Canad. Math. Bull. Vol. **41** (1), 1998 pp. 65–70

CRITERIA FOR COMMUTATIVITY IN LARGE GROUPS

A. MOHAMMADI HASSANABADI AND AKBAR RHEMTULLA

ABSTRACT. In this paper we prove the following:

1. Let $m \geq 2$, $n \geq 1$ be integers and let G be a group such that $(XY)^n = (YX)^n$ for all subsets X, Y of size m in G. Then
   a) G is abelian or a BFC-group of finite exponent bounded by a function of m and n.
   b) If $m \geq n$ then G is abelian or $|G|$ is bounded by a function of m and n.
2. The only non-abelian group G such that $(XY)^2 = (YX)^2$ for all subsets X, Y of size 2 in G is the quaternion group of order 8.
3. Let m, n be positive integers and G a group such that

$$X_1 \cdots X_n \subseteq \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

for all subsets X_i of size m in G. Then G is n-permutable or $|G|$ is bounded by a function of m and n.

1. **Introduction.** Let m, n be positive integers. Call a group G an (m,n)-group if $(XY)^n = (YX)^n$ for all subsets X, Y of size m in G. Thus $(1,1)$-groups are precisely the abelian groups and G is a $(1,n)$-group if and only if $G^n \leq Z(G)$. This easy result is proved in Lemma 2.1. In particular, groups of exponent n are $(1,n)$-groups and for large values of n, they include finitely generated infinite simple groups. We note that $(m,1)$-groups were considered in [4]. There it was proved that an $(m,1)$-group is either abelian or of order less that $2m$. Of course, every abelian group is an (m,n)-group and we shall prove that an (m,n)-group G, $m > 1$, is either abelian or a BFC-group of finite exponent bounded by a function of m and n. Recall that a group G is a BFC-group if there exists a positive integer b such that every element of G has at most b conjugates in G. We also prove that an (m,n)-group G, with $m > 1$ and the extra condition of $m \geq n$ is either abelian or of finite order bounded by a function of m and n. We note that this result no longer holds in general if $m < n$; for example let $G = Q_8 \times C$, where Q_8 is the quaternion group of order 8 and C is the direct product of an infinite number of cyclic groups of order 2. Then it is easy to see that G is a $(2,4)$-group which is neither abelian nor has bounded order. We shall also show that the only non-abelian $(2,2)$-group is the quaternion group Q_8.

Our second topic deals with a natural extension of permutable groups which have been studied by a number of people—see [1], [2], [3], [5] and [6]. Recall that a group

The first author wishes to thank the Department of Mathematics, University of Alberta for their excellent hospitality. He also thanks the University of Isfahan for its financial support.

Received by the editors June 20, 1996.

AMS subject classification: Primary: 20E34; secondary: 20F24.

https://doi.org/10.4153/CMB-1998-010-0 Published online by Cambridge University Press

G is called n-permutable if given any sequence $x_1, \ldots, x_n$ of elements of G, $x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$ for some permutation $\sigma \neq 1$ of the set $\{1, \ldots, n\}$. The main result for infinite groups in this class was obtained by Curzio, Longobardi, Maj and Robinson in [3] where it was shown that such groups are finite-by-abelian-by-finite.

Let m, n be positive integers. Call a group G, (m, n)-permutable if

$$X_1 \cdots X_n \subseteq \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

for all subsets X_i of G where $|X_i| = m$ for all $i = 1, \ldots, n$. Thus $(1, n)$-permutable groups are precisely the n-permutable groups.

We shall show that if G is (m, n)-permutable then it is n-permutable if $|G| \geq n!\,(mn)^n$. This result is another addition to many results of similar type and naturally leads to the following general question.

Let U, V be sets of words in n variables $x_1, \ldots, x_n$ and let $\mathcal{X}$ be the class of groups G such that for all $g_1, \ldots, g_n$ in G

$$\{u(g_1, \ldots, g_n) \; ; \; u \in U\} \subseteq \{v(g_1, \ldots, g_n) \; ; \; v \in V\}.$$

Next let m be a positive integer and $\mathcal{X}(m)$ the class of groups G such that for all sequences $X_1, \ldots, X_n$ of m-element subsets of G,

$$\begin{aligned} &\{u(g_1, \ldots, g_n) \; ; \; u \in U, g_i \in X_i, i = 1, \ldots, n\} \\ &\qquad \subseteq \{v(g_1, \ldots, g_n) \; ; \; v \in V, g_i \in X_i, i = 1, \ldots, n\}. \end{aligned}$$

For which sets U, V of words can one say that groups of large orders in the class $\mathcal{X}(m)$ all lie in $\mathcal{X}$? It would appear that this would be the case if the words in U and V are semigroup words—words that involve only non-negative powers of the variables $x_1, \ldots, x_n$.

The (m, n)-permutable groups may be viewed in this context, where U consists of one word $u(x_1, \ldots, x_n) = x_1 \cdots x_n$ and V consists of the words $v_\sigma(x_1, \ldots, x_n) = x_{\sigma(1)} \cdots x_{\sigma(n)}$, where σ runs through the non-identity permutations of the set $\{1, \ldots, n\}$.

The main results of this paper are as follows.

THEOREM 1. *Let $m \geq 2$, $n \geq 1$ be integers and let G be an (m, n)-group. Then*

(a) G is abelian or a BFC-*group of finite exponent bounded by a function of m and n.*

(b) If $m \geq n$ then G is abelian or $|G|$ is bounded by a function of m and n.

THEOREM 2. *The only non-abelian* $(2, 2)$-*group is the quaternion group of order* 8.

THEOREM 3. *Suppose that m, n are positive integers and let G be an (m, n)-permutable group. Then G is n-permutable or $|G|$ is bounded by a function of m and n.*

The proofs of Theorems 1, 2 and 3 are given in Sections 2, 3 and 4 respectively.

https://doi.org/10.4153/CMB-1998-010-0 Published online by Cambridge University Press

2. (m,n)-groups.

LEMMA 2.1. *G is a $(1,n)$-group if and only if $G^n \le Z(G)$, where $Z(G)$ denotes the centre of G.*

PROOF. For any x, g in G let $y = x^{-1}g$. Then $(yx)^n = (x^{-1}gx)^n = x^{-1}g^n x$. Since G is a $(1,n)$-group, $g^n = (xy)^n = (yx)^n = x^{-1}g^n x$. Thus $G^n \le Z(G)$.

The converse is equally easy. For any x, y in G, $(xy)^n = x^{-1}(xy)^n x = (yx)^n$. So G is a $(1,n)$-group.

From now on in this section we assume that $m > 1$, $n \ge 1$ and aim to prove that a non-abelian (m,n)-group is a BFC-group of finite exponent bounded by a function of m and n. Moreover under the extra condition of $m \ge n$ that such a group is finite bounded by a function of m and n. Most of the notations used are standard. For instance, we denote the centre of G by $Z(G)$; the centralizer of the set X in G by $C_G(X)$ and so on.

LEMMA 2.2. *Let G be an (m,n)-group where $m > 1$. If $x \in G$ and $|\langle x\rangle| > 4n(m-1)$ then $\langle x\rangle \le Z(G)$. In particular every element of $G/Z(G)$ has order at most $4n(m-1)$.*

PROOF. Let $X = \{1, x, \ldots, x^{m-1}\}y^{-1}$ and $Y = y\{1, x, \ldots, x^{m-1}\}$. Then $(XY)^n = \{x^k\,;\, 0 \le k \le 2n(m-1)\}$ and $(YX)^n = y(XY)^n y^{-1}$. Both the sets $(XY)^n$ and $(YX)^n$ are of size $1 + 2n(m-1)$ and there is an injective function f on $\{0, 1, \ldots, 2n(m-1)\}$ such that $y^{-1}x^k y = x^{f(k)}$, $0 \le k \le 2n(m-1)$.

Now $y^{-1}xy = (y^{-1}x^{k+1}y)(y^{-1}x^k y)^{-1} = x^{f(k+1)-f(k)}$ for all k and $f(0) = 0$. Thus $x^{f(k)} = x^{dk}$ for some integer $d \ge 1$. Since

$$\{1, 2, \ldots, 2n(m-1)\} \equiv \{d, 2d, \ldots, 2n(m-1)d\} \bmod |\langle x\rangle|,$$

$d = 1$ or $d > |\langle x\rangle| - 2n(m-1)$. In the second case $d > 2n(m-1)$. Also $d < |\langle x\rangle|$. This is not possible since such a d is not congruent to any of $1, 2, \ldots, 2n(m-1)$. Thus $y^{-1}xy = x$.

If $gZ(G)$ has order greater than $4n(m-1)$ in $G/Z(G)$, then $|\langle g\rangle| > 4n(m-1)$ and hence $g \in Z(G)$. This proves the last claim of the lemma.

LEMMA 2.3. *Let G be a non-abelian (m,n)-group where $m > 1$. Then every element of $Z(G)$ has finite order bounded above by $\big(2n(m-1)\big)^{n+1}$. In particular the exponent of G divides $\big[2\big(2n(m-1)\big)^{n+2}\big]!$.*

PROOF. Let x, y be non-commuting elements of G and let $z \in Z(G)$. Put $\alpha = 2n(m-1)$ and write z_i for z^{α^i}, $i = 0, 1, \ldots, n$.

Consider the sets $X = \{xz_i^j, j = 0, 1, \ldots, m-1\}$ and $Y = \{yz_i, x^{-1}z_i^j\,;\, j = 0, 1, \ldots, m-2\}$. Then $xyz_i \in (XY)^n = (YX)^n$ so that $xy = (yx)^r z_i^{\lambda_i}$ where $0 \le \lambda_i < 2n(m-1)$ and $r > 0$. But $\lambda_i = 0$ only if $xy = yx$ which we have ruled out. Thus $1 \le \lambda_i < 2n(m-1) = \alpha$.

Now let i run from 0 to n so that for some r, $xy = (yx)^r z_i^{\lambda_i} = (yx)^r z_j^{\lambda_j}$ where $0 \le i < j \le n$.

Since $z_j^{\lambda_j} = z^{\alpha^j \lambda_j}$, $z_i^{\lambda_i} = z^{\alpha^i \lambda_i}$ and $\alpha^j \lambda_j \geq \alpha^j \geq \alpha^{i+1} > \alpha^i \lambda_i$, it follows that $z^k = 1$ for some k, $0 < k < \alpha^{j+1} \leq \bigl(2n(m-1)\bigr)^{n+1}$ which completes the proof of the first part. The second part follows easily from this and Lemma 2.2.

LEMMA 2.4. *Suppose that G is an (m,n)-group where $m > 1$. Then G is a* BFC-*group.*

PROOF. Let x, y be a non-commuting pair in G. Choose $c_2, \ldots, c_m$ in G and consider the m-sets

$$X = \{x^{-1}, c_2x^{-1}, \ldots, c_mx^{-1}\}, \quad Y = \{xy, xc_2^{-1}, \ldots, xc_m^{-1}\}.$$

Then $y \in (XY)^n = (YX)^n \subseteq xWx^{-1}$, where W is the set of words in y, $c_2^{\pm 1}, \ldots, c_m^{\pm 1}$ of length at most $2n$. Thus $y^x \in W$ and $|W| \leq (2m)^{2n}$. Therefore each conjugacy class in G has order at most $(2m)^{2n}$, and G is a BFC-group.

Thus far the proof of part (a) of Theorem 1 is completed.

From now on we let $\lambda = (2m)^{2n}$,

$$\mu = 2[2n(m-1)]^{n+2} \quad \text{and}$$
$$\nu = \bigl(4n(m-1)\bigr)^2 \lambda^{1/2(3+5\log\lambda)}$$

where the logarithm is to the base 2.

Since by Lemma 2.4, $|G : C_G(x)| \leq \lambda$, for all elements x in an (m,n)-group G, with $m > 1$, the order of G' is bounded. Certainly $|G'| \leq \lambda^{1/2(3+5\log\lambda)}$ as was shown by P. M. Neumann and M. R. Vaughan-Lee in [7]. Let x, y be a pair of non-commuting elements of G and let $H = \langle x, y\rangle$. Notice that $|\langle x\rangle|$ and $|\langle y\rangle|$ are both at most $4n(m-1)$ since they do not lie in $Z(G)$. Also $|H'| \leq |G'|$. Thus $|H/H'| \leq \bigl(4n(m-1)\bigr)^2$ and $|H| \leq \nu$.

Suppose that $m \geq n$ and $|Z(G)| > \mu^{\nu+2(m-1)}$. Then there exists a subgroup D of $Z(G)$ that is a direct product of $2m-2$ non-trivial cyclic subgroups $C_2, \ldots, C_{2m-1}$ such that $H \cap D = 1$. Pick $1 \neq c_i$ in C_i, $i = 2, \ldots, 2m-1$ and let

$$X = \{x, xc_2, xc_4, \ldots, xc_{2m-2}\},$$
$$Y = \{y, x^{-1}c_3, x^{-1}c_5, \ldots, x^{-1}c_{2m-1}\}.$$

Then $g = xyc_2c_3\cdots c_{2n-1} \in (XY)^n = (YX)^n$. Taking the projection of g in the group D we see that if $g = y_1x_1y_2x_2\cdots y_nx_n$ then

$$\{y_1, \ldots, y_n\} = \{y, x^{-1}c_3, \ldots, x^{-1}c_{2n-1}\} \quad \text{and}$$
$$\{x_1, \ldots, x_n\} = \{x, xc_2, \ldots, xc_{2n-2}\},$$

so that only one of the y_i's equals y and we obtain $[x, y] = 1$ which is a contradiction.

We have thus shown the following result.

https://doi.org/10.4153/CMB-1998-010-0 Published online by Cambridge University Press

LEMMA 2.5. *If G is a non-abelian (m,n)-group where $m > 1$ and $m \geq n$, then* $|Z(G)| \leq \mu^{\nu+2m-2}$.

LEMMA 2.6. *Let G be a non-abelian (m,n)-group. Then the order of any abelian subgroup of G is at most $\lambda\mu^{\nu+2m-2}$.*

PROOF. Let A be any maximal abelian subgroup of G. Since G is non-abelian, $A \neq G$; and $\langle A, x\rangle$ is non-abelian if $x \in G \setminus A$. Now $|A : C_A(x)| \leq \lambda$ so that $C_A(x) \leq Z(\langle A, x\rangle)$ has order not exceeding $\mu^{\nu+2m-2}$ by Lemma 2.5. Thus $|A| \leq \lambda\mu^{\nu+2m-2}$.

PROOF OF THEOREM 1. (a) Follows directly from Lemmas 2.3 and 2.4.

(b) Let G be a non-abelian (m,n)-group where $m > 1$ and $m \geq n$. We will show that $|G| \leq \lambda\mu^{\nu+2m-2} \cdot \lambda^{\lambda\mu^{\nu+2m-2}}$.

Take any $1 \neq x_1 \in G$ and let $A_1 = \langle x_1\rangle$ and $G_1 = C_G(x_1)$. Then $|G_1| \geq |G|/\lambda$, and $|A_1| \leq \mu$ by Lemmas 2.2 and 2.3. Pick $1 \neq x_2 \in G_1 \setminus A_1$, let $A_2 = \langle x_1, x_2\rangle$ and $G_2 = C_{G_1}(x_2)$ so that $|G_2| \geq |G|/\lambda^2$ and $A_2 > A_1$. Continue this process. At i-th step, pick $1 \neq x_i \in G_{i-1} \setminus A_{i-1}$, let $A_i = \langle A_{i-1}, x_i\rangle$ and $G_i = C_{G_{i-1}}(x_i)$. Then $|G_i| \geq |G|/\lambda^i$ and A_i is an abelian group. Now $|A_i| \leq \lambda\mu^{\nu+2m-2}$ by Lemma 2.6. Thus for some integer $i < \lambda\mu^{\nu+2m-2}$, A_i must equal G_i. Therefore $|G|/\lambda^i \leq |A_i|$ and $|G| \leq \lambda\mu^{\nu+2m-2}\lambda^{\lambda\mu^{\nu+2m-2}}$.

3. $(2,2)$**-groups.** In this section we show that the only non-abelian $(2,2)$-group is the quaternion group Q_8.

Of course Q_8 itself is a $(2,2)$-group. However for any non-trivial group T, $Q_8 \times T$ is not a $(2,2)$-group. This can be seen by observing that $(XY)^2 \neq (YX)^2$ if $X = \{a, b\}$ and $Y = \{a^{-1}, a^{-1}t\}$, where a, b are generators of Q_8 and t is any element in T.

The following result about general (m,n)-groups is the key to our main result here.

LEMMA 3.1. *Let G be an (m,n)-group, where $m > 1$. If K is a subgroup of G and $|K| \geq m$ then K is normal in G. In particular if $m = 2$ then every subgroup of G is normal.*

PROOF. Let $K \leq G$ be of order m or greater. If $1 \neq x \in K$ and $y \in G$ then consider the sets

$$X = \{1, x, k_3, \ldots, k_m\}y^{-1}, \quad Y = y\{1, x, k_3, \ldots, k_m\}$$

where $k_3, \ldots, k_m$ are distinct elements from $K \setminus \{1, x\}$. Then $x \in (XY)^n = (YX)^n = y(XY)^n y^{-1}$ so that $x^y \in (XY)^n \subseteq K$. Thus $K \lhd G$.

PROOF OF THEOREM 2. Suppose that G is a non-abelian $(2,2)$-group. Then by Lemma 3.1 all subgroups of G are normal. So by the Dedekind-Baer Theorem G is isomorphic to $Q_8 \times T$, where T is some abelian group. But as we have observed above, T must be trivial. This completes the proof of Theorem 2.

4. (m, n)**-permutable groups.** We find it easier to prove the following generalized version of Theorem 3.

THEOREM 3′. *Let $m_1, \ldots, m_n$ be positive integers and let G be a group such that*

$$X_1 \cdots X_n \subseteq \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

for all subsets X_i of size m_i in G; $i = 1, \ldots, n$. Let $s = m_1 + \cdots + m_n$. Then G is n-permutable or $|G| \leq n!\, s^n$.

Theorem 3 follows from the above result by letting $m_i = m$ for all $i = 1, \ldots, n$. We shall say that G is an $(m_1, \ldots, m_n)$-permutable group if it satisfies the hypothesis of Theorem 3′.

PROOF OF THEOREM 3′. We use induction on the sum $s = m_1 + \cdots + m_n$. If $s = n$ then $m_i = 1$ for all i and G is n-permutable. So assume the result holds for all $s \leq r$. Thus if G is an $(m_1, \ldots, m_n)$-permutable group and $m_1 + \cdots + m_n = r$ then G is n-permutable or $|G| < n!\, r^n$. Now let G be an $(m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_n)$-permutable group that is not $(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_n)$-permutable. So there exist subsets X_j where $|X_j| = m_j$, $j = 1, \ldots, n$ and elements $a_j \in X_j$ such that $g = a_1 \cdots a_n \notin X_{\sigma(1)} \cdots X_{\sigma(n)}$ for all $\sigma \in S_n \setminus 1$.

Pick any $z \in G \setminus X_i$. Then $g = b_{\sigma(1)} \cdots b_{\sigma(k-1)} z b_{\sigma(k+1)} \cdots b_{\sigma(n)}$ for some $\sigma \neq 1$ and $\sigma^{-1}(i) = k$. Then

$$z \in X_{\sigma(k-1)}^{-1} \cdots X_{\sigma(1)}^{-1} g X_{\sigma(n)}^{-1} \cdots X_{\sigma(k+1)}^{-1}$$

which is a set of size bounded above by $n!\, r^n$. Thus $|G \setminus X_i| \leq n!\, r^n$ and $|G| < n!\,(r+1)^n$. So if $|G| \geq n!\,(r+1)^n$ then G is $(m_1, \ldots, m_n)$-permutable and by induction, G is n-permutable. This completes the proof of Theorem 3′.

REFERENCES

1. R. D. Blyth, *Rewriting products of group elements, I.* J. Algebra **116**(1988), 506–521.

2. ______, *Rewriting products of group elements, II.* J. Algebra **116**(1988), 246–259.

3. M. Curzio, P. Longobardi, M. Maj and D. J. S. Robinson, *A permutational property of groups.* Arch. Math. (Basel) **44**(1985), 385–389.

4. John C. Lennox, A. Mohammadi Hassanabadi and James Wiegold, *Some commutativity criteria, II.* Rend. Sem. Mat. Univ. Padova. **86**(1991), 207–211.

5. P. Longobardi, M. Maj and S. E. Stonehewer, *The classification of groups in which every product of four elements can be reordered.* Rend. Sem. Mat. Univ. Padova. **93**(1995), 7–26.

6. M. Maj and S. E. Stonehewer, *Non-nilpotent groups in which every product of four elements can be reordered.* Canad. J. Math. **42**(1990), 1053–1066.

7. P. M. Neumann and M. R. Vaughan-Lee, *An essay on* BFC-*groups.* Proc. London. Math. Soc. (3) **35**(1977), 213–237.

Department of Mathematics
University of Isfahan
Isfahan
Iran

Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta
T6G 2G1

https://doi.org/10.4153/CMB-1998-010-0 Published online by Cambridge University Press