Canad. Math. Bull. Vol. **23** (4), 1980

FIRST COUNTABLE LINDELÖF EXTENSIONS OF UNCOUNTABLE DISCRETE SPACES

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ABSTRACT. The existence of a first countable Lindelöf extension L of an uncountable discrete space D for which $L-D$ is countable is considered. Assuming CH, such extensions exist; however it is also consistent that no such spaces exist, as follows from $MA+\neg CH$.

All spaces considered are assumed to be regular and T_1. By an *extension* of a space D we mean a space containing D as a dense subspace.

Let D be an uncountable discrete space. Does there exist a first countable Lindelöf extension L of D for which $L-D$ is countable? It is easy to see that no such extension exists for which $L-D$ is countable and discrete.

If $L-D$ is required only to be countable, the situation is somewhat different. The existence of such extensions is independent of the usual axioms of set theory, as we will now show.

EXAMPLE (CH). Recall that a subset S of the real line is called a *Lusin set* if S is uncountable and has countable intersection with every nowhere dense subset of the line. It is well-known that, assuming the continuum hypothesis CH, Lusin sets exist (see e.g. [2]). Thus, assume CH and let S be a Lusin set contained in the irrationals. Let Q denote the rationals and let $L=S\cup Q$. Refine the topology on L by making the points of S isolated, while the rationals keep their usual neighborhoods. (This is equivalent to regarding L as a subspace of the Michael line—the refinement of the usual topology on the reals obtained by isolating the irrationals.)

If G is any open set containing the rationals, then the complement of G is nowhere dense in the reals, and so $S-G$ is countable. This clearly implies that L is Lindelöf. Thus L is a regular, first countable Lindelöf extension of the uncountable discrete space S with $L-S$ countable.

The fact that the above space L obtained using a Lusin set is Lindelöf, has already been observed in [1], where further properties of L are established and used.

Thus, assuming the continuum hypothesis, first countable Lindelöf extensions of uncountable discrete spaces having countable remainder exist. We now show that it is also consistent that no such extensions exist.

Received by the editors February 6, 1978 and, in revised form, November 14, 1978.
AMS(MOS) subject classification: Primary, 54020; Secondary, 54A25.
Key Words and Phrases: First countable, Lindelöf, discrete, CH and $B(\omega_1)$.

We denote the set of all functions from ω into ω by ω^ω. For f, g in ω^ω we write $f \leq g$ if $f(n) \leq g(n)$ for all n in ω, and we write $f \leq_e g$ if f is eventually less than or equal to g, that is, if there exists an $m \in \omega$ such that $f(n) \leq g(n)$ for all $n \geq m$.

Consider the following statement, which we refer to as $B(\omega_1)$. $B(\omega_1)$: If $F \subseteq \omega^\omega$ and $|F| \leq \omega_1$ then there exists $g \in \omega^\omega$ such that $f \leq_e g$ for all $f \in F$.

It is well-known that $B(\omega_1)$ follows from Martin's axiom together with the negation of the continuum hypothesis ($MA + \neg CH$), and is therefore consistent with the usual axioms of set theory. (For example, see [3]). This and other consequences of $MA + \neg CH$ have led to many interesting applications in set theory and topology; here we will use $B(\omega_1)$ to refute the existence of the Lindelöf spaces under consideration.

The following observation, pointed out to the authors by K. Kunen, will be used in conjunction with $B(\omega_1)$.

LEMMA. *Let $F \subseteq \omega^\omega$ such that $|F| = \omega_1$ and let $g \in \omega^\omega$ such that $f \leq_e g$ for all $f \in F$. Then there is a subset F_1 of F and a $g' \in \omega^\omega$ such that $|F_1| = \omega_1$ and $g \leq g'$ and $f \leq g'$ for all $f \in F_1$.*

Proof. For each f in F there is an $m \in \omega$ such that $n \geq m \to f(n) \leq g(n)$. For each $m \in \omega$, define $S_m = \{f \in F : n \geq m \to f(n) \leq g(n)\}$. Then $F = \bigcup_{m \in \omega} S_m$. Since $|F| = \omega_1$, there exists an $m_0 \in \omega$ such that $|S_{m_0}| = \omega_1$. Since there are only countably many functions from m_0 into ω, there is a subset F_1 of S_{m_0} and a function $r : m_0 \to \omega$ such that $|F_1| = \omega_1$ and such that $f \restriction m_0 = r$ for all $f \in F_1$. Define $g' \in \omega^\omega$ as follows:

$$g'(n) = \max\{r(n), g(n)\} \quad \text{if} \quad n < m_0$$

and

$$g'(n) = g(n) \quad \text{if} \quad n \geq m_0.$$

Then $g \leq g'$ and $f \leq g'$ for all f in F_1, as desired.

THEOREM. *Assume $B(\omega_1)$. Let D be an uncountable discrete space. Then D has no first countable Lindelöf extension L for which L–D is countable.*

Proof. Let L be a first countable extension of D with L–D countable. We show that L is not Lindelöf. Ennumerate the points of $L - D$ as $L - D = \{x_n : n \in \omega\}$. For each n, choose a sequence of open sets $\{G_n(m) : m \in \omega\}$ such that $\bigcap_{m \in \omega} G_n(m) = \{x_n\}$, and such that $G_n(m+1) \subseteq G_n(m)$ for all m. We may assume, without loss of generality, that $|D| = \omega_1$. (If not, choose a subset D_1 of D such that $|D_1| = \omega_1$; $D_1 \cup \{x_n : n \in \omega\}$ is closed in L, so if we show that $D_1 \cup \{x_n : n \in \omega\}$ is not Lindelöf it follows that L is not Lindelöf.)

Let $x \in D$. For each n in ω, choose $f_x(n) \in \omega$ such that $x \notin G_n(f_x(n))$. This defines a function $f_x \in \omega^\omega$ for each x in D. Let $F = \{f_x : x \in D\}$. By $B(\omega_1)$, there is a $g \in \omega^\omega$ such that $f_x \leq_e g$ for all x in D, and by the lemma, there is a $g' \in \omega^\omega$ and a subset D_1 of D such that $|D_1| = \omega_1$ and such that $f_x \leq g'$ for all x in D_1.

Let $x \in D_1$. Then for all $n \in \omega$, $f_x(n) \leq g'(n)$ and so $x \notin G_n(g'(n))$, because the sequence $\{G_n(m) : m \in \omega\}$ is descending. Thus $D_1 \cap [\bigcup_{n\in\omega} G_n(g'(n))] = \phi$. Therefore the open cover

$$\{G_n(g'(n)) : n \in \omega\} \cup \left\{\{x\} : x \in D - \bigcup_{n\in\omega} G_m(g'(n))\right\}$$

has no countable subcover, and hence L is not Lindelöf.

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https://doi.org/10.4153/CMB-1980-058-6 Published online by Cambridge University Press