_1} x^{f_2} r,$$

for each $r \in I^o$. Then the constants $\bar{0}$ and $\bar{1}$ in $\underline{\underline{B}}(A, I)$ are defined to be the mappings $A \rightarrow \{0\}$ and $A \rightarrow \{1\}$, respectively. \square

5. The first dual

In this section, we describe the functor D in the case $\mathcal{A} = \underline{\underline{P}}$, along with the category $\mathcal{X} = D(\underline{\underline{P}})$ of representation spaces.

The results of Section 3 show that, for each convex polytope A , there is a uniquely defined canonical affine space extension $\mathbb{R}(A)$. Moreover, each barycentric homomorphism $\varphi : A \rightarrow I$ of A^D extends uniquely to an affine space homomorphism $\bar{\varphi} : \mathbb{R}(A) \rightarrow \mathbb{R}(I)$ of the set

$$\bar{D}(A) = \{\bar{\varphi} : \mathbb{R}(A) \rightarrow \mathbb{R}(I) = \mathbb{R} \mid \varphi \in A^D\} \subseteq \underline{\underline{\mathbb{R}}}(\mathbb{R}(A), \mathbb{R}).$$

Note that $\mathbb{R}(A) \cong \mathbb{R}^k$ if A is k -dimensional and, by results of [11],

$$\underline{\underline{\mathbb{R}}}(\mathbb{R}(A), \mathbb{R}) \cong \mathbb{R}^{k+1}.$$

With respect to the I^o -operations, $\bar{D}(A)$ is a barycentric algebra. Moreover, the following lemma holds.

LEMMA 5.1. *The mapping*

$$\iota : A^D \rightarrow \bar{D}(A); \quad \varphi \mapsto \bar{\varphi} \tag{5.1}$$

is a barycentric algebra isomorphism.

The homomorphisms $\bar{\varphi}$ are characterized among the homomorphisms ψ of $\underline{\underline{\mathbb{R}}}(\mathbb{R}(A), \mathbb{R})$ as those satisfying $\psi(A) \subseteq I$. In other words,

$$\bar{D}(A) = \{\psi \in \underline{\underline{\mathbb{R}}}(\mathbb{R}(A), \mathbb{R}) \mid A\psi \subseteq I\}.$$

Similarly, for any barycentric subreduct S of the affine space $\mathbb{R}(A)$ that is isomorphic to a k -dimensional simplex and contains A as a subalgebra, one also has $S^D \cong \bar{D}(S)$. Moreover, the homomorphisms $\bar{\varphi}$ of $\bar{D}(S)$ that correspond to the homomorphisms φ of S^D are precisely the elements of

$$\bar{D}(S) = \{\psi \in \underline{\mathbb{R}}(\mathbb{R}(S), \mathbb{R}) \mid S\psi \subseteq I\}.$$

It is obvious that $\bar{D}(S) \subseteq \bar{D}(A)$. As $\dim \bar{D}(S) = k + 1$, it follows also that $\dim \bar{D}(A) = k + 1$.

Each homomorphism $\bar{\varphi}$ of $\bar{D}(A)$ is determined by the $(k + 1)$ -tuple $(\hat{x}_0, \dots, \hat{x}_k)$ of images $\hat{x}_i = x_i\varphi$ of the generators x_0, \dots, x_k of $\mathbb{R}(A)$. (We retain the notation of Section 3.) In particular,

$$\hat{g}_i := g_i\varphi = x_0r_{i0}\varphi + \dots + x_kr_{ik}\varphi = \hat{x}_0r_{i0} + \dots + \hat{x}_kr_{ik},$$

and $\hat{g}_i \in I$ for each $i = 0, \dots, n$. In other words, the following conditions are satisfied for each $i = 0, \dots, n$:

$$0 \leq \hat{x}_0r_{i0} + \dots + \hat{x}_kr_{ik} \leq 1.$$

In particular, for $i = 0, \dots, k$,

$$0 \leq \hat{x}_i \leq 1.$$

This leads to the definition of a barycentric algebra \hat{A} as

$$\left\{ (y_0, \dots, y_k) \in \mathbb{R}^{k+1} \mid 0 \leq \sum_{j=0}^k y_jr_{ij} \leq 1 \text{ for all } i = 0, \dots, n \right\}, \tag{5.2}$$

where the r_{ij} are the coefficients of g_i . Note that \hat{A} contains $\bar{0} = (0, \dots, 0)$ and $\bar{1} = (1, \dots, 1)$, and hence also all $\bar{r} = (r, \dots, r)$ for r in I (corresponding to the constant homomorphisms φ mapping all elements x_i to r). Note also that the inequalities

$$0 \leq y_i \leq 1, \tag{5.3}$$

for $i = 0, \dots, k$, describe the hypercube I^{k+1} . In the case $n = k$ (that is, if A is a k -dimensional simplex Δ_k), then \hat{A} is isomorphic to the hypercube \hat{I}^{k+1} , as shown in [11]. Each pair of equations

$$\sum_{j=0}^k y_jr_{ij} = 0 \quad \text{and} \quad \sum_{j=0}^k y_jr_{ij} = 1, \tag{5.4}$$

for $i = k + 1, \dots, n$, describes two parallel hyperplanes H_{i0} and H_{i1} in \mathbb{R}^{k+1} , the first containing $\bar{0}$ and the second containing $\bar{1}$. Let B_i be the subset of \mathbb{R}^{k+1} satisfying

$$0 \leq y_0r_{i0} + \dots + y_kr_{ik} \leq 1, \tag{5.5}$$

the *sandwich* bounded by the two hyperplanes H_{i_0} and H_{i_1} . It follows that \hat{A} is the intersection of the hypercube I^{k+1} and the $n - k$ sandwiches B_i for $i = k + 1, \dots, n$, each determined by one of the generators g_i . Equivalently, \hat{A} is the intersection of $n + 1$ sandwiches. In particular, \hat{A} is a convex polytope. The set \hat{A} may also be considered as the intersection of hyperparallelepipeds, each isomorphic to a $(k + 1)$ -dimensional hypercube. To see this, first note that by Theorem 3.1, the convex polytope A is the union of subalgebras A_i , each generated by a g_0 -frame of A for a fixed generator g_0 , and each isomorphic to a k -dimensional simplex. Denote such a union by $\bigoplus_{i \in I} A_i$. Then

$$A_i^D \cong \bar{D}(A_i) \cong \hat{I}^{k+1},$$

and any two of the $\bar{D}(A_i)$ have a non-empty intersection. Moreover,

$$A^D = \left(\bigoplus_{i \in I} A_i \right)^D \cong \bigcap_{i \in I} (A_i^D).$$

These observations lead to the following proposition.

PROPOSITION 5.2. *The mapping*

$$h : \bar{D}(A) \rightarrow \hat{A}; \quad \bar{\varphi} \mapsto (x_0\varphi, \dots, x_k\varphi)$$

is a barycentric algebra isomorphism preserving constants. Moreover, \hat{A} is a convex polytope, the intersection of subreducts of the affine space $\hat{\mathbb{R}}^{k+1}$ with two constants $\bar{0}$ and $\bar{1}$, each isomorphic to the hypercube \hat{I}^{k+1} .

REMARK 5.3. Note that choosing different frames of A when embedding A into $\mathbb{R}(A)$ will provide isomorphic algebras $\bar{D}(A)$. Note also that frames could be replaced by any subset of A generating a k -dimensional simplex. However, this would provide a much less transparent description of the dual space.

6. The second dual

For each k -dimensional convex polytope A , the second dual A^{DE} will be described in this section. We show that there is a full duality between the category $\mathcal{A} = \underline{\underline{P}}$ of convex polytopes with barycentric homomorphisms as morphisms, and the category $\mathcal{X} = D(\underline{\underline{P}}) =: \hat{\underline{\underline{P}}}$ of all isomorphic copies of convex polytopes with constants (as described in Section 5), with the barycentric homomorphisms preserving constants as morphisms.

We start with the following observation.

LEMMA 6.1. *The evaluation map*

$$e_A : A \rightarrow A^{DE}; \quad a \mapsto (ae_A : x \mapsto ax)$$

is injective.

PROOF. For a, b in A , and r in I^o , and for x in A^D ,

$$x(abr)e_A = (abr)x = ax \underline{bx} \underline{r} = xae_A xbe_A \underline{r} = x(ae_A be_A \underline{r}),$$

where \underline{r} is calculated as in the proof of Lemma 4.2. This shows that e_A is a barycentric algebra homomorphism. The injectivity of e_A follows by the fact that each convex polytope is a subalgebra of a direct power of I (compare [14, Ch. 7]). Thus distinct elements a, b of A are separated by some homomorphism in A^D . \square

In what follows, we use the symbol $\hat{}$ to denote algebras with constants and homomorphisms preserving constants. In particular, we identify the first dual A^D with \hat{A} , and denote by $\underline{\hat{\mathbb{R}}}$ the category of real affine spaces with constants $\bar{0}$ and $\bar{1}$.

According to our convention, we may identify the set A^{DE} with $\bar{D}(A)^E$ or \hat{A}^E consisting of all $\underline{\hat{B}}$ -homomorphisms $\hat{\varphi} : \hat{A} \rightarrow \hat{I}$. As in the case of the first dual, \hat{A}^E is a cancellative barycentric algebra, and is isomorphic to the barycentric algebra $\bar{E}(\hat{A})$ consisting of those affine space homomorphisms

$$\hat{\varphi} := \bar{\varphi} : \hat{\mathbb{R}}^{k+1} \cong \hat{\mathbb{R}}(\hat{A}) \longrightarrow \hat{\mathbb{R}}$$

with $\hat{\varphi}(\hat{A}) \subseteq I$, $\hat{\varphi}(\bar{0}) = 0$ and $\hat{\varphi}(\bar{1}) = 1$ that are unique extensions of the homomorphisms $\hat{\varphi} : \hat{A} \rightarrow \hat{I}$. In fact, $\hat{\varphi}(\hat{A}) = I$. Recall that

$$\underline{\hat{\mathbb{R}}}(\hat{\mathbb{R}}^{k+1}, \hat{\mathbb{R}}) \in \underline{\hat{\mathbb{R}}} \quad \text{and} \quad \underline{\hat{\mathbb{R}}}(\hat{\mathbb{R}}^{k+1}, \hat{\mathbb{R}}) \cong \mathbb{R}^k \tag{6.1}$$

[11, Proposition 5.3], and note that

$$\bar{E}(\hat{A}) \subseteq \underline{\hat{\mathbb{R}}}(\hat{\mathbb{R}}^{k+1}, \hat{\mathbb{R}}). \tag{6.2}$$

This may be summarized as follows.

PROPOSITION 6.2. *To within isomorphism, the barycentric algebra $\bar{E}(\hat{A})$ is an I^o -subreduct of the k -dimensional affine space \mathbb{R}^k .*

Let

$$\{e_0 = \bar{0}, e_1 = (1, 0, \dots, 0), \dots, e_{k+1} = (0, \dots, 0, 1)\}$$

be the standard set of generators of the affine space \mathbb{R}^{k+1} . With e_0 as zero, \mathbb{R}^{k+1} is also a vector space with basis $\{e_1, \dots, e_{k+1}\}$. As in the proof of [11, Proposition 5.3], the conditions $\hat{\varphi}(\bar{0}) = 0$ and $\hat{\varphi}(\bar{1}) = 1$ imply that the elements of $\underline{\hat{\mathbb{R}}}(\hat{\mathbb{R}}^{k+1}, \hat{\mathbb{R}})$ are (idempotent) vector space projections taking $\bar{1}$ to 1. Each such element may be described as

$$\hat{\varphi} = \check{x}_1 r_1 + \dots + \check{x}_{k+1} r_{k+1}, \tag{6.3}$$

with $\sum_{i=1}^{k+1} r_i = 1$, where for $i = 1, \dots, k + 1$ the map

$$\check{x}_i : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$$

is defined by $e_i \mapsto 1$ and $e_j \mapsto 0$ for $j \neq i$. If x_0, \dots, x_k are the free generators of the affine space \mathbb{R}^k , then the mapping $x_i \mapsto \check{x}_{i+1}$, for $i = 0, \dots, k$, gives the required isomorphism of the affine space \mathbb{R}^k with the affine space $\underline{\mathbb{R}}(\hat{\mathbb{R}}^{k+1}, \hat{\mathbb{R}})$. Note that all the mappings \check{x}_i belong to $\bar{E}(\hat{A})$, and also that each homomorphism $\hat{\varphi}$ is determined by the k -dimensional subspace $\text{Ker } \hat{\varphi}$ of \mathbb{R}^{k+1} , or by its normal vector. In particular, $\text{Ker } \check{x}_i$ is the hyperplane $y_{i-1} = 0$. No homomorphism $\hat{\varphi}$ has its kernel containing the line through $\bar{0}$ and $\bar{1}$. (Indeed, note that such a kernel has an equation

$$y_0 r_0 + \dots + y_k r_k = 0,$$

with $r_0 + \dots + r_k = 0$.)

Let A be a k -dimensional convex polytope as above, with a fixed frame $\{x_0 = g_0, \dots, x_k = g_k\}$. For each $a = \sum_{j=0}^k x_j r_{aj}$ in A , define the mapping

$$\hat{\varphi}_a : \hat{\mathbb{R}}^{k+1} \rightarrow \hat{\mathbb{R}}; \quad (y_0, \dots, y_k) \mapsto \sum_{j=0}^k y_j r_{aj}.$$

(Note that $y\hat{\varphi}_{x_i} = y_i = y\check{x}_{i+1}$ for $i = 0, \dots, k$ and $y = (y_0, \dots, y_k) \in \hat{\mathbb{R}}^{k+1}$.) Define $\hat{\Phi}$ to be the set $\{\hat{\varphi}_a \mid a \in A\}$.

LEMMA 6.3. *Let A be a convex polytope as above. Then $\hat{\Phi}$ embeds into the barycentric algebra $\bar{E}(\hat{A})$, and is isomorphic to A and hence also to $e_A(A)$.*

PROOF. By Lemma 6.1, $e_A(A)$ is a subalgebra of A^{DE} . As $A^{DE} \cong E(\hat{A})$, in the evaluation mapping e_A of Lemma 6.1, we can replace A^{DE} by $E(\hat{A})$, and the mapping $A^D \rightarrow \hat{I}$ by $\hat{A} \rightarrow \hat{I}$. Recall that $E(\hat{A}) \cong \bar{E}(\hat{A})$. Let

$$\bar{e}_A : A \rightarrow \bar{E}(\hat{A}); \quad a \mapsto \bar{e}_A(a) : \hat{\mathbb{R}}(\hat{A}) \rightarrow \hat{\mathbb{R}}(\hat{I}).$$

Then the map

$$e_A(A) \rightarrow \bar{e}_A(A) : e_A(a) \mapsto \bar{e}_A(a)$$

is a barycentric algebra isomorphism. Now recall that $\hat{\mathbb{R}}(\hat{A}) \cong \hat{\mathbb{R}}^{k+1}$ and $\hat{\mathbb{R}}(\hat{I}) \cong \hat{\mathbb{R}}$. Now for each $a = \sum_{j=0}^k x_j r_{aj}$ in A , we have $\bar{e}_A(a)(y) = \hat{\varphi}_a(y)$. It follows that any two of the barycentric algebras $A, e_A(A), \bar{e}_A(A)$ and $\hat{\Phi}$ are isomorphic. \square

Note that for the generators g_0, \dots, g_n of A as above, the barycentric algebra $\hat{\Phi}$ is generated by the $\hat{\varphi}_{g_i}$. Moreover, $\text{Ker } \hat{\varphi}_{g_i}$ is the hyperplane H_{i0} defined by $\sum_{j=0}^k y_j r_{ij} = 0$.

PROPOSITION 6.4. *The barycentric algebras $\bar{E}(\hat{A})$ and $\hat{\Phi}$ are isomorphic.*

PROOF. We will identify $\hat{\Phi}$ with its image $\bar{e}_A(A)$ in $\bar{E}(\hat{A})$, and simply write $\hat{\Phi} \subseteq \bar{E}(\hat{A})$. We will show that the converse inclusion holds.

Let $\hat{\varphi} \in \bar{E}(\hat{A})$. We may assume that $\hat{\varphi} : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ with $\hat{\varphi}(\hat{A}) = I$, and that for $y = (y_0, \dots, y_k) \in \hat{A}$,

$$y\hat{\varphi} = y\check{x}_1r_1 + \dots + y\check{x}_{k+1}r_{k+1} = y_0r_1 + \dots + y_kr_{k+1}.$$

Define the element a of A by

$$a := \sum_{j=0}^k x_j \frac{1}{k+1}.$$

Write $\hat{\sigma} := \hat{\varphi}_a$. Note that for $y = (y_0, \dots, y_k) \in \hat{A}$,

$$\hat{\sigma}(y) = \sum_{j=0}^k y_j \frac{1}{k+1} = \sum_{j=0}^k y\check{x}_{j+1} \frac{1}{k+1}.$$

Since $\hat{\varphi}_{x_i} = \check{x}_{i+1} \in \hat{\Phi}$, it follows that $\hat{\sigma} \in \hat{\Phi}$.

Now suppose that

$$\hat{\varphi} \in \bar{E}(\hat{A}) \setminus \hat{\Phi},$$

and let $\hat{\theta}$ be an element of the intersection $\hat{\Phi} \cap [\hat{\sigma}, \hat{\varphi}]$ lying on the boundary of $\hat{\Phi}$. This element belongs to a proper wall W of $\hat{\Phi}$. Suppose that W is generated by the set $\{\hat{\varphi}_{g_{i_1}}, \dots, \hat{\varphi}_{g_{i_p}}\}$. Choose a frame $F(W) = \{z_0, \dots, z_j\}$ of W . Note that

$$F(W) \subseteq \{\hat{\varphi}_{g_{i_1}}, \dots, \hat{\varphi}_{g_{i_p}}\}.$$

Since the wall W is proper, $j < k$. The frame $F(W)$ can be extended to a frame

$$F(\hat{\Phi}) = \{z_0, \dots, z_j, \dots, z_k\} \subseteq \{\hat{\varphi}_{g_0}, \dots, \hat{\varphi}_{g_n}\},$$

of $\hat{\Phi}$. Without loss of generality, we may assume that

$$x_0 = \iota^{-1}(z_0), \dots, x_j = \iota^{-1}(z_j), \quad x_{j+1} = \iota^{-1}(z_{j+1}), \dots, x_k = \iota^{-1}(z_k),$$

where $\iota : A \rightarrow \hat{\Phi}; a \mapsto \hat{\varphi}_a$ is the isomorphism described in the proof of Lemma 6.3.

Let $\eta : \mathbb{R}(A) \rightarrow \mathbb{R}$ be the affine space homomorphism that is defined on the generators x_i by $\eta(x_i) = 0$ if $i = 0, \dots, j$, and otherwise by $\eta(x_i) = 1$. Since A is a convex polytope, there is a generator g_i of A and a real number $q > 0$ such that

$$q = \eta(g_i) = \max_{x \in A} \eta(x).$$

Now define $\eta' : \mathbb{R}(A) \rightarrow \mathbb{R}$ by

$$\eta'(x) = \frac{1}{q} \eta(x).$$

Note that

$$\max_{x \in A} \eta'(x) = 1$$

and

$$\min_{x \in A} \eta'(x) = 0,$$

whence $\eta' \in \bar{D}(A)$. By Proposition 5.2, η' may be identified with the element $\hat{t} := (x_0\eta', \dots, x_k\eta')$ of \hat{A} , and this element is not zero.

Now, for each $s \in \mathbb{R}$, define

$$\hat{\xi}_s := \hat{\sigma} \hat{\theta} \underline{s}.$$

Then $\hat{\xi}_0 = \hat{\sigma}$, $\hat{\xi}_1 = \hat{\theta}$ and, for some $s_0 > 1$, $\hat{\xi}_{s_0} = \hat{\varphi}$. Define f to be the mapping

$$f : \mathbb{R} \longrightarrow \mathbb{R}; \quad s \mapsto \hat{\xi}_s(\hat{t}).$$

Note that

$$f(s) = \hat{\sigma}(\hat{t}) \hat{\theta}(\hat{t}) \underline{s} = f(0)(1 - s) + f(1)s = (f(0) - f(1))s + f(1),$$

so that f is a linear function. Since $j < k$, the definition of η' implies that $f(0) = \hat{\sigma}(\hat{t}) > 0$. Moreover, $f(1) = \hat{\theta}(\hat{t}) = 0$. By the linearity of f , we have $f(s_0) = \hat{\varphi}(\hat{t}) < 0$. This contradicts the assumption that $\hat{\varphi} \in \bar{E}(\hat{A})$. The required equality $\hat{\Phi} = \bar{E}(\hat{A})$ follows. □

THEOREM 6.5 (Duality theorem). *There is a full duality between the categories $\underline{\underline{P}}$ and $\underline{\underline{\hat{P}}}$ given by the schizophrenic object I .*

PROOF. The existence of a duality between the categories $\underline{\underline{P}}$ and $\underline{\underline{\hat{P}}}$ follows directly by the previous results, with the natural isomorphism e given by

$$e_A : A \rightarrow A^{DE}; \quad a \mapsto \kappa(\hat{\varphi}_a),$$

where $\kappa : \hat{\Phi} \rightarrow e_A(A)$ is the isomorphism described in the proof of Lemma 6.3.

This duality is full. First recall that each member of the dual category $\underline{\underline{\hat{P}}}$ has the form A^D for some convex polytope A . As $A \cong A^{DE}$, we may assume that each element of A^{DE} is of the form $\hat{\varphi}_a$ for precisely one $a \in A$. Hence, $e_A(a) = \hat{\varphi}_a$ and

$$e_A^{-1} : A^{DE} \rightarrow A; \quad \hat{\varphi}_a \mapsto a.$$

By the definition of the functor D ,

$$D(e_A^{-1}) : A^D \longrightarrow A^{DED}; \quad \psi \mapsto \psi \circ e_A^{-1}$$

and

$$D(e_A^{-1})(\psi)(\hat{\varphi}_a) = \psi(e_A^{-1}(\hat{\varphi}_a)) = \psi(a).$$

Note also that for ψ in A^D ,

$$\psi(a) = e_A(a)(\psi) = \hat{\varphi}_a(\psi). \quad (6.4)$$

Now the natural isomorphism ε of ED with the identity functor is given by the evaluation

$$\varepsilon_{AD} : A^D \rightarrow A^{DED}; \quad \psi \mapsto \varepsilon_{AD}(\psi),$$

and by equation (6.4),

$$\varepsilon_{AD}(\psi)(\hat{\varphi}_a) = \hat{\varphi}_a(\psi) = \psi(a).$$

It follows that

$$\varepsilon_{AD} = D(e_A^{-1}). \quad \square$$

Recall that for a k -dimensional convex polytope $A = \bigoplus_{i \in I} A_i$ partitioned into k -dimensional simplices A_i as in Section 4,

$$\hat{A} \cong A^D = \left(\bigoplus_{i \in I} A_i \right)^D \cong \bigcap_{i \in I} (A_i^D).$$

Dualizing $\hat{\mathbb{R}}(\hat{A})$ in parallel with \hat{A} provides corresponding dual spaces $A_i^{DE} \cong A_i$ of $\hat{A}_i \cong A_i^D$.

Acknowledgements

The second author acknowledges the hospitality of Department of Mathematics, Iowa State University, Ames, Iowa, during summer 2004 and winter 2005. We are grateful to a referee for many helpful comments on an earlier version of this paper.

References

- [1] A. Brøndsted, *An Introduction to Convex Polytopes* (Springer, New York, 1983).
- [2] D. M. Clark and B. A. Davey, *Natural Dualities for the Working Algebraist* (Cambridge University Press, Cambridge, 1998).
- [3] B. A. Davey, 'Duality theory on ten dollars a day', in: *Algebras and Orders* (eds. I. G. Rosenberg and G. Sabidussi) (Kluwer Academic, Dordrecht, 1993), pp. 71–111.
- [4] B. A. Davey and H. Werner, 'Dualities and equivalences for varieties of algebras', *Colloq. Math. Soc. János Bolyai* **33** (1983), 101–275.
- [5] B. Grünbaum, *Convex Polytopes*, 2nd edn (Springer, New York, 2003).
- [6] K. H. Hofmann, M. Mislove and A. Stralka, *Pontryagin Duality of Compact 0-Dimensional Semilattices and its Applications*, Lecture Notes in Mathematics, 396 (Springer, Berlin, 1974).
- [7] P. T. Johnstone, *Stone Spaces* (Cambridge University Press, Cambridge, 1982).
- [8] S. Mac Lane, *Categories for the Working Mathematician* (Springer, Berlin, 1971).
- [9] W. D. Neumann, 'On the quasivariety of convex subsets of affine spaces', *Arch. Math.* **21** (1970), 11–16.
- [10] K. J. Pszczoła, 'Duality for affine spaces over finite fields', *Contributions to General Algebra* **13** (2001), 285–293.
- [11] K. J. Pszczoła, A. B. Romanowska and J. D. H. Smith, 'Duality for some free modes', *Discuss. Math. Gen. Algebra Appl.* **23** (2003), 45–61.

- [12] ———, 'Duality for quadrilaterals', *Contributions to General Algebra* **14** (2004), 127–134.
- [13] A. B. Romanowska and J. D. H. Smith, *Modal Theory* (Heldermann, Berlin, 1985).
- [14] ———, *Modes* (World Scientific, Singapore, 2002).
- [15] J. D. H. Smith and A. B. Romanowska, *Post-modern Algebra* (Wiley, New York, 1999).

A. B. ROMANOWSKA, Faculty of Mathematics and Information Sciences,
Warsaw University of Technology, 00-661 Warsaw, Poland
e-mail: aroman@mini.pw.edu.pl

P. ŚLUSARSKI, Faculty of Mathematics and Information Sciences,
Warsaw University of Technology, 00-661 Warsaw, Poland
e-mail: P.Slusarski@mini.pw.edu.pl

J. D. H. SMITH, Department of Mathematics, Iowa State University, Ames,
Iowa 50011, USA
e-mail: jdsmith@math.iastate.edu