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Application of capacities to space–time fractional dissipative equations I: regularity and the blow-up set

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Abstract. We apply capacities to explore the space–time fractional dissipative equation:

$$\begin{cases} \partial_t^\beta u(t,x) = -\nu(-\Delta)^{\alpha/2}u(t,x) + f(t,x), & (t,x)\in\mathbb{R}^{1+n}_+, \\ u(0,x)=\varphi(x),\ x\in\mathbb{R}^n, \end{cases} \tag{0.1}$$

where $\alpha > n$ and $\beta \in (0,1)$. In this paper, we focus on the regularity and the blow-up set of mild solutions to (0.1). First, we establish the Strichartz-type estimates for the homogeneous term $R_{\alpha,\beta}(\varphi)$ and inhomogeneous term $G_{\alpha,\beta}(g)$, respectively. Second, we obtain some space–time estimates for $G_{\alpha,\beta}(g)$. Based on these estimates, we prove that the continuity of $R_{\alpha,\beta}(\varphi)(t,x)$ and the Hölder continuity of $G_{\alpha,\beta}(g)(t,x)$ on $\mathbb{R}^{1+n}_+$, which implies a Moser–Trudinger-type estimate for $G_{\alpha,\beta}$. Then, for a newly introduced $L^q_tL^p_x$-capacity related to the space–time fractional dissipative operator $\partial_t^\beta + (-\Delta)^{\alpha/2}$, we perform the geometric-measure-theoretic analysis and establish its basic properties. Especially, we estimate the capacity of fractional parabolic balls in $\mathbb{R}^{1+n}_+$ by using the Strichartz estimates and the Moser–Trudinger-type estimate for $G_{\alpha,\beta}$. A strong-type estimate of the $L^q_tL^p_x$-capacity and an embedding of Lorentz spaces are also derived. Based on these results, especially the Strichartz-type estimates and the $L^q_tL^p_x$-capacity of fractional parabolic balls, we deduce the size, i.e., the Hausdorff dimension, of the blow-up set of solutions to (0.1).

1 Introduction

We will study the following space–time fractional equation:

$$\begin{cases} \partial_t^\beta u(t,x) = -\nu(-\Delta)^{\alpha/2}u(t,x) + f(t,x), & (t,x)\in\mathbb{R}^{1+n}_+ := [0,\infty)\times\mathbb{R}^n, \\ u(0,x)=\varphi(x),\ x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

with the initial condition $\varphi(\cdot)$ and the nonhomogeneous term $f(\cdot,\cdot)$. Here, $(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian, and the symbol ∂_t^β denotes the Caputo fractional derivative defined as

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$$\partial_t^\beta u(t,x) = \frac{1}{\Gamma(1-\beta)} \int_0^t \partial_r u(r,x) \frac{dr}{(t-r)^\beta}, \quad \beta \in (0,1).$$

When $\beta = 1$ and $\alpha = 2$, (1.1) is the classical heat equation, which is extremely important in many areas of mathematics, physics, fluid dynamics, and engineering. When $\beta = 1$ and $\alpha \in (0,2)$, (1.1) reduces to the space-fractional heat equation, which has been applied to the research of fluid dynamics (see [18, 48, 52, 60, 62] and the references therein). When $\beta \in (0,1)$ and $\alpha = 2$, (1.1) becomes the time-fractional heat equation

$$\partial_t^\beta u(t,x) - \Delta u(t,x) = 0, \tag{1.2}$$

which exhibits the subdiffusive behavior and is related with anomalous diffusion, or diffusion in nonhomogeneous media, with random fractal structures (cf. [51]).

The time–space fractional dissipative operator

$$L_{\alpha,\beta} := \partial_t^\beta + \nu(-\Delta)^\alpha, \quad \alpha > 0 \ \& \ \beta \in (0,1),$$

has the salient significance and backgrounds in mathematical physics. The fractional Laplacian $(-\Delta)^\alpha$ plays a significant role in many areas of mathematics, such as harmonic analysis and PDEs. In addition, the fractional Laplacian has been applied to study a wide class of physical systems and engineering problems, including Lévy flights, stochastic interfaces, and anomalous diffusion problems. For example, in fluid mechanics, $(-\Delta)^\alpha$ is often applied to describe many complicated phenomena via partial differential equations. Caffarelli and Silvestre showed in [10] that any fractional power of the Laplacian can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem. This characterization of $(-\Delta)^\alpha$ via the local (degenerate) PDE was first used in [9] to get regularity estimates of the obstacle problem for the fractional Laplacian. We also refer the reader to [11, 12, 34, 56] for further information on applications of the fractional Laplacian in PDEs.

The Caputo fractional derivative ∂_t^β was introduced by Caputo [13] when studying some anelastic materials and soon became a popular tool in engineering (see also [8, 30, 41, 45] for generalizations of Caputo derivatives). Similar to the ordinary derivative ∂_t, the Caputo derivative is suitable for initial value problems, and is extremely important in physical systems (cf. [46]) since the derivatives paired with fractional Brownian noise must be Caputo derivatives in physical systems which are different from those in the financial model (see [21]). For this reason, the time-fractional calculus is widely used in a rather large number of scientific branches, such as statistical mechanics, theoretical physics, theoretical neuroscience, the theory of complex chemical reactions, fluid dynamics, hydrology, and mathematical finance (see, e.g., [40] for an extensive list of references).

In recent years, fractional partial differential equations with Caputo time derivatives have attracted the attention of many researchers. There exist many related results on this topic. In [6], Allen et al. established a De Giorgi–Nash–Moser Hölder regularity theorem for solutions and also proved results regarding the existence, the uniqueness, and higher regularities in time. Eidelman and Kochubei in [22] provided fundamental solutions of the Cauchy problem of fractional diffusion equations. Chen et al. [17] proved the existence and uniqueness of solutions to a class of SPDEs with time-fractional derivatives. Li and Liu in [44] developed some compactness criteria that are analogies of the Aubin–Lions lemma for the existence of weak solutions to time-fractional PDEs. In [5], Allen proved the uniqueness for weak solutions to abstract parabolic equations with fractional Caputo or Marchaud time derivatives. Interested readers can also refer to [21, 23, 24, 26, 27, 31, 32, 55, 57, 59, 61].

Compared with the aforesaid achievements, the study of space–time fractional PDEs with the Caputo time derivative and the fractional Laplacian on spatial variables is relatively few. In [42], Kolokoltsov and Veretennikova studied the Cauchy problem for nonlinear in time and space pseudo-differential equations and analyzed the well-posedness and smoothing properties of the corresponding linear equation. For a nonlocal heat equation with fractional order both in space and time, Kemppainen et al. [39] proved a representation formula for classical solutions, a quantitative decay rate at which the solution tends to the fundamental solution, an optimal L^2-decay of mild solutions in all dimensions, and L^2-decay of weak solutions via energy methods. For a system of nonlinear space–time fractional SPDEs, Mijena and Nane in [53] proved the existence and the uniqueness of the mild solution, and the bounds for intermittency fronts solutions to these equations were investigated in [54]. For space–time fractional SPDEs in a Gaussian noisy environment, Chen et al. in [16] proved the existence and the uniqueness of solutions. Foondun and Nane [25] studied the asymptotic properties of space–time fractional SPDEs. Time-fractional Hamilton–Jacobi equations and the notion of viscosity solutions have been discussed in [28, 43].

Different from the abovementioned works on space–time fractional equations, in this paper, we aim to investigate the regularity properties and the blow-up set of solutions to equation (1.1) via capacities. This work is closely motivated by [15, 35, 36, 64, 65]. Using the fractional Duhamel principle, the mild solution of (1.1) is represented as

$$u(t,x)=\int_{\mathbb{R}^n}G_t(x-y)\varphi(y)dy+\int_0^t\int_{\mathbb{R}^n}G_{t-s}(x-y)\partial_s^{1-\beta}f(s,y)dyds.$$

Let $f(t,x)=I_t^{1-\beta}g(t,x)$, where $I_t^{1-\beta}$ denotes the fractional integral corresponding to the time variable t. Then

$$u(t,x)=\int_{\mathbb{R}^n}G_t(x-y)\varphi(y)dy+\int_0^t\int_{\mathbb{R}^n}G_{t-s}(x-y)g(s,y)dyds. \tag{1.3}$$

Set

$$\begin{cases}R_{\alpha,\beta}(\varphi)(t,x):=\int_{\mathbb{R}^n}G_t(x-y)\varphi(y)dy,\\ G_{\alpha,\beta}(g)(t,x):=\int_0^t\int_{\mathbb{R}^n}G_{t-s}(x-y)g(s,y)dyds.\end{cases}$$

Under the assumption that $\alpha>n$ and $\beta\in(0,1)$, we first establish the homogeneous and inhomogeneous Strichartz-type estimates for (1.1), some other space–time estimates of $G_{\alpha,\beta}(g)$, and the regularity of $R_{\alpha,\beta}(\varphi)$ and $G_{\alpha,\beta}(g)$, respectively. Strichartz-type estimates are significant tools for PDEs, such as nonlinear wave equations and Schrödinger equations (see [29, 37, 38, 52, 58, 66]). In [25], Foondun and Nane proved that the space–time fractional heat kernel $G_t(\cdot)$ satisfies the following estimate:

$$G_t(x) \sim \frac{t^\beta}{(|x| + t^{\beta/\alpha})^{n+\alpha}}, \quad \alpha > n \ \& \ \beta \in (0,1]. \tag{1.4}$$

It follows from (1.4) and the Young inequality that

$$\|R_{\alpha,\beta}(\varphi)\|_{L^p_x(\mathbb{R}^n)} \lesssim t^{-n\beta(1/r-1/p)/\alpha} \|\varphi\|_{L^r_x(\mathbb{R}^n)}, \quad 1 \le r \le p \le \infty. \tag{1.5}$$

Inequality (1.5) allows us, in Section 2.2, to deduce the Strichartz-type estimates and the space–time estimates for $R_{\alpha,\beta}$ and $G_{\alpha,\beta}$ related to $L^q_t(I; L^p_x(\mathbb{R}^n))$, respectively. Here, the mixed norm Lebesgue space $L^q_t(I; L^p_x(\mathbb{R}^n))$, $1 \le p, q \le \infty$, is defined as the set of all measurable functions $g(\cdot,\cdot)$ over an interval $I \subseteq (0,\infty)$ satisfying

$$\|g\|_{L^q_t(I;L^p_x(\mathbb{R}^n))} := \left(\int_I \left(\int_{\mathbb{R}^n} |g(t,x)|^p dx \right)^{q/p} dt \right)^{1/q} < \infty.$$

Specially, for $I = (0,\infty)$, we denote $L^q_t((0,\infty); L^p_x(\mathbb{R}^n))$ by $L^q_t L^p_x(\mathbb{R}^{1+n}_+)$. These space–time estimates obtained in Section 2.2 will be used to compute the lower bound of the capacities of fractional parabolic balls. Moreover, we investigate the regularities of $R_{\alpha,\beta}$ and $G_{\alpha,\beta}$. By the aid of fractional heat kernels $K_{\alpha,t}(\cdot)$, in Proposition 2.11, we prove that there exist positive constants C and δ such that for $|h| < t^{\beta/\alpha}$,

$$\left|G_t(x+h) - G_t(x)\right| \le C \left(\frac{|h|}{t^{\beta/\alpha}} \right)^\delta \frac{t^\beta}{(t^{\beta/\alpha} + |x|)^{n+\alpha}},$$

which, together with the estimate

$$\begin{aligned} &\left|R_{\alpha,\beta}(\varphi)(t_1,x) - R_{\alpha,\beta}(\varphi)(t_2,x)\right| \\ &\quad \lesssim \|\varphi\|_{L^p_x(\mathbb{R}^n)} \left| t_1^{\beta(1-1/\alpha-n/\alpha p)} - t_2^{\beta(1-1/\alpha-n/\alpha p)} \right|, \quad 1 \le p \le \infty, \end{aligned}$$

implies that $R_{\alpha,\beta}(\varphi)$ is continuous on $\mathbb{R}^{1+n}_+$ for $\varphi \in L^p_x(\mathbb{R}^n)$ (see Theorem 2.12). Let $p \in [1,\infty)$, $1 < q < \infty$, $n\beta/p + \alpha/q < \alpha$, $(t,x) \in \mathbb{R}^{1+n}_+$, and $\|g\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} < \infty$. For the inhomogeneous part $G_{\alpha,\beta}$, under the assumption that (t,x) is sufficiently close to (t_0,x_0), the following Hölder continuity holds, precisely:

$$\begin{aligned} &\left|G_{\alpha,\beta}(g)(t,x) - G_{\alpha,\beta}(g)(t_0,x_0)\right| \\ &\quad \lesssim \|g\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} \left(|t-t_0|^{1-1/q-\beta n/p\alpha} + |x-x_0|^{\alpha(1-1/q)/\beta - n/p} \right) \end{aligned}$$

(see Theorem 2.13).

Theorems 2.12 and 2.13 indicate that the blow-up phenomenon of mild solutions to (1.1) merely occurs on the nonlinear part of (1.3), i.e., $G_{\alpha,\beta}(g)$ for $n\beta/p + \alpha/q > \alpha$. Based on this observation, in Section 3, we introduce the following blow-up set, denoted by $\mathcal{B}[G_{\alpha,\beta}(g), p, q]$, of solutions to equation (1.1):

$$\mathcal{B}[G_{\alpha,\beta}(g), p, q] := \left\{ (t,x) \in \mathbb{R}^{1+n}_+ : \ G_{\alpha,\beta}(g)(t,x) = \infty \right\}$$

for nonnegative functions $g(\cdot,\cdot) \in L^q_t L^p_x(\mathbb{R}^{1+n}_+)$. We apply capacities to measure the size, i.e., the Hausdorff dimension, of $\mathcal{B}[G_{\alpha,\beta}(g), p, q]$. In the literature, capacities related to operators and given function spaces are widely applied in the research of the potential theory and partial differential equations. For example, the Besov-type capacities $cap(\cdot; \dot{\Lambda}_{\alpha}^{p,q})$ were used to establish the embedding of homogeneous Besov spaces into Lorentz spaces with respect to nonnegative Borel measures (see [4, 50, 63, 65]). The embedding of Sobolev spaces via heat equations and the p-variational capacity was due to Xiao [64, 65]. In [20], Dafni et al. introduced a class of measures generated by Riesz, or Bessel, or Besov capacities, and established geometric characterizations of these measures. For further information on this topic, we refer the reader to [1, 15, 19, 47, 67] and the references therein.

To measure the Hausdorff dimension of the blow-up set of the wave equation, in [2], Adams introduced a class of capacities related to the wave operator $\Box$ and $L_t^q L_x^p$-norm spaces, and studied the size (in terms of Hausdorff content) of the blow-up sets of weak solutions to the nonhomogeneous wave equation in three space dimensions. By a similar idea, Jiang et al. [35] applied the $L_t^q L_x^p$-type capacities to investigate the blow-up set of a weak solution to the special case $\beta = 1$ of equation (1.1). Following the idea of [2, 35], we introduce the following $L_t^q L_x^p$-capacity associated with $G_{\alpha,\beta}$.

Definition 1.1 Let $1 \le p, q < \infty$. Denote by $p \wedge q := \min\{p, q\}$. For any set $E \subset \mathbb{R}_+^{n+1}$, define

$$C_{p,q}^{(\alpha,\beta)}(E) := \inf\left\{ \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p\wedge q} : \ g \ge 0 \ \& \ G_{\alpha,\beta}(g) \ge 1_E \right\}$$

be the $L_t^q L_x^p$-capacity of E for the space–time fractional dissipative operator, where 1_E is the characteristic function of E.

In Sections 3.1–3.3, we study the dual form, the basic properties of the $L_t^q L_x^p$-capacity $C_{p,q}^{(\alpha,\beta)}(\cdot)$, and further, utilize Theorem 2.6 to estimate the $L_t^q L_x^p$-capacities of fractional parabolic balls $B_{r_0}^{(\alpha,\beta)}(t_0, x_0)$. In Section 3.4, denote by E_λ with $\lambda > 0$, the distribution set of $G_{\alpha,\beta}$, i.e.,

$$E_\lambda := \left\{ (t,x) \in \mathbb{R}_+^{1+n} : \quad G_{\alpha,\beta}(g)(t,x) > \lambda \right\}.$$

Let $1 \le p, q < \infty$, $\alpha > n$ and $\beta \in (0,1)$. We obtain the following capacitary strong-type inequality:

$$\int_0^\infty \lambda^{p\wedge q} C_{p,q}^{(\alpha,\beta)}(E_\lambda) \frac{d\lambda}{\lambda} \lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p\wedge q} \quad \forall\, g(\cdot,\cdot) \in L_t^q L_x^p(\mathbb{R}_+^{1+n})$$

(see Theorem 3.8). As a corollary of Theorem 3.8, in Theorem 3.10, we deduce an equivalent condition of the embedding from $L_t^q L_x^p(\mathbb{R}_+^{1+n})$ to Lorentz spaces $L^{(r,s)}(\mathbb{R}_+^{1+n}, \mu)$. By use of the results obtained in Sections 3.2 and 3.3, we obtain that the Hausdorff dimension of $\mathcal{B}[G_{\alpha,\beta}(g), p, q]$ is dominated by $n\beta - \alpha(p \wedge q - 1)$ under the assumption that $1 \le p < \infty$ and $1 < q < \infty$ with $\alpha > n$ and $(p \wedge q)(n\beta/p + \alpha/q - \alpha) > 0$ (see Theorem 4.2).

Remark 1.2

(i) In the main results of this paper, we restrict the scope of the index (α, β) to $(n, \infty) \times (0, 1)$. For equation (1.1), there are many important cases concerning $\alpha \le n$. However, for the characterization of (1.1) via the capacity, we need the

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convolution kernel $G_t(\cdot)$ satisfies the upper bound estimate:

$$G_t(x) \le \frac{Ct^{\beta}}{(|x| + t^{\beta/\alpha})^{n+\alpha}},$$

which is true for $\alpha > n$.

(ii) The results in the paper are stated for assumptions on $g(\cdot,\cdot) \in L^q_t L^p_x(\mathbb{R}^{n+1}_+)$. In fact, we can replace this assumption by the assumption that the nonhomogeneous term $f(\cdot,\cdot)$ satisfies

$$\left(\int_0^\infty \left(\int_{\mathbb{R}^n} |\partial_t^{1-\beta} f(t,x)|^p dx\right)^{q/p} dt\right)^{1/q} < \infty,$$

which indicates $g(\cdot,\cdot) \in L^q_t L^p_x(\mathbb{R}^{n+1}_+)$.

Some notations:

- Let $\Omega \subseteq \mathbb{R}^n$. Throughout this article, we use $C(\Omega)$ to denote the space of all continuous functions on Ω. Let $k \in \mathbb{N}_+ \cup \{\infty\}$. The symbol $C^k(\Omega)$ denotes the class of all functions $f: \Omega \to \mathbb{R}$ with k continuous partial derivatives. Let $C_0^\infty(\Omega)$ stand for all infinitely smooth functions with compact supports in Ω.
- For $1 \le p \le \infty$, denote by p' the conjugate number of p, i.e., $1/p + 1/p' = 1$. $\mathsf{U} \simeq \mathsf{V}$ represents that there is a constant $c > 0$ such that $c^{-1}\mathsf{V} \le \mathsf{U} \le c\mathsf{V}$ whose right inequality is also written as $\mathsf{U} \lesssim \mathsf{V}$. Similarly, one writes $\mathsf{V} \gtrsim \mathsf{U}$ for $\mathsf{V} \ge c\mathsf{U}$.
- For convenience, the positive constant C may change from one line to another and usually depends on the dimension n, α, β, and other fixed parameters. For $f \in \mathscr{S}(\mathbb{R}^n)$, $\widehat{f}$ means the Fourier transform of f.

2 Regularity estimates

In this section, we investigate the regularity of solutions to (1.1). We first state some preliminaries which will be used in the sequel. For further information, we refer the reader to [25] and the references therein.

2.1 Basic estimates of the space–time fractional heat kernel

Let X_t denote a symmetric α stable process with the density function denoted by $K_{\alpha/2,t}(\cdot)$. This is characterized through the Fourier transform, which is given by

$$\widehat{K_{\alpha/2,t}}(\xi) = e^{-\nu t|\xi|^\alpha}.$$

Let $D = \{D_r, r \ge 0\}$ denote a β-stable subordinator, and let E_t be its first passage time. It is well known that the density of the time changed X_{E_t} is given by $G_t(x)$. By conditioning, we have

$$G_t(x) = \int_0^\infty K_{\alpha/2,s}(x) f_{E_t}(s)ds, \tag{2.1}$$

where

$$f_{E_t}(s) = t\beta^{-1}s^{-1-1/\beta}g_\beta(ts^{-1/\beta}).$$

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Here, $g_\beta(\cdot)$ is the density function of D_1 and is infinitely differentiable on the entire real line, with $g_\beta(u) = 0$ for $u \le 0$. Moreover,

$$g_\beta(u) \sim \begin{cases} K(\beta/u)^{(1-\beta/2)/(1-\beta)} \exp\Big\{ -|1-\beta|(u/\beta)^{\beta/(\beta-1)}\Big\}, & u \to 0+, \\ \dfrac{\beta}{\Gamma(1-\beta)} u^{-\beta-1}, & u \to \infty. \end{cases} \tag{2.2}$$

Another explicit description of the heat kernel $G_t(\cdot)$ is as follows. Denote by $\widetilde{(\cdot)}$ the Laplace transform. Then

$$\widetilde{\widehat{G}}_t(x) = \frac{\lambda^{\beta-1}}{\lambda^\beta + \nu|\xi|^\alpha}.$$

Inverting the Laplace transform yields the Fourier transform of $G_t(\cdot)$ is $\widehat{G}_t(\xi) = E_\beta(-\nu|\xi|^\beta t^\beta)$, where $E_\beta(\cdot)$ is the Mittag–Leffler function, which is defined as

$$\begin{cases} E_\beta(x) = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\beta k)}, \\ \dfrac{1}{1+\Gamma(1-\beta)} \le E_\beta(-x) \le \dfrac{1}{1+\Gamma(1+\beta)^{-1}x}, \quad x > 0. \end{cases}$$

Let $H^{m,n}_{p,q}$ denote the H-function given in [49, Definition 1.9.1, p. 55]. By the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} f(\xi)d\xi = \frac{1}{\pi}\int_0^\infty f(\xi)\cos(\xi x)d\xi,$$

it can be deduced from the cosine transform of the H-function (cf. [33, equation (12.9)]) that

$$G_t(x) = \frac{1}{|x|} H^{2,1}_{3,3}\left[\frac{|x|^\alpha}{\nu t^\beta}\middle|^{(1,1),(1,\beta),(1,\alpha/2)}_{(1,\alpha),(1,1),(1,\alpha/2)}\right].$$

Specially, by reduction formula for the H-function, we can get, for $\alpha = 2$,

$$G_t(x) = \begin{cases} \dfrac{1}{|x|} H^{1,0}_{1,1}\left[\dfrac{|x|^2}{\nu t^\beta}\middle|^{(1,\beta)}_{(1,2)}\right], & \beta \in (0,1). \\ \dfrac{1}{(4\nu\pi t)^{1/2}} \exp(-|x|^2/4\nu t), & \beta = 1. \end{cases}$$

Foondun and Nane [25] obtained the following estimate for $G_t(\cdot)$.

Proposition 2.1 [25, Lemma 2.1] *Let $\beta \in (0,1)$.*

(i) *There exists a positive constant C_1 such that for all $x \in \mathbb{R}^n$,*

$$G_t(x) \ge C_1 \min\left\{t^{-\beta n/\alpha}, \frac{t^\beta}{|x|^{n+\alpha}}\right\}. \tag{2.3}$$

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(ii) *If we further suppose that* $\alpha > n$, *then there exists a positive constant* C_2 *such that for all* $x \in \mathbb{R}^n$,

$$G_t(x) \le C_2 \min\left\{t^{-\beta n/\alpha}, \frac{t^\beta}{|x|^{n+\alpha}}\right\}. \tag{2.4}$$

The following is an immediate corollary of Proposition 2.1.

Corollary 2.2 *Let* $\alpha > n$ *and* $\beta \in (0,1)$. *Then*

$$\begin{cases} G_t(x) \simeq \dfrac{t^\beta}{(|x| + t^{\beta/\alpha})^{n+\alpha}}, & (x,t) \in \mathbb{R}^{1+n}_+, \\ \displaystyle\int_{\mathbb{R}^n} G_t(x)dx \lesssim 1. \end{cases} \tag{2.5}$$

Proof Below, we always assume that $\alpha > n$. By (i) of Proposition 2.1, it holds

$$G_t(x) \gtrsim \begin{cases} \dfrac{t^\beta}{|x|^{n+\alpha}} \gtrsim \dfrac{t^\beta}{(|x| + t^{\beta/\alpha})^{n+\alpha}}, & |x| \ge t^{\beta/\alpha}, \\ \dfrac{1}{t^{n\beta/\alpha}} \gtrsim \dfrac{t^\beta}{(|x| + t^{\beta/\alpha})^{n+\alpha}}, & |x| < t^{\beta/\alpha}. \end{cases}$$

On the other hand, it can be deduced from (ii) of Proposition 2.1 that

$$G_t(x) \lesssim \begin{cases} \dfrac{t^\beta}{|x|^{n+\alpha}} \lesssim \dfrac{t^\beta}{(|x| + t^{\beta/\alpha})^{n+\alpha}}, & |x| \ge t^{\beta/\alpha}, \\ \dfrac{1}{t^{\beta n/\alpha}} \lesssim \dfrac{t^\beta}{(|x| + t^{\beta/\alpha})^{n+\alpha}}, & |x| < t^{\beta/\alpha}. \end{cases}$$

Finally,

$$G_t(x) \simeq \frac{t^\beta}{(|x| + t^{\beta/\alpha})^{n+\alpha}},$$

which, via a direct computation, gives

$$\int_{\mathbb{R}^n} G_t(x)dx \simeq \int_{\mathbb{R}^n} \frac{t^\beta}{(|x| + t^{\beta/\alpha})^{n+\alpha}} dx \lesssim 1. \qquad \blacksquare$$

In the following, we assume $\alpha > n$. We can deduce the following lemma from Corollary 2.2.

Lemma 2.3 *Let* $1 \le r \le p \le \infty$ *and* $\varphi \in L^r(\mathbb{R}^n)$. *For* $\alpha > n$, $\beta \in (0,1)$, *and* $t > 0$,

$$\|R_{\alpha,\beta}(\varphi)\|_{L^p_x(\mathbb{R}^n)} \lesssim t^{-n\beta(1/r-1/p)/\alpha} \|\varphi\|_{L^r_x(\mathbb{R}^n)}.$$

Proof Let q obey $1/r + 1/q = 1/p + 1$. By Young's inequality,

$$\|R_{\alpha,\beta}(\varphi)\|_{L^p_x(\mathbb{R}^n)} = \|G_t * \varphi\|_{L^p_x(\mathbb{R}^n)} \lesssim \|\varphi\|_{L^r_x(\mathbb{R}^n)} \|G_t(\cdot)\|_{L^q_x(\mathbb{R}^n)}.$$

It follows from Corollary 2.2 that

$$\|G_t(\cdot)\|_{L^q_x(\mathbb{R}^n)} \simeq \left(\int_{\mathbb{R}^n} \frac{t^{\beta q}}{(|x| + t^{\beta/\alpha})^{q(n+\alpha)}} dx\right)^{1/q} \lesssim t^{\beta n(1/q-1)/\alpha},$$

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which implies

$$\|R_{\alpha,\beta}(\varphi)\|_{L^p_x(\mathbb{R}^n)} \lesssim t^{\beta n(1/q-1)/\alpha}\|\varphi\|_{L^r_x(\mathbb{R}^n)} = t^{-\beta n(1/r-1/p)/\alpha}\|\varphi\|_{L^r_x(\mathbb{R}^n)}. \qquad \blacksquare$$

2.2 Strichartz-type estimates

In this section, we establish homogeneous and inhomogeneous Strichartz-type estimates.

Definition 2.4 Let $\mathbb{X}$ be a Banach space, and let $I = [0, T)$.

(i) The space $C_\sigma(I, \mathbb{X})$ is defined as the set of all $f \in C(I; \mathbb{X})$ such that

$$\|f\|_{C_\sigma(I;\mathbb{X})} := \sup_{t\in I} t^{1/\sigma}\|f(t,\cdot)\|_{\mathbb{X}} < \infty.$$

(ii) The space $C_0(I; \mathbb{X})$ is defined as the set of all bounded continuous functions from I to $\mathbb{X}$.

For $R_{\alpha,\beta}(\varphi)$, we can prove the following Strichartz-type estimates.

Theorem 2.5 *Assume that $\alpha > n$ and $0 < \beta < 1$. Let $1 \le r \le p < \infty$ satisfying $1/q = \beta n(1/r - 1/p)/\alpha$. Given $\varphi \in L^r(\mathbb{R}^n)$ and $I = [0, T)$ with $0 < T \le \infty$.*

(i) *$R_{\alpha,\beta}(\varphi) \in L^q_t(I; L^p_x(\mathbb{R}^n)) \cap C_0(I; L^r_x(\mathbb{R}^n))$ with the estimate*

$$\|R_{\alpha,\beta}(\varphi)\|_{L^q_t(I;L^p_x(\mathbb{R}^n))} \lesssim \|\varphi\|_{L^r_x(\mathbb{R}^n)}.$$

(ii) *$R_{\alpha,\beta}(\varphi) \in C_q(I; L^p_x(\mathbb{R}^n)) \cap C_0(I; L^r_x(\mathbb{R}^n))$ with the estimate*

$$\|R_{\alpha,\beta}(\varphi)\|_{C_q(I;L^p_x(\mathbb{R}^n))} \lesssim \|\varphi\|_{L^r_x(\mathbb{R}^n)}.$$

Proof (i) We divide the argument into two cases.

Case 1: $p = r$ and $q = \infty$. By Lemma 2.3, we obtain

$$\|R_{\alpha,\beta}(\varphi)\|_{L^\infty_t(I;L^r_x(\mathbb{R}^n))} \lesssim \sup_{t>0} t^{-n\beta(1/r-1/r)/\alpha}\|\varphi\|_{L^r_x(\mathbb{R}^n)} \lesssim \|\varphi\|_{L^r_x(\mathbb{R}^n)}.$$

Case 2: $p \ne r$. Denote $F(t)(\varphi) = \|R_{\alpha,\beta}(\varphi)\|_{L^p_x(\mathbb{R}^n)}$. Since $1/q = \beta n(1/r - 1/p)/\alpha$, a further use of Lemma 2.3 can deduce that

$$F(t)(\varphi) = \|R_{\alpha,\beta}(\varphi)\|_{L^p_x(\mathbb{R}^n)} \lesssim t^{-n\beta(1/r-1/p)/\alpha}\|\varphi\|_{L^r_x(\mathbb{R}^n)} = t^{-1/q}\|\varphi\|_{L^r_x(\mathbb{R}^n)}. \tag{2.6}$$

It follows from (2.6) that $F(t)$ is a weak (r, q)-type operator since

$$\left|\left\{t : |F(t)(\varphi)| > \tau\right\}\right| \le \left|\left\{t : t < \left(\frac{\|\varphi\|_{L^r_x(\mathbb{R}^n)}}{\tau}\right)^q\right\}\right| \lesssim \left(\frac{\|\varphi\|_{L^r_x(\mathbb{R}^n)}}{\tau}\right)^q.$$

On the other hand, the inequality

$$|R_{\alpha,\beta}(\varphi)(x)| \lesssim \int_{\mathbb{R}^n} \frac{t^\beta}{(|x-y| + t^{\beta/\alpha})^{n+\alpha}}|\varphi(y)|dy$$

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implies that

$$\|F(t)(\varphi)\|_{L^\infty_t(I)} = \sup_{t>0} \|R_{\alpha,\beta}(\varphi)\|_{L^p_x(\mathbb{R}^n)} \lesssim t^{-n\beta(1/p-1/p)/\alpha}\|\varphi\|_{L^p_x(\mathbb{R}^n)},$$

which means that $F(t)$ is a (p,∞)-type operator.

We can find another triplet (q_1, p, r_1) such that $q_1 < q < \infty$ and $r_1 < r < p$ satisfying

$$\begin{cases} 1/q = \theta/q_1 + (1-\theta)/\infty, \\ 1/r = \theta/r_1 + (1-\theta)/p, \\ 1/q_1 = \beta n(1/r_1 - 1/p)/\alpha. \end{cases}$$

The Marcinkiewicz interpolation theorem implies that $F(t)$ is a strong (r,q)-type operator and

$$\|R_{\alpha,\beta}(\varphi)\|_{L^q_t(I;L^p_x(\mathbb{R}^n))} \lesssim \|\varphi\|_{L^r_x(\mathbb{R}^n)}.$$

(ii) The argument can be also divided into two cases.

Case 3: $p = r$ and $q = \infty$. We have

$$\|R_{\alpha,\beta}(\varphi)\|_{L^\infty_t(I;L^p_x(\mathbb{R}^n))} = \sup_{t>0}\|R_{\alpha,\beta}(\varphi)\|_{L^p_x(\mathbb{R}^n)} \lesssim \|\varphi\|_{L^p_x(\mathbb{R}^n)}.$$

Case 4: $p \neq r$. Because $1/q = \beta n(1/r - 1/p)/\alpha$, upon taking q^* such that $1/p + 1 = 1/r + 1/q^*$, we obtain

$$\|R_{\alpha,\beta}(\varphi)\|_{C_q(I;L^p_x(\mathbb{R}^n))} \lesssim \sup_{t>0} t^{1/q} t^{-\beta n(1/r-1/p)/\alpha}\|\varphi\|_{L^r_x(\mathbb{R}^n)} \lesssim \|\varphi\|_{L^r_x(\mathbb{R}^n)}.$$

On the other hand, for $t \in I$, $\|R_{\alpha,\beta}(\varphi)\|_{L^r_x(\mathbb{R}^n)} \lesssim \|\varphi\|_{L^r_x(\mathbb{R}^n)}$. Consequently, $R_{\alpha,\beta}(\varphi) \in C_0(I; L^r_x(\mathbb{R}^n))$. ■

We then give the following Strichartz-type estimate for $G_{\alpha,\beta}(g)$.

Theorem 2.6 *Assume that $\alpha > n$ and $0 < \beta < 1$. If (q,p) and $(\widetilde{q},\widetilde{p})$ satisfy*

$$\begin{cases} 1 \le \widetilde{p} < p \le \infty \ \& \ 1 < \widetilde{q} < q < \infty, \\ (1/\widetilde{q} - 1/q) + \beta n(1/\widetilde{p} - 1/p)/\alpha = 1, \end{cases}$$

then

$$\left\| \int_0^t \int_{\mathbb{R}^n} G_{t-s}(x-y) g(s,y)\,dy\,ds \right\|_{L^q_t(I;L^p_x(\mathbb{R}^n))} \lesssim \|g\|_{L^{\widetilde{q}}_t(I;L^{\widetilde{p}}_x(\mathbb{R}^n))}.$$

Proof An application of Lemma 2.3 yields

$$\left\| \int_{\mathbb{R}^n} G_{t-s}(x-y) g(s,y)\,dy \right\|_{L^p_x(\mathbb{R}^n)} \lesssim |t-s|^{-n\beta(1/\widetilde{p}-1/p)/\alpha}\|g(s,\cdot)\|_{L^{\widetilde{p}}_x(\mathbb{R}^n)}.$$

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It follows from the boundedness of fractional integrals that

$$\begin{aligned}
&\left\|\int_0^t \int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dyds\right\|_{L_t^q(I;L_x^p(\mathbb{R}^n))}\\
&\lesssim \left\|\int_0^t \left\|\int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dy\right\|_{L_x^p(\mathbb{R}^n)} ds\right\|_{L_t^q(I)}\\
&\lesssim \left\|\int_0^t |t-s|^{-n\beta(1/\widetilde{p}-1/p)/\alpha}\|g(s,\cdot)\|_{L_x^{\widetilde{p}}(\mathbb{R}^n)}ds\right\|_{L_t^q(I)}\\
&\lesssim \|g\|_{L_t^{\widetilde{q}}(I;L_x^{\widetilde{p}}(\mathbb{R}^n))},
\end{aligned}$$

which finishes the proof. ∎

2.3 Other space–time estimates for $G_{\alpha,\beta}$

We will establish the following space–time estimate for $G_{\alpha,\beta}$.

Theorem 2.7 *Given $\alpha > n$ and $0 < \beta < 1$. For $b > 0$ and $T > 0$, let $r_0 = n\beta b/\alpha$ and $I = [0,T)$. Assume that $r \geq r_0 > 1$ and that (q,p,r) is a triplet satisfying $1 \leq r \leq p < \infty$, $1/q = \beta n(1/r - 1/p)/\alpha$, and $p > b+1$.*

(i) *If $g(\cdot,\cdot) \in L_t^{q/(b+1)}(I; L_x^{p/(b+1)}(\mathbb{R}^n))$, then*

$$\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))} \lesssim \begin{cases} T^{1-\beta nb/(r\alpha)}\|g\|_{L_t^{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))}, & p < r(b+1),\\ T^{1-n\beta b/(r\alpha)}\left\||g|^{1/(b+1)}\right\|^{\theta(b+1)}_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}\cdot\left\||g|^{1/(b+1)}\right\|^{(1-\theta)(b+1)}_{L_t^q(I;L_x^p(\mathbb{R}^n))}, & p \geq r(b+1),\end{cases}$$

where $\theta = (p/(b+1)-r)/(p-r)$.

(ii) *If $g(\cdot,\cdot) \in L_t^{q/(b+1)}(I; L_x^{p/(b+1)}(\mathbb{R}^n))$, then*

$$\|G_{\alpha,\beta}(g)\|_{L_t^q(I;L_x^p(\mathbb{R}^n))} \lesssim \begin{cases} T^{1-nb\beta/(r\alpha)}\|g\|_{L_t^{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))}, & p < r(b+1),\\ T^{1-nb\beta/(r\alpha)}\left\||g|^{1/(b+1)}\right\|^{\theta(b+1)}_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}\cdot\left\||g|^{1/(b+1)}\right\|^{(1-\theta)(b+1)}_{L_t^q(I;L_x^p(\mathbb{R}^n))}, & p \geq r(b+1),\end{cases}$$

where $\theta = (p/(b+1)-r)/(p-r)$.

Proof (i) For the case $p < r(b+1)$, we have

$$\begin{aligned}
\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))} &= \sup_{t\in I}\left\|\int_0^t \int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dyds\right\|_{L_x^r(\mathbb{R}^n)}\\
&\lesssim \sup_{t\in I}\int_0^t \left\|\int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dy\right\|_{L_x^r(\mathbb{R}^n)} ds.
\end{aligned}$$

Take q^* such that $(b+1)/p + 1/q^* = 1 + 1/r$. Then, by Lemma 2.3,

$$\left\|\int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dy\right\|_{L_x^r(\mathbb{R}^n)} \lesssim (t-s)^{-n\beta(1-1/q^*)/\alpha}\|g(s,\cdot)\|_{L_x^{p/(b+1)}(\mathbb{R}^n)},$$

which implies that

$$\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L^r(\mathbb{R}^n))} \lesssim \sup_{t\in I}\int_0^t (t-s)^{-n\beta((b+1)/p-1/r)/\alpha}\|g(s,\cdot)\|_{L_x^{p/(b+1)}(\mathbb{R}^n)}ds.$$

Let $\widetilde{q}$ be the conjugate of $q/(b+1)$, i.e., $(b+1)/q+1/\widetilde{q}=1$. Because $r>r_0:=b\beta n/\alpha$, then

$$0<1-n\beta((b+1)/p-1/r)\widetilde{q}/\alpha=\widetilde{q}(1-b\beta n/r\alpha)<1.$$

A direct computation, together with change of variables, gives

$$\left(\int_0^t (t-s)^{-n\beta((b+1)/p-1/r)\widetilde{q}/\alpha}ds\right)^{1/\widetilde{q}} \lesssim T^{1/\widetilde{q}-n\beta((b+1)/p-1/r)/\alpha}.$$

Then we obtain

$$\begin{aligned}
&\|G_{\alpha,\beta}(g)\|_{L^\infty(I;L^r)}\\
&\lesssim \sup_{t\in I}\left(\int_0^t (t-s)^{-n\beta((b+1)/p-1/r)\widetilde{q}/\alpha}ds\right)^{1/\widetilde{q}}\left(\int_0^t \|g(s,\cdot)\|^{q/(b+1)}_{L_x^{p/(b+1)}(\mathbb{R}^n)}ds\right)^{(b+1)/q}\\
&\lesssim T^{1/\widetilde{q}-n\beta((b+1)/p-1/r)/\alpha}\|g\|_{L_t^{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))}\\
&\lesssim T^{1-b\beta n/r\alpha}\|g\|_{L_t^{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))},
\end{aligned}$$

where in the last inequality we have used the fact that

$$1/\widetilde{q}-n\beta[(b+1)/p-1/r]/\alpha=1-b\beta n/r\alpha.$$

When $p\geq r(b+1)$, Lemma 2.3 gives

$$\begin{aligned}
\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))} &\lesssim \sup_{t\in I}\int_0^t \left\|\int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dy\right\|_{L_x^r(\mathbb{R}^n)}ds\\
&\lesssim \sup_{t\in I}\int_0^t \|g(s,\cdot)\|_{L_x^r(\mathbb{R}^n)}ds.
\end{aligned}$$

Notice that $\|g(s,\cdot)\|_{L_x^r(\mathbb{R}^n)}=\||g(s,\cdot)|^{1/(b+1)}\|^{b+1}_{L_x^{r(b+1)}(\mathbb{R}^n)}$. Hence,

$$\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))} \lesssim \sup_{t\in I}\int_0^t \left\||g(s,\cdot)|^{1/(b+1)}\right\|^{b+1}_{L_x^{r(b+1)}(\mathbb{R}^n)}ds.$$

Take $\theta\in(0,1)$ such that $1/(rb+r)=\theta/r+(1-\theta)/p$. Let $p_1=(b(1+\theta))^{-1}$ and $q_1=p/(r(b+1)(1-\theta))$ such that $1/p_1+1/q_1=1$. Applying Hölder's inequality on the spatial variable, we obtain

$$\begin{aligned}
&\||g(s,\cdot)|^{1/(b+1)}\|^{b+1}_{L_x^{r(b+1)}(\mathbb{R}^n)}\\
&\lesssim \left\{\int_{\mathbb{R}^n}|g(s,x)|^{r\theta p_1}dx\right\}^{1/(rp_1)}\left\{\int_{\mathbb{R}^n}|g(s,x)|^{r(1-\theta)q_1}dx\right\}^{1/(rq_1)}\\
&\lesssim \left\{\int_{\mathbb{R}^n}|g(s,x)|^{r/(b+1)}dx\right\}^{\theta(b+1)/r}\left\{\int_{\mathbb{R}^n}|g(s,x)|^{p/(b+1)}dx\right\}^{(b+1)(1-\theta)/p},
\end{aligned}$$

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which indicates that

$$\begin{aligned}
&\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}\\
&\lesssim \sup_{t\in I}\int_0^t \big\||g(s,\cdot)|^{1/(b+1)}\big\|_{L_x^r(\mathbb{R}^n)}^{\theta(b+1)}\cdot\big\||g(s,\cdot)|^{1/(b+1)}\big\|_{L_x^p(\mathbb{R}^n)}^{(1-\theta)(b+1)}ds\\
&\lesssim \big\||g(\cdot,\cdot)|^{1/(b+1)}\big\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}^{(b+1)\theta}\sup_{t\in I}\int_0^t \big\||g(s,\cdot)|^{1/(b+1)}\big\|_{L_x^p(\mathbb{R}^n)}^{(1-\theta)(b+1)}ds\\
&\lesssim \big\||g(\cdot,\cdot)|^{1/(b+1)}\big\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}^{(b+1)\theta}\sup_{t\in I}\left\{\left(\int_0^t 1ds\right)^{1-(b+1)(1-\theta)/q}\right.\\
&\qquad\times\left.\left(\int_0^t \big\||g(s,\cdot)|^{1/(b+1)}\big\|_{L_x^p(\mathbb{R}^n)}^{(1-\theta)(b+1)q/(1-\theta)(1+b)}ds\right)^{(b+1)(1-\theta)/q}\right\}\\
&\lesssim T^{1-(b+1)(1-\theta)/q}\big\||g(\cdot,\cdot)|^{1/(b+1)}\big\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}^{(b+1)\theta}\big\||g|^{1/(b+1)}\big\|_{L_t^q(I;L^p(\mathbb{R}^n))}^{(b+1)(1-\theta)}.
\end{aligned}$$

Since $1/q=\beta n(1/r-1/p)/\alpha$ and $\theta=(p-rb-r)/((p-r)(b+1))$, then

$$\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}\lesssim T^{1-n\beta b/(r\alpha)}\Big\||g|^{1/(b+1)}\Big\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}^{\theta(b+1)}\Big\||g|^{1/(b+1)}\Big\|_{L_t^q(I;L_x^p(\mathbb{R}^n))}^{(1-\theta)(b+1)}.$$

This completes the proof of (i).

(ii) For the case $p<r(b+1)$, using Minkowski's inequality and Lemma 2.3, we obtain

$$\begin{aligned}
&\|G_{\alpha,\beta}(g)\|_{L_t^q(I;L_x^p(\mathbb{R}^n))}\\
&\lesssim\left\{\int_0^T\left(\int_0^t\Big\|\int_{\mathbb{R}^n}G_{t-s}(x-y)g(s,y)dy\Big\|_{L_x^p(\mathbb{R}^n)}ds\right)^q dt\right\}^{1/q}\\
&\lesssim\left\{\int_0^T\left(\int_0^t(t-s)^{-n\beta((b+1)/p-1/p)/\alpha}\|g(s,\cdot)\|_{L_x^{p/(b+1)}(\mathbb{R}^n)}ds\right)^q dt\right\}^{1/q}\\
&=\left\|\int_0^t(t-s)^{-n\beta b/p\alpha}\|g(s,\cdot)\|_{L_x^{p/(b+1)}(\mathbb{R}^n)}ds\right\|_{L_t^q(I)}.
\end{aligned}$$

Let χ be the number such that $1/q+1=(1+b)/q+1/\chi$. An application of Young's inequality gives

$$\begin{aligned}
\|G_{\alpha,\beta}(g)\|_{L_t^q(I;L_x^p(\mathbb{R}^n))}&\lesssim\|g(\cdot,\cdot)\|_{L_t^{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))}\left(\int_0^T t^{-n\beta b\chi/p\alpha}dt\right)^{1/\chi}\\
&\lesssim T^{1-n\beta b/r\alpha}\|g(\cdot,\cdot)\|_{L_t^{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))},
\end{aligned}$$

where in the last inequality we have used the fact that $1/q=n\beta(1/r-1/p)/\alpha$.

For the case $p \geq r(b+1)$, we apply Lemma 2.3 again to get

$$\|G_{\alpha,\beta}(g)\|_{L_t^q(I;L_x^p(\mathbb{R}^n))} \lesssim \left\| \int_0^t \left\| \int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dy \right\|_{L_x^p(\mathbb{R}^n)} ds \right\|_{L_t^q(I)}$$

$$\lesssim \left\| \int_0^t (t-s)^{-n\beta(1/r-1/p)/\alpha} \left\| |g(s,\cdot)|^{1/(b+1)} \right\|_{L_x^p(\mathbb{R}^n)} ds \right\|_{L_t^q(I)}.$$

Choose $\theta \in (0,1)$ such that $1/(b+1) = \theta + (1-\theta)r/p$. Letting $p_2 = (b+b\theta)^{-1}$ and $q_2 = p/(r(b+1)(1-\theta))$, we use Hölder's inequality on the spatial variable to deduce

$$\int_0^t (t-s)^{-\beta(1/r-1/p)/\alpha} \left\| |g(s,\cdot)|^{1/(b+1)} \right\|_{L_x^{r(b+1)}(\mathbb{R}^n)}^{b+1} ds$$

$$\lesssim \int_0^t (t-s)^{-n\beta(1/r-1/p)/\alpha} \left(\int_{\mathbb{R}^n} |g(s,x)|^{r\theta p_2} dx \right)^{1/(rp_2)} \left(\int_{\mathbb{R}^n} |g(s,x)|^{r(1-\theta)q_2} dx \right)^{1/(rq_2)} ds$$

$$\lesssim \int_0^t (t-s)^{-n\beta(1/r-1/p)/\alpha} \left\| |g(s,\cdot)|^{1/(b+1)} \right\|_{L_x^r(\mathbb{R}^n)}^{\theta(b+1)} \left\| |g(s,\cdot)|^{1/(b+1)} \right\|_{L_x^p(\mathbb{R}^n)}^{(1-\theta)(b+1)} ds$$

$$\lesssim \left\| |g(s,\cdot)|^{1/(b+1)} \right\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}^{\theta(b+1)} \int_0^t (t-s)^{-n\beta(1/r-1/p)/\alpha} \left\| |g(s,\cdot)|^{1/(b+1)} \right\|_{L_x^p(\mathbb{R}^n)}^{(1-\theta)(b+1)} ds,$$

which gives

$$\|G_{\alpha,\beta}(g)\|_{L_t^q(I;L_x^p(\mathbb{R}^n))} \lesssim \left\| |g(\cdot,\cdot)|^{1/(b+1)} \right\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}^{\theta(b+1)} \times \left\| \int_0^t (t-s)^{-n\beta(1/r-1/p)/\alpha} \left\| |g(s,\cdot)|^{1/(b+1)} \right\|_{L_x^p(\mathbb{R}^n)}^{(1-\theta)(b+1)} ds \right\|_{L_t^q(I)}.$$

Suppose that $\widetilde{\chi}$ obeys $1/q + 1 = (1+b)(1-\theta)/q + 1/\widetilde{\chi}$. Young's inequality on the time variable gives

$$\begin{aligned}
&\|G_{\alpha,\beta}(g)\|_{L_t^q(I;L_x^p(\mathbb{R}^n))} \\
&\quad \lesssim \left\| |g(\cdot,\cdot)|^{1/(b+1)} \right\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}^{\theta(b+1)} \left\| |g(\cdot,\cdot)|^{1/(b+1)} \right\|_{L_t^q(I;L_x^p(\mathbb{R}^n))}^{(b+1)(1-\theta)} \\
&\quad\quad \times \left(\int_0^T t^{-n\beta(1/r-1/p)\widetilde{\chi}/\alpha} dt \right)^{1/\widetilde{\chi}} \\
&\quad \lesssim T^{1-b\beta n/(r\alpha)} \left\| |g(\cdot,\cdot)|^{1/(b+1)} \right\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}^{\theta(b+1)} \left\| |g(\cdot,\cdot)|^{1/(b+1)} \right\|_{L_t^q(I;L_x^p(\mathbb{R}^n))}^{(b+1)(1-\theta)},
\end{aligned}$$

where in the last inequality we have used the fact that $1/\widetilde{\chi} - n\beta(1/r-1/p)/\alpha = 1 - b\beta n/(r\alpha)$. This completes the proof of Theorem 2.7. ■

Theorem 2.8 *Let $\alpha > n$ and $0 < \beta < 1$. For $b > 0$ and $T > 0$, let $r_0 = bn\beta/\alpha$, $I = [0,T)$. Assume that $r \geq r_0 > 1$ and (q,p,r) is a triplet satisfying $1 \leq r \leq p < \infty$, $1/q = \beta n(1/r - 1/p)/\alpha$, and $p > b+1$.*

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(i) *If* $g \in C_{q/(b+1)}(I; L_x^{p/(b+1)}(\mathbb{R}^n))$, *then*

$$\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))} \lesssim \begin{cases} T^{1-b\beta n/(r\alpha)}\|g\|_{C_{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))}, & p < r(b+1), \\ T^{1-bn\beta/(r\alpha)}\||g|^{1/(b+1)}\|^{\theta(b+1)}_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}\||g|^{1/(b+1)}\|^{(1-\theta)(b+1)}_{C_q(I;L_x^r(\mathbb{R}^n))}, & p \geq r(b+1), \end{cases}$$

where $\theta = (p/(b+1) - r)/(p - r)$.

(ii) *If* $g \in C_{q/(b+1)}(I; L_x^{p/(b+1)}(\mathbb{R}^n))$, *then*

$$\|G_{\alpha,\beta}(g)\|_{C_q(I;L_x^p(\mathbb{R}^n))} \lesssim \begin{cases} T^{1-b\beta n/(r\alpha)}\|g\|_{C_{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))}, & p < r(b+1), \\ T^{1-bn\beta/(r\alpha)}\||g|^{1/(b+1)}\|^{\theta(b+1)}_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}\||g|^{1/(b+1)}\|^{(1-\theta)(b+1)}_{C_q(I;L_x^p(\mathbb{R}^n))}, & p \geq r(b+1), \end{cases}$$

where $\theta = (p/(b+1) - r)/(p - r)$.

Proof (i) We first consider the case $p < r(b+1)$, and it follows from Lemma 2.3 that

$$\left\| \int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dy \right\|_{L_x^r(\mathbb{R}^n)} \lesssim (t-s)^{-n\beta[(b+1)/p-1/r]/\alpha}\|g(s,\cdot)\|_{L_x^{p/(b+1)}(\mathbb{R}^n)},$$

which implies that

$$\begin{aligned} &\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))} \\ &\quad\lesssim \sup_{t\in[0,T)}\left\{\int_0^t \left\|\int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dy\right\|_{L_x^r(\mathbb{R}^n)} ds\right\} \\ &\quad\lesssim \sup_{t\in[0,T)}\left\{\int_0^t (t-s)^{-n\beta[(b+1)/p-1/r]/\alpha}\|g(s,\cdot)\|_{L_x^{p/(b+1)}(\mathbb{R}^n)} ds\right\} \\ &\lesssim \|g\|_{C_{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))} \sup_{t\in[0,T)}\left\{\int_0^t (t-s)^{-n\beta[(b+1)/p-1/r]/\alpha} s^{-(b+1)/q} ds\right\}. \end{aligned}$$

A direct computation, together with the change of variables: $u = s/t$, gives

$$\int_0^t (t-s)^{-n\beta[(b+1)/p-1/r]/\alpha} s^{-(b+1)/q} ds \lesssim t^{1-n\beta b/(r\alpha)},$$

which indicates that

$$\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))} \lesssim T^{1-n\beta b/r\alpha}\|g\|_{C_{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))}.$$

Now, we consider the case $p \geq (b+1)r$. By Lemma 2.3, we have

$$\left\| \int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dy \right\|_{L_x^r(\mathbb{R}^n)} \lesssim \|g(s,\cdot)\|_{L_x^r(\mathbb{R}^n)}.$$

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If $\theta \in (0,1)$, $p_3 = \frac{1}{(b+1)\theta}$, and $q_3 = \frac{p}{r(b+1)(1-\theta)}$, then we obtain

$$\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))} = \sup_{t\in I}\left\| \int_0^t \int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dyds \right\|_{L_x^r(\mathbb{R}^n)}$$
$$\lesssim \sup_{t\in I} \int_0^t \|g(s,\cdot)\|_{L_x^r(\mathbb{R}^n)} ds.$$

Notice that $1-(b+1)(1-\theta)/q = 1 - b\beta n/(r\alpha)$. An application of Hölder's inequality gives

$$\begin{aligned}
&\|G_{\alpha,\beta}(g)\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))} \\
&\lesssim \sup_{t\in I} \int_0^t \left\||g(s,\cdot)|^{1/(b+1)}\right\|^{b+1}_{L_x^{r(b+1)}(\mathbb{R}^n)} ds \\
&\lesssim \sup_{t\in[0,T)} \left\{ \int_0^t \left\||g(s,\cdot)|^{1/(b+1)}\right\|^{\theta(b+1)}_{L_x^r(\mathbb{R}^n)} \left\||g(s,\cdot)|^{1/(b+1)}\right\|^{(1-\theta)(b+1)}_{L_x^p(\mathbb{R}^n)} ds \right\} \\
&\lesssim \left\||g|^{1/(b+1)}\right\|^{\theta(b+1)}_{L_t^\infty(I;L_x^r(\mathbb{R}^n))} \left\||g|^{1/(b+1)}\right\|^{(1-\theta)(b+1)}_{C_q(I;L_x^p(\mathbb{R}^n))} \sup_{t\in[0,T)} \left\{ \int_0^t s^{-(b+1)(1-\theta)/q} ds \right\} \\
&\lesssim T^{1-(b+1)(1-\theta)/q} \left\||g|^{1/(b+1)}\right\|^{\theta(b+1)}_{L_t^\infty(I;L_x^r(\mathbb{R}^n))} \left\||g|^{1/(b+1)}\right\|^{(1-\theta)(b+1)}_{C_q(I;L_x^p(\mathbb{R}^n))}.
\end{aligned}$$

We begin to prove (ii). Let $g \in C_{q/(b+1)}(I; L_x^{p/(b+1)}(\mathbb{R}^n))$. For the case $p < r(b+1)$, we have

$$\left\| \int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dy \right\|_{L_x^p(\mathbb{R}^n)} \lesssim (t-s)^{-n\beta b/p\alpha} \|g(s,\cdot)\|_{L_x^{p/(b+1)}(\mathbb{R}^n)},$$

and hence

$$\begin{aligned}
\left\|G_{\alpha,\beta}(g)\right\|_{C_q(I;L_x^p(\mathbb{R}^n))} &\lesssim \sup_{t\in[0,T)} t^{1/q} \int_0^t \left\| \int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dy \right\|_{L_x^p(\mathbb{R}^n)} ds \\
&\lesssim \sup_{t\in[0,T)} t^{1/q} \int_0^t (t-s)^{-n\beta b/p\alpha} \|g(s,\cdot)\|_{L_x^{p/(b+1)}(\mathbb{R}^n)} ds \\
&\lesssim \|g\|_{C_{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))} \sup_{t\in[0,T)} t^{1/q} \int_0^t (t-s)^{-n\beta b/p\alpha} s^{-(b+1)/q} ds \\
&\lesssim \|g\|_{C_{q/(b+1)}(I;L_x^{p/(b+1)}(\mathbb{R}^n))} T^{1-\beta b n/p\alpha}.
\end{aligned}$$

For the case $p \geq r(b+1)$, it holds

$$\begin{aligned}
\left\| \int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dy \right\|_{L_x^p(\mathbb{R}^n)} &\lesssim (t-s)^{-n\beta(1/r-1/p)/\alpha} \|g(s,\cdot)\|_{L_x^r(\mathbb{R}^n)} \\
&\lesssim (t-s)^{-n\beta(1/r-1/p)/\alpha} \left\||g(s,\cdot)|^{1/(b+1)}\right\|^{b+1}_{L_x^{r(b+1)}(\mathbb{R}^n)}.
\end{aligned}$$

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Taking $\theta = (p - r(b+1))/(p-r)$, we use the Hölder inequality to deduce

$$\begin{aligned}\left\|G_{\alpha,\beta}(g)\right\|_{C_q(I;L_x^p(\mathbb{R}^n))} &\lesssim \sup_{t\in[0,T)} t^{1/q}\int_0^t (t-s)^{-\beta n(1/r-1/p)/\alpha}\left\||g(s,\cdot)|^{1/(b+1)}\right\|_{L_x^r(\mathbb{R}^n)}^{(b+1)\theta} \\ &\quad \times \left\||g(s,\cdot)|^{1/(b+1)}\right\|_{L_x^p(\mathbb{R}^n)}^{(b+1)(1-\theta)} ds \\ &\lesssim \left\||g|^{1/(b+1)}\right\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}^{(b+1)\theta}\left\||g|^{1/(b+1)}\right\|_{C_q(I;L_x^p(\mathbb{R}^n))}^{(b+1)(1-\theta)} \\ &\quad \times \sup_{t\in[0,T)}\left\{t^{1/q}\int_0^t (t-s)^{-\beta n(1/r-1/p)/\alpha} s^{-(b+1)(1-\theta)/q} ds\right\} \\ &\lesssim T^{1-\beta bn/r\alpha}\left\||g|^{1/(b+1)}\right\|_{L_t^\infty(I;L_x^r(\mathbb{R}^n))}^{(b+1)\theta}\left\||g|^{1/(b+1)}\right\|_{C_q(I;L_x^p(\mathbb{R}^n))}^{(b+1)(1-\theta)}.\end{aligned}$$

This completes the proof of Theorem 2.8. ∎

2.4 Regularities of $R_{\alpha,\beta}$ and $G_{\alpha,\beta}$

At first, let us recall the following well-known estimates for the fractional heat kernel $K_{\alpha,t}(\cdot)$.

Proposition 2.9 [52, Lemmas 2.1 and 2.2] *Let* $\alpha > 0$.

(i) *There exists a constant C such that*

$$\left|K_{\alpha,t}(x)\right| \simeq C\min\left\{t^{-n/2\alpha}, \frac{t}{|x|^{n+2\alpha}}\right\} \simeq \frac{Ct}{(t^{1/2\alpha}+|x|)^{n+2\alpha}}.$$

(ii) *For* $\varepsilon > 0$, *there exists a constant* $C > 0$ *such that for* $|h| < t^{1/2\alpha}$,

$$\left|K_{\alpha,t}(x+h) - K_{\alpha,t}(x)\right| \leq \frac{Ct}{(t^{1/2\alpha}+|x|)^{n+2\alpha}}\left(\frac{|h|}{t^{1/2\alpha}}\right)^{\varepsilon}.$$

Remark 2.10 It is easy to see that in (ii) of Proposition 2.9, the condition $|h| < t^{1/2\alpha}$ can be replaced by $|h| < |x|/2$.

Below, we investigate the Hölder continuity of the kernel $G_t(\cdot)$.

Proposition 2.11 *Assume that* $\alpha > n$ *and* $0 < \beta < 1$. *For* $\varepsilon \in (0, \alpha - n)$, *there exists a positive constant C such that for all* $x \in \mathbb{R}^n$ *and* $|h| < t^{\beta/\alpha}$,

$$\left|G_t(x+h) - G_t(x)\right| \leq C\left(\frac{|h|}{t^{\beta/\alpha}}\right)^{\varepsilon}\frac{t^{\beta}}{(t^{\beta/\alpha}+|x|)^{n+\alpha}}. \tag{2.7}$$

Proof We first verify (2.7) for $|h| < |x|/2$. Recall that

$$G_t(x) = \int_0^\infty K_{\alpha/2,s}(x) f_{E_t}(s)ds,$$

where $f_{E_t}(x) = t\beta^{-1}x^{-1-1/\beta}g_\beta(tx^{-1/\beta})$. An application of change of variables gives

$$G_t(x) = \int_0^\infty K_{\alpha/2,(t/u)^\beta}(x)g_\beta(u)du,$$

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which, together with Proposition 2.9(i) with $\varepsilon \in (0, \alpha - n)$, implies that

(2.8)

$$\begin{aligned}\left|G_t(x+h) - G_t(x)\right| &\lesssim \int_0^\infty \left|K_{\alpha/2,(t/u)^\beta}(x+h) - K_{\alpha/2,(t/u)^\beta}(x)\right| g_\beta(u)du \\ &\lesssim \int_0^\infty \left(\frac{|h|}{(t/u)^{\beta/\alpha}}\right)^\varepsilon \frac{(t/u)^\beta}{((t/u)^{\beta/\alpha} + |x|)^{n+\alpha}} g_\beta(u)du \\ &\lesssim |h|^\varepsilon \int_0^\infty \left(\frac{t}{u}\right)^{\beta - \beta\varepsilon/\alpha} \frac{1}{((t/u)^{\beta/\alpha} + |x|)^{n+\alpha}} g_\beta(u)du \\ &\lesssim \frac{|h|^\varepsilon}{t^{\beta(n+\varepsilon)/\alpha}} \int_0^\infty u^{\beta(n+\varepsilon)/\alpha} g_\beta(u)du \\ &\lesssim \left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon \frac{1}{t^{n\beta/\alpha}} \left\{\int_0^1 u^{\beta(n+\varepsilon)/\alpha} du + \int_1^\infty u^{\beta(n+\varepsilon)/\alpha - \beta - 1} du\right\} \\ &\lesssim \left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon \frac{1}{t^{n\beta/\alpha}},\end{aligned}$$

where in the last inequality we have used the facts that $\alpha > n$ and $0 < \varepsilon < \alpha - n$.

On the other hand, by Proposition 2.9,

$$\left|K_{\alpha/2,s}(x+h) - K_{\alpha/2,s}(x)\right| \lesssim \frac{s}{(s^{1/\alpha} + |x|)^{n+\alpha}} \left(\frac{|h|}{s^{1/\alpha}}\right)^\varepsilon \lesssim \left(\frac{|h|}{s^{1/\alpha}}\right)^\varepsilon \frac{s}{|x|^{n+\alpha}},$$

and we have

$$\begin{aligned}\left|G_t(x+h) - G_t(x)\right| &\lesssim \int_0^\infty \frac{(t/u)^\delta}{|x|^{n+\alpha}} \left(\frac{|h|}{(t/u)^{\beta/\alpha}}\right)^\varepsilon g_\beta(u)du \\ &\lesssim \frac{|h|^\varepsilon}{|x|^{n+\alpha}} t^{\beta - \beta\varepsilon/\alpha} \int_0^\infty u^{-\beta(1-\varepsilon/\alpha)} g_\beta(u)du \\ &\lesssim \frac{t^\beta}{|x|^{n+\alpha}} \left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon.\end{aligned} \tag{2.9}$$

Finally, it can be deduced from (2.8) and (2.9) that

$$\left|G_t(x+h) - G_t(x)\right| \lesssim \min\left\{\left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon \frac{1}{t^{n\beta/\alpha}}, \frac{t^\beta}{|x|^{n+\alpha}} \left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon\right\};$$

equivalently,

$$\left|G_t(x+h) - G_t(x)\right| \le C \left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon \frac{t^\beta}{(t^{\beta/\alpha} + |x|)^{n+\alpha}}.$$

Now, we assume that $|h| < t^{\beta/\alpha}$. If $|h| < |x|/2 < t^{\beta/\alpha}$ or $|h| < t^{\beta/\alpha} < |x|/2$, it is obvious that (2.7) holds. Now, we consider the case $|x|/2 < |h| < t^{\beta/\alpha}$. We split $|G_t(x+h) -$

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$G_t(x)| \le S_1 + S_2$, where

$$\begin{cases} S_1 := \int_{u:|h|<(t/u)^{\beta/\alpha}} \left|K_{\alpha/2,(t/u)^\beta}(x+h) - K_{\alpha/2,(t/u)^\beta}(x)\right| g_\beta(u)du, \\ S_2 := \int_{u:|h|\ge(t/u)^{\beta/\alpha}} \left|K_{\alpha/2,(t/u)^\beta}(x+h) - K_{\alpha/2,(t/u)^\beta}(x)\right| g_\beta(u)du. \end{cases}$$

For S_1, since $|h| < (t/u)^{\beta/\alpha}$, it follows from (ii) of Proposition 2.9 that

$$\begin{aligned} S_1 &\lesssim \int_{u:|h|<(t/u)^{\beta/\alpha}} \left(\frac{|h|}{(t/u)^{\beta/\alpha}}\right)^\varepsilon \frac{(t/u)^\beta}{((t/u)^{\beta/\alpha}+|x|)^{n+\alpha}} g_\beta(u)du \\ &\lesssim \left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon \frac{1}{t^{n\beta/\alpha}} \left\{\int_0^1 u^{\beta(n+\varepsilon)/\alpha}du + \int_1^\infty u^{\beta(n+\varepsilon)/\alpha-\beta-1}du\right\} \\ &\lesssim \left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon \frac{1}{t^{n\beta/\alpha}}. \end{aligned}$$

We further divide S_2 as $S_2 \le S_{2,1} + S_{2,2}$, where

$$\begin{cases} S_{2,1} := \int_{u:|h|\ge(t/u)^{\beta/\alpha}} \left|K_{\alpha/2,(t/u)^\beta}(x)\right| g_\beta(u)du, \\ S_{2,2} := \int_{u:|h|\ge(t/u)^{\beta/\alpha}} \left|K_{\alpha/2,(t/u)^\beta}(x+h)\right| g_\beta(u)du. \end{cases}$$

Noticing that $|h| \ge (t/u)^{\beta/\alpha}$, we deduce from (2.5) that

$$\begin{aligned} S_{2,1} &\lesssim \int_{u:|h|\ge(t/u)^{\beta/\alpha}} \left(\frac{|h|}{(t/u)^{\beta/\alpha}}\right)^\varepsilon \frac{(t/u)^\beta}{(|x|+(t/u)^{\beta/\alpha})^{n+\alpha}} g_\beta(u)du \\ &\lesssim \left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon \frac{1}{t^{n\beta/\alpha}} \int_0^\infty u^{\beta(n+\varepsilon)/\alpha} g_\beta(u)du \\ &\lesssim \left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon \frac{1}{t^{n\beta/\alpha}}. \end{aligned}$$

The term $S_{2,2}$ can be dealt with similar to $S_{2,1}$. Finally, due to $|x|/2 < t^{\beta/\alpha}$, it holds

$$\left|G_t(x+h) - G_t(x)\right| \lesssim \left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon \frac{1}{t^{n\beta/\alpha}} \lesssim \left(\frac{|h|}{t^{\beta/\alpha}}\right)^\varepsilon \frac{t^\beta}{(t^{\beta/\alpha}+|x|)^{n+\alpha}},$$

which completes the proof of Proposition 2.11. ■

Theorem 2.12 *Assume that $\alpha > n$ and $0 < \beta < 1$. If $p \in [1, \infty]$ and $\varphi \in L^p(\mathbb{R}^n)$, then $R_{\alpha,\beta}(\varphi)(\cdot,\cdot)$ is continuous on $\mathbb{R}^{1+n}_+$.*

Proof At first, for fixed $t_0 > 0$, choose $(t_0, x), (t_0, x_0) \in \mathbb{R}^{1+n}_+$. By Proposition 2.9 and Hölder's inequality, we can get

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$$
\begin{aligned}
&\left|R_{\alpha,\beta}(\varphi)(t_0,x)-R_{\alpha,\beta}(\varphi)(t_0,x_0)\right| \\
&\quad\lesssim \int_{\mathbb{R}^n}\left\{\int_0^\infty \left|K_{\alpha/2,(t_0/u)^\beta}(x-y)-K_{\alpha/2,(t_0/u)^\beta}(x_0-y)\right|g_\beta(u)du\right\}|\varphi(y)|dy \\
&\quad\lesssim \int_0^\infty\left\{\int_{\mathbb{R}^n}\left|K_{\alpha/2,(t_0/u)^\beta}(x-y)-K_{\alpha/2,(t_0/u)^\beta}(x_0-y)\right|\cdot|\varphi(y)|dy\right\}g_\beta(u)du \\
&\quad\lesssim M_1+M_2,
\end{aligned}
$$

where

$$
\begin{cases}
M_1 := \int_{u:(t_0/u)^{\beta/\alpha}>|x-x_0|}\left\{\int_{\mathbb{R}^n}\left|K_{\alpha/2,(t_0/u)^\beta}(x-y)-K_{\alpha/2,(t_0/u)^\beta}(x_0-y)\right|\cdot|\varphi(y)|dy\right\}g_\beta(u)du,\\
M_2 := \int_{u:(t_0/u)^{\beta/\alpha}\le|x-x_0|}\left\{\int_{\mathbb{R}^n}\left|K_{\alpha/2,(t_0/u)^\beta}(x-y)-K_{\alpha/2,(t_0/u)^\beta}(x_0-y)\right|\cdot|\varphi(y)|dy\right\}g_\beta(u)du.
\end{cases}
$$

For M_1, since $|x-x_0|<(t_0/u)^{\beta/\alpha}$, it follows from (ii) of Proposition 2.9 that for $\varepsilon\in(0,\alpha-n/p)$,

$$
\left|K_{\alpha/2,(t_0/u)^\beta}(x-y)-K_{\alpha/2,(t_0/u)^\beta}(x_0-y)\right|\lesssim\left(\frac{|x-x_0|}{(t_0/u)^{\beta/\alpha}}\right)^\varepsilon\frac{(t_0/u)^\beta}{((t_0/u)^{\beta/\alpha}+|x-y|)^{n+\alpha}},
$$

and hence

(2.10)

$$
\begin{aligned}
M_1 &\lesssim \int_{u:(t_0/u)^{\beta/\alpha}>|x-x_0|}\left\{\int_{\mathbb{R}^n}\left(\frac{|x-x_0|}{(t_0/u)^{\beta/\alpha}}\right)^\varepsilon\frac{(t_0/u)^\beta}{((t_0/u)^{\beta/\alpha}+|x-y|)^{n+\alpha}}|\varphi(y)|dy\right\}g_\beta(u)du\\
&\lesssim \int_{u:(t_0/u)^{\beta/\alpha}>|x-x_0|}\left(\frac{|x-x_0|}{(t_0/u)^{\beta/\alpha}}\right)^\varepsilon\frac{(t_0/u)^\beta}{(t_0/u)^{\beta(1+n/\alpha)}}\|\varphi\|_{L^p_x}(t_0/u)^{\beta n/(\alpha p')}g_\beta(u)du\\
&\lesssim \|\varphi\|_{L^p_x}\frac{|x-x_0|^\varepsilon}{t_0^{\beta(\varepsilon+n/p)/\alpha}}\int_0^\infty u^{\beta(\varepsilon+n/p)/\alpha}g_\beta(u)du\\
&\lesssim \|\varphi\|_{L^p_x}t_0^{-\beta(\varepsilon+n/p)/\alpha}|x-x_0|^\varepsilon,
\end{aligned}
$$

where in the last inequality we have applied (2.2) to estimate the integral as

$$
\int_0^\infty u^{\beta(\varepsilon+n/p)/\alpha}g_\beta(u)du\lesssim\int_0^1 u^{\beta(\varepsilon+n/p)/\alpha}du+\int_1^\infty u^{\beta(\varepsilon+n/p)/\alpha-\beta-1}du<\infty.
$$

For M_2, noticing that $|x-x_0|\ge(t/u)^{\beta/\alpha}$, we can use (i) of Proposition 2.9 to get

$$
\begin{aligned}
&\left|K_{\alpha/2,(t_0/u)^\beta}(x-y)-K_{\alpha/2,(t_0/u)^\beta}(x_0-y)\right|\\
&\quad\lesssim\left(\frac{|x-x_0|}{(t_0/u)^{\beta/\alpha}}\right)^\varepsilon\left\{\frac{(t_0/u)^\beta}{((t_0/u)^{\beta/\alpha}+|x-y|)^{n+\alpha}}+\frac{(t_0/u)^\beta}{((t_0/u)^{\beta/\alpha}+|x_0-y|)^{n+\alpha}}\right\}.
\end{aligned}
$$

We split $M_2\le M_{2,1}+M_{2,2}$, where

$$
\begin{cases}
M_{2,1} := \int_{u:(t_0/u)^{\beta/\alpha}>|x-x_0|}\left\{\int_{\mathbb{R}^n}\left(\frac{|x-x_0|}{(t_0/u)^{\beta/\alpha}}\right)^\varepsilon\frac{(t_0/u)^\beta}{((t_0/u)^{\beta/\alpha}+|x-y|)^{n+\alpha}}|\varphi(y)|dy\right\}g_\beta(u)du,\\
M_{2,2} := \int_{u:(t_0/u)^{\beta/\alpha}>|x-x_0|}\left\{\int_{\mathbb{R}^n}\left(\frac{|x-x_0|}{(t_0/u)^{\beta/\alpha}}\right)^\varepsilon\frac{(t_0/u)^\beta}{((t_0/u)^{\beta/\alpha}+|x-y|)^{n+\alpha}}|\varphi(y)|dy\right\}g_\beta(u)du.
\end{cases}
$$

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We can follow the procedure of (2.10) to deduce that

$$M_{2,1} + M_{2,2} \lesssim \|g\|_{L^p_x} t_0^{-\beta(\varepsilon+n/p)/\alpha} |x - x_0|^{\varepsilon}.$$

This means

$$\left|R_{\alpha,\beta}(\varphi)(t_0, x) - R_{\alpha,\beta}(\varphi)(t_0, x_0)\right| \lesssim \|g\|_{L^p_x} t_0^{-\beta(\varepsilon+n/p)/\alpha} |x - x_0|^{\varepsilon}, \tag{2.11}$$

which indicates that $R_{\alpha,\beta}(\varphi)(t_0, x)$ is continuous with respect to the spatial variable x. Now, we investigate the continuity respect to the time variable t. For fixed $x \in \mathbb{R}^n$,

$$\begin{aligned}
&R_{\alpha,\beta}(\varphi)(t, x) - R_{\alpha,\beta}(\varphi)(t_0, x)\\
&= \int_{\mathbb{R}^n} \left(G_t(x - y) - G_{t_0}(x - y)\right) \varphi(y) dy\\
&= \int_{\mathbb{R}^n} \left\{\int_0^\infty \left(K_{\alpha/2,(t/u)^\beta}(x - y) - K_{\alpha/2,(t_0/u)^\beta}(x - y)\right) g_\beta(u) du\right\} \varphi(y) dy\\
&= \int_0^\infty \left\{\int_{\mathbb{R}^n} \left(K_{\alpha/2,(t/u)^\beta}(x - y) - K_{\alpha/2,(t_0/u)^\beta}(x - y)\right) \varphi(y) dy\right\} g_\beta(u) du\\
&= \int_0^\infty \left(e^{-(t/u)^\beta(-\Delta)^{\alpha/2}} \varphi(x) - e^{-(t_0/u)^\beta(-\Delta)^{\alpha/2}} \varphi(x)\right) g_\beta(u) du,
\end{aligned}$$

which gives

$$\begin{aligned}
&\left|R_{\alpha,\beta}(\varphi)(t, x) - R_{\alpha,\beta}(\varphi)(t_0, x)\right|\\
&\lesssim \int_0^\infty \left|e^{-(t/u)^\beta(-\Delta)^{\alpha/2}} \varphi(x) - e^{-(t_0/u)^\beta(-\Delta)^{\alpha/2}} \varphi(x)\right| g_\beta(u) du.
\end{aligned}$$

Case 1: $p \in [1, \infty)$. We have

$$\begin{aligned}
&\left|e^{-(t/u)^\beta(-\Delta)^{\alpha/2}} \varphi(x) - e^{-(t_0/u)^\beta(-\Delta)^{\alpha/2}} f(x)\right|\\
&\lesssim \|\varphi\|_{L^p_x(\mathbb{R}^n)} \left|(t/u)^{\beta(1-1/\alpha-n/\alpha p)} - (t_0/u)^{\beta(1-1/\alpha-n/\alpha p)}\right|\\
&\lesssim \|\varphi\|_{L^p_x(\mathbb{R}^n)} u^{-\beta(1-1/\alpha-n/\alpha p)} \left|t^{\beta(1-1/\alpha-n/\alpha p)} - t_0^{\beta(1-1/\alpha-n/\alpha p)}\right|.
\end{aligned}$$

Because $\alpha > n$ and $p \geq 1$, then $1 + n/p \leq 1 + n < \alpha$. The above estimate gives

$$\begin{aligned}
&\left|R_{\alpha,\beta}(\varphi)(t, x) - R_{\alpha,\beta}(\varphi)(t_0, x)\right|\\
&\lesssim \|\varphi\|_{L^p_x(\mathbb{R}^n)} \left|t^{\beta(1-1/\alpha-n/\alpha p)} - t_0^{\beta(1-1/\alpha-n/\alpha p)}\right| \int_0^\infty g_\beta(u) u^{-\beta(1-1/\alpha-n/\alpha p)} du.
\end{aligned}$$

It can be seen from (2.2) that if $u \in (0, 1)$, for any $N > 0$,

$$\begin{aligned}
g_\beta(u) &\sim K(\beta/u)^{(1-\beta/2)/(1-\beta)} \exp\{-|1 - \beta|(u/\beta)^{\beta/(\beta-1)}\}\\
&\lesssim u^{-(1-\beta/2)/(1-\beta)} \frac{1}{u^{\beta N/(\beta-1)}}\\
&\lesssim u^{(N\beta-1+\beta/2)/(1-\beta)}.
\end{aligned}$$

Then, we split the integral

$$\int_0^\infty g_\beta(u) u^{-\beta(1-1/\alpha-n/\alpha p)} du \lesssim I_1 + I_2,$$

where

$$\begin{cases} I_1 := \int_0^1 u^{(N\beta-1+\beta/2)/(1-\beta)} u^{-\beta(1-1/\alpha-n/\alpha p)} du, \\ I_2 := \int_1^\infty u^{-\beta-1} u^{-\beta(1-1/\alpha-n/\alpha p)} du. \end{cases}$$

Taking N large enough, we have $I_1 < \infty$. For I_2, because $\beta \in (0,1)$ and $\alpha > n$, then $1/\alpha + n/\alpha p - 2 < 0$ and

$$I_2 = \int_1^\infty u^{-\beta(2-1/\alpha-n/\alpha p)-1} du < \infty,$$

which indicates that

(2.12)
$$\left|R_{\alpha,\beta}(\varphi)(t,x) - R_{\alpha,\beta}(\varphi)(t_0,x)\right| \lesssim \|\varphi\|_{L^p_x(\mathbb{R}^n)} \left|t^{\beta(1-1/\alpha-n/\alpha p)} - t_0^{\beta(1-1/\alpha-n/\alpha p)}\right|.$$

Case 2: $p = \infty$. We can also get

(2.13)
$$\begin{aligned} &\left|R_{\alpha,\beta} f(t,x) - R_{\alpha,\beta} f(t_0,x)\right| \\ &\lesssim \|\varphi\|_{L^\infty_x(\mathbb{R}^n)} \left|t^{\beta(1-1/\alpha)} - t_0^{\beta(1-1/\alpha)}\right| \left(\int_0^\infty g_\beta(u) u^{-(1-1/\alpha)\beta} du\right) \\ &\lesssim \|\varphi\|_{L^\infty_x(\mathbb{R}^n)} \left|t^{\beta(1-1/\alpha)} - t_0^{\beta(1-1/\alpha)}\right|. \end{aligned}$$

Now, fix $(t_0, x_0) \in \mathbb{R}^{1+n}_+$. Since

$$\begin{aligned} &\left|R_{\alpha,\beta}\varphi(t,x) - R_{\alpha,\beta}\varphi(t_0,x_0)\right| \\ &\le \left|R_{\alpha,\beta}\varphi(t,x) - R_{\alpha,\beta}\varphi(t_0,x)\right| + \left|R_{\alpha,\beta} f(t_0,x) - R_{\alpha,\beta}\varphi(t_0,x_0)\right|, \end{aligned}$$

it follows from (2.11)–(2.13) that

$$\lim_{(t,x)\to(t_0,x_0)} R_{\alpha,\beta}\varphi(t,x) = R_{\alpha,\beta}\varphi(t_0,x). \qquad \blacksquare$$

Theorem 2.13 *Assume that* $(\alpha,\beta,p,q) \in (n,\infty) \times (0,1) \times [1,\infty) \times (1,\infty)$ *satisfying* $n\beta/p + \alpha/q < \alpha$. *For* $g \in L^q_t L^p_x(\mathbb{R}^{1+n}_+)$, $G_{\alpha,\beta}(g)$ *is Hölder continuous in the sense that for any two sufficient close points* $(t,x), (t_0,x_0) \in \mathbb{R}^{1+n}_+$,

$$\left|G_{\alpha,\beta}(g)(t,x) - G_{\alpha,\beta}(t_0,x_0)\right| \lesssim \|g\|_{L^q_t L^p_x(\mathbb{R}^n)} \left\{|t-t_0|^{1-1/q-n\beta/(\alpha p)} + |x-x_0|^{\alpha(1-1/q)/\beta - n/p}\right\}.$$

Proof Given a point $(t_0,x_0) \in \mathbb{R}^{1+n}_+$, let $x \in \mathbb{R}^n$ be sufficiently close to x_0 such that $\delta = |x - x_0|$ small enough. Then $|G_{\alpha,\beta}(g)(t_0,x) - G_{\alpha,\beta}(t_0,x_0)| \lesssim I + II$, where

$$\begin{cases} I := \int_0^{t_0} \int_{B(x_0,3\delta)} \left|G_{t_0-s}(x_0-y) - G_{t_0-s}(x-y)\right| |g(s,y)| dy ds, \\ II := \int_0^{t_0} \int_{\mathbb{R}^n \setminus B(x_0,3\delta)} \left|G_{t_0-s}(x_0-y) - G_{t_0-s}(x-y)\right| |g(s,y)| dy ds. \end{cases}$$

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We first estimate I. Write

$$I \lesssim \int_0^{t_0}\int_{B(x_0,3\delta)}\Big|G_{t_0-s}(x_0-y)\Big||g(s,y)|dyds+\int_0^{t_0}\int_{B(x_0,3\delta)}\Big|G_{t_0-s}(x-y)\Big||g(s,y)|dyds.$$

We further split

$$\int_0^{t_0}\int_{B(x_0,3\delta)}\Big|G_{t_0-s}(x_0-y)\Big||g(s,y)|dyds \lesssim I_1+I_2,$$

where

$$\begin{cases} I_1 := \displaystyle\int_0^{t_0-(2\delta)^{\alpha/\beta}}\int_{B(x_0,3\delta)}\Big|G_{t_0-s}(x_0-y)\Big||g(s,y)|dyds,\\ I_2 := \displaystyle\int_{t_0-(2\delta)^{\alpha/\beta}}^{t_0}\int_{B(x_0,3\delta)}\Big|G_{t_0-s}(x_0-y)\Big||g(s,y)|dyds. \end{cases}$$

Applying Hölder's inequality to the variables y and s, respectively, we have

$$\begin{aligned} I_1 &\lesssim \int_0^{t_0-(2\delta)^{\alpha/\beta}}\int_{B(x_0,3\delta)}\frac{|t_0-s|^{\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{n+\alpha}}|g(s,y)|dyds\\ &\lesssim \int_0^{t_0-(2\delta)^{\alpha/\beta}}\frac{1}{|t_0-s|^{\beta n/\alpha}}\left(\int_{B(x_0,3\delta)}|g(y,s)|^p dy\right)^{1/p}\left(\int_{B(x_0,3\delta)}1dy\right)^{1-1/p}ds\\ &\lesssim \int_0^{t_0-(2\delta)^{\alpha/\beta}}\frac{1}{|t_0-s|^{\beta n/\alpha}}\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\delta^{n(1-1/p)}ds\\ &\lesssim \|g(\cdot,\cdot)\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\delta^{\alpha(1-1/q)/\beta-n/p}. \end{aligned}$$

Similarly, for I_2, we have

$$\begin{aligned} I_2 &\lesssim \int_{t_0-(2\delta)^{\alpha/\beta}}^{t_0}\int_{B(x_0,3\delta)}\frac{|t_0-s|^{\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{n+\alpha}}|g(s,y)|dyds\\ &\lesssim \int_{t_0-(2\delta)^{\alpha/\beta}}^{t_0}\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\left(\int_{B(x_0,3\delta)}\frac{|t_0-s|^{p'\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)p'}}dy\right)^{1/p'}ds\\ &\lesssim \int_{t_0-(2\delta)^{\alpha/\beta}}^{t_0}\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n}\frac{|t_0-s|^{p'\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)p'}}dy\right)^{1/p'}ds. \end{aligned}$$

Split

$$\begin{aligned} &\int_{\mathbb{R}^n}\frac{|t_0-s|^{p'\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)p'}}dy\\ &\lesssim \int_{|y-x_0|<|t_0-s|^{\beta/\alpha}}\frac{|t_0-s|^{p'\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)p'}}dy\\ &\quad+\sum_{k=0}^{\infty}\int_{2^k|t_0-s|^{\beta/\alpha}\le|y-x_0|<2^{k+1}|t_0-s|^{\beta/\alpha}}\frac{|t_0-s|^{p'\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)p'}}dy \end{aligned}$$

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$$\lesssim \int_{|y-x_0|<|t_0-s|^{\beta/\alpha}} \frac{|t_0-s|^{p'\beta}}{|t_0-s|^{\beta(n+\alpha)p'/\alpha}} dy$$

$$+ \sum_{k=0}^{\infty} \int_{2^k|t_0-s|^{\beta/\alpha} \le |y-x_0| < 2^{k+1}|t_0-s|^{\beta/\alpha}} \frac{|t_0-s|^{p'\beta}}{(2^k|t_0-s|^{\beta/\alpha})^{(n+\alpha)p'}} dy$$

$$\lesssim \frac{|t_0-s|^{p'\beta}}{|t_0-s|^{\beta(n+\alpha)p'/\alpha}} |t_0-s|^{n\beta/\alpha} + \sum_{k=0}^{\infty} \frac{|t_0-s|^{p'\beta}}{(2^k|t_0-s|)^{\beta(n+\alpha)p'/\alpha}} (2^{k+1}|t_0-s|^{\beta/\alpha})^n$$

$$\lesssim \frac{1}{|t_0-s|^{n\beta(p'-1)/\alpha}}.$$

Then we obtain

$$I_2 \lesssim \int_{t_0-(2\delta)^{\alpha/\beta}}^{t_0} \|g(s,\cdot)\|_{L_x^p(\mathbb{R}^n)} \frac{ds}{|t_0-s|^{n\beta/\alpha p}}$$

$$\lesssim \|g(\cdot,\cdot)\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \left(\int_{t_0-(2\delta)^{\alpha/\beta}}^{t_0} \frac{ds}{|t_0-s|^{n\beta q'/\alpha p}} \right)^{1/q'}$$

$$\lesssim \|g(\cdot,\cdot)\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \delta^{\alpha(1-1/q)/\beta - n/p}.$$

Now, we estimate II. It follows from (ii) of Proposition 2.9 that there exists a constant $\varepsilon > 0$ such that for $|x-x_0| < |x|/2$,

$$|K_{\alpha/2,s}(x) - K_{\alpha/2,s}(x_0)| \lesssim \frac{s|x-x_0|^{\varepsilon}}{(s^{1/\alpha}+|x|)^{n+\alpha+\varepsilon}} \lesssim \frac{s|x-x_0|^{\varepsilon}}{|x|^{n+\alpha+\varepsilon}}.$$

We have

$$\left|G_t(x) - G_t(x_0)\right| \lesssim \int_0^{\infty} \frac{(t/u)^{\beta}}{|x|^{n+\alpha+\varepsilon}} |x-x_0|^{\varepsilon} g_{\beta}(u) du$$

$$\lesssim \frac{t^{\beta}|x-x_0|^{\varepsilon}}{|x|^{n+\alpha+\varepsilon}} \int_0^{\infty} u^{-\beta} g_{\beta}(u) du$$

$$\lesssim \frac{t^{\beta}|x-x_0|^{\varepsilon}}{|x|^{n+\alpha+\varepsilon}},$$

which implies that

$$\left|G_t(x) - G_t(x_0)\right| \lesssim \min\left\{ \left(\frac{|x-x_0|}{t^{\beta/\alpha}} \right)^{\varepsilon} \frac{1}{t^{n\beta/\alpha}}, \frac{t^{\beta}|x-x_0|^{\varepsilon}}{|x|^{n+\alpha+\varepsilon}} \right\}.$$

If $|x| \le t^{\beta/\alpha}$, we have

$$\left|G_t(x) - G_t(x_0)\right| \lesssim \frac{|x-x_0|^{\varepsilon} t^{\beta}}{t^{\beta(n+\alpha+\varepsilon)/\alpha}} \lesssim \frac{|x-x_0|^{\varepsilon} t^{\beta}}{(t^{\beta/\alpha}+|x|)^{n+\alpha+\varepsilon}}.$$

If $|x| > t^{\beta/\alpha}$, it still holds

$$\left|G_t(x) - G_t(x_0)\right| \lesssim \frac{|x-x_0|^{\varepsilon} t^{\beta}}{|x|^{n+\alpha+\varepsilon}} \lesssim \frac{|x-x_0|^{\varepsilon} t^{\beta}}{(t^{\beta/\alpha}+|x|)^{n+\alpha+\varepsilon}}. \tag{2.14}$$

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Applying (2.14), we can deduce that

$$\begin{aligned} II &\lesssim \int_0^{t_0}\int_{\mathbb{R}^n\setminus B(x_0,3\delta)} \frac{\delta^\varepsilon|t_0-s|^\beta}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{n+\alpha+\varepsilon}}|g(s,y)|dyds\\ &\lesssim \int_0^{t_0-(2\delta)^{\alpha/\beta}}\int_{\mathbb{R}^n\setminus B(x_0,3\delta)} \frac{\delta^\varepsilon|t_0-s|^\beta}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{n+\alpha+\varepsilon}}|g(s,y)|dyds\\ &\quad+\int_{t_0-(2\delta)^{\alpha/\beta}}^{t_0}\int_{\mathbb{R}^n\setminus B(x_0,3\delta)} \frac{\delta^\varepsilon|t_0-s|^\beta}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{n+\alpha+\varepsilon}}|g(s,y)|dyds\\ &:= II_1+II_2, \end{aligned}$$

where

$$\begin{aligned} II_1 &\lesssim \int_0^{t_0-(2\delta)^{\alpha/\beta}}\int_{\mathbb{R}^n\setminus B(x_0,3\delta)} \frac{\delta^\varepsilon|t_0-s|^\beta}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{n+\alpha+\varepsilon}}|g(s,y)|dyds\\ &\lesssim \int_0^{t_0-(2\delta)^{\alpha/\beta}} \delta^\varepsilon|t_0-s|^\beta\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n\setminus B(x_0,3\delta)}\frac{dy}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{p'(n+\alpha+\varepsilon)}}\right)^{1/p'}ds \end{aligned}$$

and, similarly,

$$II_2 \lesssim \int_{t_0-(2\delta)^{\alpha/\beta}}^{t_0} \delta^\varepsilon|t_0-s|^\beta\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n\setminus B(x_0,3\delta)}\frac{dy}{|x_0-y|^{p'(n+\alpha+\varepsilon)}}\right)^{1/p'}ds.$$

We first estimate II_1. The following integral can be estimated as

$$\begin{aligned} \int_{\mathbb{R}^n\setminus B(x_0,3\delta)}\frac{\delta^{p'\varepsilon}|t_0-s|^{p'\beta}dy}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{p'(n+\alpha+\varepsilon)}} &\lesssim \sum_{j=1}^\infty \frac{\delta^{p'\varepsilon}|t_0-s|^{p'\beta}(3^j\delta)^n}{(|t_0-s|^{\beta/\alpha}+3^j\delta)^{p'(n+\alpha+\varepsilon)}}\\ &\lesssim \sum_{j=1}^\infty\frac{1}{3^j}\int_{3^j\delta}^{3^{j+1}\delta}\frac{\delta^{p'\varepsilon-1}|t_0-s|^{p'\beta}}{(|t_0-s|^{\beta/\alpha}+r)^{p'(n+\alpha+\varepsilon)-n}}dr\\ &\lesssim \int_{3\delta}^\infty \frac{\delta^{p'\varepsilon-1}|t_0-s|^{p'\beta}}{(|t_0-s|^{\beta/\alpha}+r)^{p'(n+\alpha+\varepsilon)-n}}dr. \end{aligned}$$

A direct computation gives

$$\begin{aligned} \int_{3\delta}^\infty \frac{\delta^{p'\varepsilon-1}|t_0-s|^{p'\beta}}{(|t_0-s|^{\beta/\alpha}+r)^{p'(n+\alpha+\varepsilon)-n}}dr &\lesssim \delta^{p'\varepsilon-1}|t_0-s|^{p'\beta}\int_{3\delta+|t_0-s|^{\beta/\alpha}}^\infty\frac{1}{u^{p'(n+\alpha+\varepsilon)-n}}du\\ &\lesssim \frac{\delta^{p'\varepsilon-\beta}|t_0-s|^{p'\beta}}{(3\delta+|t_0-s|^{\beta/\alpha})^{p'(n+\alpha+\varepsilon)-n-1}}, \end{aligned}$$

which yields

$$\begin{aligned} II_1 &\lesssim \int_0^{t_0-(2\delta)^{\alpha/\beta}}\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\frac{\delta^{\varepsilon-1/p'}|t_0-s|^\beta}{(\delta+|t_0-s|^{\beta/\alpha})^{(n+\alpha+\varepsilon)-(n+1)/p'}}ds\\ &\lesssim \int_0^{t_0-(2\delta)^{\alpha/\beta}}\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\frac{\delta^{\varepsilon-1/p'}}{(\delta+|t_0-s|^{\beta/\alpha})^{n/p+\varepsilon-1/p'}}ds \end{aligned}$$

$$\lesssim \delta^{\varepsilon-1/p'}\|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\left(\int_0^{t_0-(2\delta)^\alpha}\frac{1}{(\delta+|t_0-s|^{\beta/\alpha})^{(n/p+\varepsilon-1/p')q'}}ds\right)^{1/q'}$$
$$\lesssim \delta^{\alpha(1-1/q)/\beta-n/p}\|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}.$$

For the term II_2, an application of Hölder's inequality yields

$$II_2 \lesssim \int_{t_0-(2\delta)^{\alpha/\beta}}^{t_0}|t_0-s|^\beta\delta^\varepsilon\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n\setminus B(x_0,3\delta)}\frac{dy}{|x_0-y|^{p'(n+\varepsilon+\alpha)}}\right)^{p'}ds$$
$$\lesssim \int_{t_0-(2\delta)^{\alpha/\beta}}^{t_0}\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}|t_0-s|^\beta\delta^{-(n+\varepsilon+\alpha)+n/p'}\delta^\varepsilon ds$$
$$\lesssim \delta^{-(n+\varepsilon+\alpha)+n/p'}\delta^\varepsilon\|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\left(\int_{t_0-(2\delta)^{\alpha/\beta}}^{t_0}|t_0-s|^{\beta q'}ds\right)^{1/q'}$$
$$\lesssim \|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\delta^{\alpha(1-1/q)/\beta-n/p}.$$

The estimates for I and II imply that

$$\left|G_{\alpha,\beta}(g)(t_0,x_0)-G_{\alpha,\beta}(g)(t_0,x)\right|\lesssim\|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\delta^{\alpha(1-1/q)/\beta-n/p}.$$

For fixed $x\in\mathbb{R}^n$ and $t_1>t_2$, we can see that

$$\left|G_{\alpha,\beta}(g)(t_1,x)-G_{\alpha,\beta}(g)(t_2,x)\right|\lesssim III+IV,$$

where

$$\begin{cases} III := \left|\int_0^{t_2}\int_{\mathbb{R}^n}\left(G_{t_1-s}(x-y)-G_{t_2-s}(x-y)\right)g(s,y)dsdy\right|,\\ IV := \left|\int_{t_2}^{t_1}\int_{\mathbb{R}^n}G_{t_1-s}(x-y)g(s,y)dyds\right|.\end{cases}$$

It follows from Hölder's inequality that

$$IV \lesssim \int_{t_2}^{t_1}\int_{\mathbb{R}^n}|G_{t_1-s}(x-y)||g(s,y)|dyds$$
$$\lesssim \int_{t_2}^{t_1}\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\left\{\int_{\mathbb{R}^n}\frac{|t_1-s|^{\beta p'}}{(|t_1-s|^{\beta/\alpha}+|x-y|)^{(n+\alpha)p'}}dy\right\}^{1/p'}ds$$
$$\lesssim \int_{t_2}^{t_1}\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}|t_1-s|^{-\beta n/(\alpha p)}ds$$
$$\lesssim \|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\left(\int_{t_2}^{t_1}|t_1-s|^{-\beta nq'/(\alpha p)}ds\right)^{1/q'}$$
$$\lesssim \|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}|t_1-t_2|^{1-1/q-n\beta/(\alpha p)}.$$

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To estimate III, we first investigate the Lipschitz continuity of $G_t(\cdot)$ on the time variable t. Because

$$G_{t_1-s}(x-y) - G_{t_2-s}(x-y) = \int_0^\infty \left(K_{\alpha,\left(\frac{t_1-s}{u}\right)^\beta}(x-y) - K_{\alpha,\left(\frac{t_2-s}{u}\right)^\beta}(x-y) \right) g_\beta(u)du,$$

it holds

$$\begin{aligned} III &= \left| \int_0^{t_2} \int_{\mathbb{R}^n} \left\{ \int_0^\infty \left(K_{\alpha/2,\left(\frac{t_1-s}{u}\right)^\beta}(x-y) - K_{\alpha/2,\left(\frac{t_2-s}{u}\right)^\beta}(x-y) \right) g_\beta(u)du \right\} g(s,y)dsdy \right| \\ &= \left| \int_0^{t_2} \int_0^\infty \left\{ \int_{\mathbb{R}^n} \left(K_{\alpha/2,\left(\frac{t_1-s}{u}\right)^\beta}(x-y) - K_{\alpha/2,\left(\frac{t_2-s}{u}\right)^\beta}(x-y) \right) g(s,y)dy \right\} g_\beta(u)dsdu \right| \\ &= \left| \int_0^\infty \int_0^{t_2} \left\{ e^{-\left(\frac{t_1-s}{u}\right)^\beta(-\Delta)^{\alpha/2}}(\varphi)(s,x) - e^{-\left(\frac{t_2-s}{u}\right)^\beta(-\Delta)^{\alpha/2}}(\varphi)(s,x) \right\} g_\beta(u)dsdu \right|. \end{aligned}$$

Noticing that

$$\begin{aligned} &\left| e^{-\left(\frac{t_1-s}{u}\right)^\beta(-\Delta)^{\alpha/2}}(\varphi)(s,x) - e^{-\left(\frac{t_2-s}{u}\right)^\beta(-\Delta)^{\alpha/2}}(\varphi)(s,x) \right| \\ &\quad = \left| \int_{\left(\frac{t_2-s}{u}\right)^\beta}^{\left(\frac{t_1-s}{u}\right)^\beta} \frac{\partial}{\partial_r} e^{-r(-\Delta)^{\alpha/2}}(\varphi)(s,x)dr \right| \\ &\quad \lesssim \int_{\left(\frac{t_2-s}{u}\right)^\beta}^{\left(\frac{t_1-s}{u}\right)^\beta} \left| (-\Delta)^{\alpha/2} e^{-r(-\Delta)^{\alpha/2}}(\varphi)(s,x) \right| dr \\ &\quad \lesssim \int_{\left(\frac{t_2-s}{u}\right)^\beta}^{\left(\frac{t_1-s}{u}\right)^\beta} r^{-1-n/(\alpha p)} \|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)} dr, \end{aligned}$$

we can obtain

$$\begin{aligned} III &\lesssim \int_0^\infty \left\{ \int_0^{t_2} \left(\int_{\left(\frac{t_2-s}{u}\right)^\beta}^{\left(\frac{t_1-s}{u}\right)^\beta} r^{-1-\frac{n}{\alpha p}} \|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)} dr \right) ds \right\} g_\beta(u)du \\ &\lesssim \int_0^\infty \left\{ \int_0^{t_2} \left(\int_{\left(\frac{t_2-s}{u}\right)^\beta}^{\left(\frac{t_1-s}{u}\right)^\beta} r^{-1-\frac{n}{\alpha p}} dr \right) \|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)} ds \right\} g_\beta(u)du \\ &\lesssim \int_0^\infty \left\{ \int_0^{t_2} \left(\int_0^{\left(\frac{t_1-s}{u}\right)^\beta - \left(\frac{t_2-s}{u}\right)^\beta} \frac{dr}{\left(r + \left(\frac{t_2-s}{u}\right)^\beta\right)^{1+\frac{n}{\alpha p}}} \right) \|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)} ds \right\} g_\beta(u)du. \end{aligned}$$

Since for $\beta \in (0,1)$,

$$(t_1/u - s/u)^\beta - (t_2/u - s/u)^\beta \le (t_1/u - t_2/u)^\beta,$$

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applying Fubini's theorem and Hölder's inequality, we can get

$$
\begin{aligned}
III &\lesssim \int_0^\infty \left\{ \int_0^{t_2} \left(\int_0^{(\frac{t_1-t_2}{u})^\beta} \frac{dr}{\left(r+\left(\frac{t_2-s}{u}\right)^\beta\right)^{1+\frac{n}{\alpha p}}} \right) \|g(s,\cdot)\|_{L_x^p(\mathbb{R}^n)} ds \right\} g_\beta(u)du \\
&\lesssim \int_0^\infty \left\{ \int_0^{(\frac{t_1-t_2}{u})^\beta} \left(\int_0^{t_2} \frac{\|g(s,\cdot)\|_{L_x^p(\mathbb{R}^n)}}{\left(r+\left(\frac{t_2-s}{u}\right)^\beta\right)^{1+\frac{n}{\alpha p}}} ds \right) dr \right\} g_\beta(u)du \\
&\lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \int_0^\infty \left\{ \int_0^{(\frac{t_1-t_2}{u})^\beta} \left(\int_0^{t_2} \frac{ds}{\left(r+\left(\frac{t_2-s}{u}\right)^\beta\right)^{(1+\frac{n}{\alpha p})q'}} \right)^{1/q'} dr \right\} g_\beta(u)du.
\end{aligned}
$$

By change of variables, we obtain

$$
\int_0^{t_2} \frac{ds}{\left(r+\left(\frac{t_2-s}{u}\right)^\beta\right)^{(1+\frac{n}{\alpha p})q'}} = u \int_0^{(t_2/u)^\beta} \frac{s_1^{1/\beta-1} ds_1}{(r+s_1)^{(1+\frac{n}{\alpha p})q'}} \lesssim u r^{-(1+\frac{n}{\alpha p})q'+\frac{1}{\beta}}.
$$

Finally, we have, upon $1/q'\beta - n/(\alpha p) > 0$,

$$
\begin{aligned}
III &\lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \int_0^\infty \left\{ \int_0^{(t_1/u-t_2/u)^\beta} r^{-(1+\frac{n}{\alpha p})+\frac{1}{q'\beta}} dr \right\} u^{1/q'} g_\beta(u)du \\
&\lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \int_0^\infty \left(\frac{t_1-t_2}{u} \right)^{(\frac{1}{q'}-\frac{\beta n}{\alpha p})} u^{1/q'} g_\beta(u)du \\
&\lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} |t_1-t_2|^{1-1/q-n\beta/(\alpha p)},
\end{aligned}
$$

where in the last inequality we have used the following estimate: $\int_0^\infty u^{\beta n/\alpha p} g_\beta(u)du < \infty$. In fact, by (2.2), $g_\beta(u) \lesssim u^{(N\beta-1+\beta/2)/(1-\beta)}$ for $u \in (0,1)$ and N being large enough, which indicates that

$$
\int_0^1 u^{\beta n/\alpha p} g_\beta(u)du \lesssim \int_0^1 u^{\beta n/\alpha p} u^{(N\beta-1+\beta/2)/(1-\beta)} du < \infty.
$$

On the other hand, under the assumption that $\alpha > n$ and $\beta \in (0,1)$, we can see that $\beta n/(\alpha p) - \beta - 1 < -1$ and therefore

$$
\int_1^\infty u^{\beta n/(\alpha p)} g_\beta(u)du \lesssim \int_1^\infty u^{\beta n/(\alpha p)} u^{-\beta-1} du < \infty. \qquad \blacksquare
$$

Theorems 2.12 and 2.13 show that, for the case $\alpha(1-1/q) - \beta n/p > 0$, the solution for (1.2) is continuous. For the endpoint case $\alpha(1-1/q) - \beta n/p = 0$, we can obtain the following Moser–Trudinger-type estimate for $G_{\alpha,\beta}$.

Theorem 2.14 *Given $\alpha > n$ and $0 < \beta < 1$. Assume that $p \in [1,\infty)$, $1 < q < \infty$, and $n\beta/(\alpha p) + 1/q = 1$. Let $(t_0, x_0) \in \mathbb{R}_+^{1+n}$, $r_0 = t_0^{1/(2\alpha)}$, and $0 < \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}$. There exists*

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a constant $C>0$ such that

$$\frac{1}{r^{2\alpha}|B^{(\alpha)}_{r_0}(t_0,x_0)|}\iint_{B^{(\alpha)}_{r_0}(t_0,x_0)}\exp\left(\frac{G_{\alpha,\beta}(g)(t,x)}{C\|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}}\right)^{q/(q-1)}dxdt\lesssim 1.$$

Proof Let $(t,x)\in\mathbb{R}^{1+n}_+$ be fixed. Then $|G_{\alpha,\beta}(g)(t,x)|\lesssim I+II$, where

$$\begin{cases} I:=\int_0^r\int_{\mathbb{R}^n}G_{t-s}(x-y)|g(s,y)|dyds,\\ II:=\int_r^t\int_{\mathbb{R}^n}G_{t-s}(x-y)|g(s,y)|dyds.\end{cases}$$

Below, we deal with the terms I and II, separately. By Hölder's inequality, we get

$$\begin{aligned} I&\lesssim\int_0^r\int_{\mathbb{R}^n}\frac{(t-s)^\beta}{(|t-s|^{\beta/\alpha}+|x-y|)^{n+\alpha}}g(s,y)dyds\\ &\lesssim\int_0^r(t-s)^\beta\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\left\{\int_{\mathbb{R}^n}\frac{dy}{(|t-s|^{\beta/\alpha}+|x-y|)^{p(n+\alpha)/(p-1)}}\right\}^{1-1/p}ds.\end{aligned}$$

It can be deduced from the integral

$$\int_{\mathbb{R}^n}\frac{dy}{(|t-s|^{\beta/\alpha}+|x-y|)^{p(n+\alpha)/(p-1)}}\lesssim\frac{|t-s|^{n\beta/\alpha}}{|t-s|^{\beta(n/\alpha+1)p/(p-1)}}$$

that

$$I\lesssim\int_0^r(t-s)^\beta\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\frac{|t-s|^{n\beta(1-/p)/\alpha}}{|t-s|^{\beta(n+\alpha)/\alpha}}ds\lesssim\int_0^r\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\frac{ds}{|t-s|^{n\beta/(\alpha p)}}.$$

By the fact that $1/q+n\beta/(\alpha p)=1$, we apply Hölder's inequality on s to obtain

$$I\lesssim\|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\left(\int_0^r\frac{ds}{|t-s|^{n\beta q'/(\alpha p)}}\right)^{1-1/q}\lesssim\|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\left(\ln\frac{t}{t-r}\right)^{1-1/q}.$$

Denote by $M_{\mathbb{R}}$ the Hardy–Littlewood maximal function on $\mathbb{R}$. Then, since $1-n\beta/(p\alpha)=1/q$, it holds

$$\begin{aligned} II&\lesssim\sum_{k=-\infty}^{0}\int_{t-2^k|t-r|}^{t-2^{k-1}|t-r|}\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}\frac{ds}{|t-s|^{n\beta/(p\alpha)}}\\ &\lesssim\sum_{k=-\infty}^{0}\frac{1}{(2^k|t-r|)^{n\beta/(p\alpha)}}\int_{t-2^k|t-r|}^{t}\|g(s,\cdot)\|_{L^p_x(\mathbb{R}^n)}ds\\ &\lesssim\sum_{k=-\infty}^{0}(2^k|t-r|)^{1/q}M_{\mathbb{R}}(\|g(\cdot,\cdot)\|_{L^p_x(\mathbb{R}^n)})(t)\\ &\lesssim|t-r|^{1/q}M_{\mathbb{R}}(\|g(\cdot,\cdot)\|_{L^p_x(\mathbb{R}^n)})(t).\end{aligned}$$

The estimates for I and II indicate that

$$|G_{\alpha,\beta}(g)(t,x)|\lesssim\|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\left(\ln\frac{t}{t-r}\right)^{1-1/q}+|t-r|^{1/q}M_{\mathbb{R}}(\|g(\cdot,\cdot)\|_{L^p_x(\mathbb{R}^n)})(t).$$

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Via choosing $r \in (0, t)$ such that

$$|t-r|^{1/q} = \min\left\{ t^{1/q}, \ \frac{\|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}}{M_{\mathbb{R}}(\|g(\cdot,\cdot)\|_{L_x^p(\mathbb{R}^n)})(t)} \right\},$$

we get

$$|G_{\alpha,\beta}(g)(t,x)| \lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \left\{ 1 + \left(\ln \frac{t}{t-r} \right)^{1-1/q} \right\}.$$

Case 1: $|t-r|^{1/q} = t^{1/q}$. Then

$$|G_{\alpha,\beta}(g)(t,x)| \lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}.$$

Case 2: $|t-r|^{1/q} = \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} M_{\mathbb{R}}^{-1}(\|g(\cdot,\cdot)\|_{L_x^p(\mathbb{R}^n)})(t)$. We have

$$\begin{aligned}
|G_{\alpha,\beta}(g)(t,x)| &\lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \left\{ 1 + \left(\ln \frac{t\left(M_{\mathbb{R}}(\|g(\cdot,\cdot)\|_{L_x^p(\mathbb{R}^n)})(t)\right)^q}{\|g\|^q_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} \right)^{1-1/q} \right\} \\
&\lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \left\{ 1 + \left(q \ln \frac{t^{1/q} M_{\mathbb{R}}(\|g(\cdot,\cdot)\|_{L_x^p(\mathbb{R}^n)})(t)}{\|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} \right)^{1-1/q} \right\}.
\end{aligned}$$

Let $r_0 = t_0^{1/2\alpha}$. Then

$$\begin{aligned}
&\iint_{B_{r_0}^{(\alpha)}(t_0,x_0)} \exp\left(\frac{G_{\alpha,\beta}(g)(t,x)}{C\|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} \right)^{q/(q-1)} dxdt \\
&\lesssim \iint_{B_{r_0}^{(\alpha)}(t_0,x_0)} \exp\left\{ \frac{\|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \left\{ 1 + \left(q \ln \frac{t^{1/q} M_{\mathbb{R}}(\|g(\cdot,\cdot)\|_{L_x^p(\mathbb{R}^n)})(t)}{\|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} \right)^{1-1/q} \right\}}{C\|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} \right\}^{q/(q-1)} dxdt \\
&\lesssim \iint_{B_{r_0}^{(\alpha)}(t_0,x_0)} \exp\left\{ 1 + \left(\ln \frac{t^{1/q} M_{\mathbb{R}}(\|g(\cdot,\cdot)\|_{L_x^p(\mathbb{R}^n)})(t)}{\|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} \right)^{1-1/q} \right\}^{q/(q-1)} dxdt \\
&\lesssim \iint_{B_{r_0}^{(\alpha)}(t_0,x_0)} \exp\left(\ln \frac{t^{1/q} M_{\mathbb{R}}(\|g(\cdot,\cdot)\|_{L_x^p(\mathbb{R}^n)})(t)}{\|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} \right) dxdt \\
&\lesssim \iint_{B_{r_0}^{(\alpha)}(t_0,x_0)} \frac{t^{1/q} M_{\mathbb{R}}(\|g(\cdot,\cdot)\|_{L_x^p(\mathbb{R}^n)})(t)}{\|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} dxdt \\
&\lesssim |B_{r_0}^{(\alpha)}(t_0,x_0)| t_0^{1/q} \int_0^{2t_0} \frac{M_{\mathbb{R}}(\|g(\cdot,\cdot)\|_{L_x^p(\mathbb{R}^n)})(t)}{\|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} dt \lesssim r_0^{2\alpha} |B_{r_0}^{(\alpha)}(t_0,x_0)|.
\end{aligned}$$

This completes the proof of Theorem 2.14. ■

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3 $L_t^q L_x^p$-capacities associated with space–time fractional equations

In this section, we study the $L_t^q L_x^p$-capacity associated with $G_{\alpha,\beta}$ defined in Definition 1.1. We establish the dual form, some basic properties, and the $L_t^q L_x^p$-capacity of fractional parabolic balls in $\mathbb{R}_+^{n+1}$. As a by-product, a strong-type inequality is also obtained.

3.1 The dual form of the $L_t^q L_x^p$-capacity

First, let us introduce the adjoint operator of $G_{\alpha,\beta}$. For any $f, g \in C_0^\infty(\mathbb{R}_+^{1+n})$, since

$$\begin{aligned}
&\iint_{\mathbb{R}_+^{1+n}} G_{\alpha,\beta}(f)(t,x)g(t,x)dxdt\\
&= \iint_{\mathbb{R}_+^{1+n}} \left(\int_0^t \int_{\mathbb{R}^n} G_{t-s}(x-y)f(s,y)dyds \right) g(t,x)dxdt\\
&= \int_0^\infty \int_{\mathbb{R}^n} \left(\int_0^t \int_{\mathbb{R}^n} G_{t-s}(x-y)f(s,y)dyds \right) g(t,x)dxdt\\
&= \int_0^\infty \int_{\mathbb{R}^n} f(t,x) \left(\int_t^\infty \int_{\mathbb{R}^n} G_{s-t}(y-x)g(s,y)dsdy \right) dxdt,
\end{aligned}$$

the adjoint operator $G_{\alpha,\beta}^*$ of $G_{\alpha,\beta}$ is defined via

$$G_{\alpha,\beta}^*(g)(t,x) := \int_t^\infty \int_{\mathbb{R}^n} G_{s-t}(y-x)g(s,y)dyds. \tag{3.1}$$

The definition of $G_{\alpha,\beta}^*$ can be extended to the family of Borel measures μ with compact supports in $\mathbb{R}_+^{1+n}$. Let $\|\mu\|_1$ denote the total variants of μ. If F is continuous and compactly supported in $\mathbb{R}_+^{1+n}$, then

$$\begin{aligned}
\left| \iint_{\mathbb{R}_+^{1+n}} G_{\alpha,\beta}(g)(t,x)d\mu(t,x) \right| &\le \iint_{\mathbb{R}_+^{1+n}} \left| \int_0^t \int_{\mathbb{R}^n} G_{t-s}(x-y)g(s,y)dyds \right| d\mu(t,x)\\
&\le \iint_{\mathbb{R}_+^{1+n}} \left\{ \int_0^t \int_{\mathbb{R}^n} |G_{t-s}(x-y)| \cdot |g(s,y)|dyds \right\} d\mu(t,x)\\
&\le \sup_{(t,x)\in\mathbb{R}_+^{1+n}} \{|g(t,x)|\} \cdot \|\mu\|_1.
\end{aligned}$$

Hence, an application of the Riesz representation theorem yields that there exists a Borel measure $\nu(\cdot,\cdot)$ on $\mathbb{R}_+^{1+n}$ such that

$$\iint_{\mathbb{R}_+^{1+n}} G_{\alpha,\beta}(g)(t,x)d\mu(t,x) = \iint_{\mathbb{R}_+^{1+n}} g(t,x)d\nu(t,x).$$

Proposition 3.1 *For a compact subset $K \subset \mathbb{R}_+^{1+n}$, let $\mathcal{M}_+(K)$ be the class of all positive measures on $\mathbb{R}_+^{1+n}$ supported by K. If $1 < p, q < \infty$, $p' = p/(p-1)$, and $q' = q/(q-1)$, then*

$$C_{p,q}^{(\alpha,\beta)}(K) = \sup\left\{ \|\mu\|_1^{p\wedge q} : \mu \in \mathcal{M}_+(K) \ \&\ \|G_{\alpha,\beta}^*\mu\|_{L_t^{q'}L_x^{p'}(\mathbb{R}_+^{1+n})} \le 1 \right\}.$$

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Proof Set

$$\widetilde{C_{p,q}^{(\alpha,\beta)}}(K) := \sup\left\{\|\mu\|_1^{p\wedge q} : \mu \in \mathcal{M}_+(K) \;\&\; \|G^*_{\alpha,\beta}\mu\|_{L_t^{q'}L_x^{p'}(\mathbb{R}_+^{1+n})} \le 1\right\}.$$

For any $g \ge 0$ such that $G_{\alpha,\beta}(g) \ge 1_K$, by (3.1) and Hölder's inequality, we have

$$\begin{aligned}
\|\mu\|_1 = \mu(K) &= \iint_K 1 d\mu(t,x)\\
&\le \iint_K G_{\alpha,\beta}(g)(t,x)d\mu(t,x)\\
&\le \iint_{\mathbb{R}_+^{1+n}} G_{\alpha,\beta}(g)(t,x)d\mu(t,x)\\
&= \iint_{\mathbb{R}_+^{1+n}} g(t,x)G^*_{\alpha,\beta}(\mu)(t,x)dxdt\\
&\le \|g\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})}\|G^*_{\alpha,\beta}\mu\|_{L_t^{q'}L_x^{p'}(\mathbb{R}_+^{1+n})},
\end{aligned}$$

which implies that if μ satisfies $\|G^*_{\alpha,\beta}\mu\|_{L_t^{q'}L_x^{p'}(\mathbb{R}_+^{1+n})} \le 1$, then $\|\mu\|_1^{p\wedge q} \le \|g\|^{p\wedge q}_{L_t^qL_x^p(\mathbb{R}_+^{1+n})}$, i.e.,

$$\widetilde{C_{p,q}^{(\alpha,\beta)}}(K) \le C_{p,q}^{(\alpha,\beta)}(K).$$

Moreover, let

$$\begin{cases}
\mathcal{X} := \left\{\mu : \mu \in \mathcal{M}_+(K) \;\&\; \mu(K) = 1\right\},\\
\mathcal{Y} := \left\{g : 0 \le g \in L_t^qL_x^p(\mathbb{R}_+^{1+n}) \;\&\; \|g\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})} \le 1\right\},\\
\mathcal{Z} := \left\{g : 0 \le g \in L_t^qL_x^p(\mathbb{R}_+^{1+n}) \;\&\; G_{\alpha,\beta}(g) \ge 1_K\right\}.
\end{cases}$$

We have

$$\begin{aligned}
\min_{\mu\in\mathcal{M}_+(K)}\left\{(\mu(K))^{-1}\|G^*_{\alpha,\beta}\mu\|_{L_t^{q'}L_x^{p'}(\mathbb{R}_+^{1+n})}\right\} &= \min_{\mu\in\mathcal{X}}\left\{(\mu(K))^{-1}\|G^*_{\alpha,\beta}\mu\|_{L_t^{q'}L_x^{p'}(\mathbb{R}_+^{1+n})}\right\}\\
&= \min_{\mu\in\mathcal{X}}\sup_{g\in\mathcal{Y}}\left\{\iint_{\mathbb{R}_+^{1+n}} g(t,x)(G^*_{\alpha,\beta}\mu)(t,x)dxdt\right\}.
\end{aligned}$$

By [3, Theorem 2.4.1], we can deduce that

$$\begin{aligned}
&\min_{\mu\in\mathcal{X}}\sup_{g\in\mathcal{Y}}\left\{\iint_{\mathbb{R}_+^{1+n}} g(t,x)(G^*_{\alpha,\beta}\mu)(t,x)dxdt\right\}\\
&= \sup_{g\in\mathcal{Y}}\min_{\mu\in\mathcal{X}}\left\{\iint_{\mathbb{R}_+^{1+n}} G_{\alpha,\beta}(g)(t,x)d\mu(t,x)\right\}\\
&= \sup_{0\le g\in L_t^qL_x^p}\frac{1}{\|g\|_{L_t^qL_x^p}}\min_{\mu\in\mathcal{X}}\left\{\iint_{\mathbb{R}_+^{1+n}} G_{\alpha,\beta}(g)(t,x)d\mu(t,x)\right\},
\end{aligned}$$

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which implies

$$\min_{\mu\in\mathcal{M}_+(K)}\left\{\frac{1}{\mu(K)}\|G^*_{\alpha,\beta}\mu\|_{L_t^{q'}L_x^{p'}(\mathbb{R}^{1+n}_+)}\right\}\le \sup_{0\le g\in L_t^qL_x^p}\frac{1}{\|g\|_{L_t^qL_x^p}}\min_{(t,x)\in K}\left\{G_{\alpha,\beta}(g)(t,x)\right\}$$
$$=\sup_{g\in\mathcal{Z}}\frac{1}{\|g\|_{L_t^qL_x^p}}=\left(C^{(\alpha,\beta)}_{p,q}(K)\right)^{-1/(p\wedge q)}.$$

Hence, it follows

$$C^{(\alpha,\beta)}_{p,q}(K)=\left(\frac{1}{\min\limits_{\mu\in\mathcal{M}_+(K)}\|G^*_{\alpha,\beta}\mu\|_{L_t^qL_x^p}/\mu(K)}\right)^{p\wedge q}$$
$$=\left(\sup_{\mu\in\mathcal{M}_+(K)}\frac{\|\mu\|_1}{\|G^*_{\alpha,\beta}\mu\|_{L_t^qL_x^p}}\right)^{p\wedge q}\lesssim\widetilde{C^{(\alpha,\beta)}_{p,q}}(K).\qquad\blacksquare$$

3.2 Basic properties of the $L_t^qL_x^p$-capacity

Now, we are ready to establish the following basic properties of the $L_t^qL_x^p$-capacities.

Theorem 3.2 *Assume that $\alpha>n$ and $0<\beta<1$. Let $1\le p<\infty$ and $1<q<\infty$.*

(i) $C^{(\alpha,\beta)}_{p,q}(\varnothing)=0$. *Moreover, under $K\subset\mathbb{R}^{1+n}_+$ and $K\ne\varnothing$, $C^{(\alpha,\beta)}_{p,q}(K)=0$ if and only if there exists $0\le g\in L_t^qL_x^p(\mathbb{R}^{1+n}_+)$ such that*

$$K\subseteq\left\{(t,x)\in\mathbb{R}^{1+n}_+:\ G_{\alpha,\beta}(g)(t,x)=\infty\right\}.$$

(ii) *If $E_1\subseteq E_2\subset\mathbb{R}^{1+n}_+$,*

$$C^{(\alpha,\beta)}_{p,q}(E_1)\le C^{(\alpha,\beta)}_{p,q}(E_2).$$

(iii) *Let $\{E_j\}_{j=1}^\infty$ be any sequence of subsets of $\mathbb{R}^{1+n}_+$,*

$$C^{(\alpha,\beta)}_{p,q}\left(\bigcup_{j=1}^\infty E_j\right)\le\sum_{j=1}^\infty C^{(\alpha,\beta)}_{p,q}(E_j).$$

(iv) *For any $K\subset\mathbb{R}^{1+n}_+$ and any $x_0\in\mathbb{R}^n$,*

$$C^{(\alpha,\beta)}_{p,q}(K+(0,x_0))=C^{(\alpha,\beta)}_{p,q}(K).$$

Proof (i) If $K=\varnothing$, then for $g(t,x)\equiv0, 1_K\equiv0$ and $G_{\alpha,\beta}(g)=1_K=0$, which indicates that $C^{(\alpha,\beta)}_{p,q}(\varnothing)=0$. Set

$$K_\lambda=\left\{(t,x)\in\mathbb{R}^{1+n}_+:\ g\ge0\ \&\ G_{\alpha,\beta}(g)\ge\lambda\right\}.$$

We can see that

$$C^{(\alpha,\beta)}_{p,q}\left(\left\{(t,x)\in\mathbb{R}^{1+n}_+:\ g\ge0\ \&\ G_{\alpha,\beta}(g)\ge\lambda\right\}\right):=\inf\left\{\|g\|^{p\wedge q}_{L_t^qL_x^p}:\ G\ge0\ \&\ G_{\alpha,\beta}(G)\ge1_{K_\lambda}\right\}.$$

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Take $g \geq 0$ and $G_{\alpha,\beta}(g) \geq \lambda$ on K_λ such that the function $G = g/\lambda$ satisfies $G_{\alpha,\beta}(G) \geq (G_{\alpha,\beta}(g))/\lambda \geq 1_{K_\lambda}$, which means

$$C_{p,q}^{(\alpha,\beta)}(K_\lambda) \leq \|g/\lambda\|_{L_t^q L_x^p}^{p\wedge q} \leq \lambda^{-p\wedge q}\|g\|_{L_t^q L_x^p}^{p\wedge q}.$$

For any $(t,x) \in \mathcal{B}(G_{\alpha,\beta}(g),p,q)$, $G_{\alpha,\beta}(g)(t,x) = \infty$. For any j, $G_{\alpha,\beta}(g)(t,x) > 2^j$. Then

$$(t,x) \in \left\{(t,x) \in \mathbb{R}_+^{1+n} : \ g \geq 0 \ \& \ G_{\alpha,\beta}(g)(t,x) > 2^j\right\},$$

that is,

$$\mathcal{B}(G_{\alpha,\beta}(g),p,q) \subset \left\{(t,x) \in \mathbb{R}_+^{1+n} : \ g \geq 0 \ \& \ G_{\alpha,\beta}(g)(t,x) > 2^j\right\}.$$

We obtain

$$\begin{aligned} C_{p,q}^{(\alpha,\beta)}\left(\mathcal{B}(G_{\alpha,\beta}(g),p,q)\right) &\leq C_{p,q}^{(\alpha,\beta)}\left(\left\{(t,x) \in \mathbb{R}_+^{1+n} : \ g \geq 0 \ \& \ G_{\alpha,\beta}(g)(t,x) > 2^j\right\}\right) \\ &\leq 2^{-j(p\wedge q)}\|g\|_{L_t^q L_x^p}^{p\wedge q}. \end{aligned}$$

Letting $j \to \infty$ gives $C_{p,q}^{(\alpha,\beta)}(\mathcal{B}(G_{\alpha,\beta}(g),p,q)) = 0$. This shows that $C_{p,q}^{(\alpha,\beta)}(K) = 0$ if $K \subseteq \mathcal{B}(G_{\alpha,\beta}(g),p,q)$.

Conversely, if $C_{p,q}^{(\alpha,\beta)}(K) = 0$, from the definition of $C_{p,q}^{(\alpha,\beta)}(\cdot)$, we can see that

$$\inf\left\{\|g\|_{L_t^q L_x^p}^{p\wedge q} : \ g \geq 0 \ \& \ G_{\alpha,\beta}(g) \geq 1_K\right\} = 0.$$

Then, for any $j \in \mathbb{Z}$, there exists g_j such that $g_j \geq 0$ and $G_{\alpha,\beta}(g_j) \geq 1$ on K satisfying $\|g_j\|_{L_t^q L_x^p}^{p\wedge q} < 2^{-j}$. Let $g = \sum_{j=1}^\infty g_j$. It is obvious that $g \geq 0$ and $G_{\alpha,\beta}(g) = \infty$ on K, which implies that $K \subset \mathcal{B}(G_{\alpha,\beta}, g, p, q)$.

(ii) Let $K_1 \subseteq K_2$. Then $1_{K_1}(t,x) \leq 1_{K_2}(t,x)$ and

$$\left\{\|g\|_{L_t^q L_x^p}^{p\wedge q} : \ g \geq 0 \ \& \ G_{\alpha,\beta}(g) \geq 1_{K_2}\right\} \subseteq \left\{\|g\|_{L_t^q L_x^p}^{p\wedge q} : \ g \geq 0 \ \& \ G_{\alpha,\beta}(g) \geq 1_{K_1}\right\}. \tag{3.2}$$

Taking the infimum on both sides of (3.2) yields $C_{p,q}^{(\alpha,\beta)}(K_1) \leq C_{p,q}^{(\alpha,\beta)}(K_2)$.

(iii) The proof of this assertion is divided into two cases.

Case 1: $p \geq q$, i.e., $q = p \wedge q$. Let $\{g_j\}$ be a sequence of functions such that $G_{\alpha,\beta}(g_j) \geq 1$ on E_j. Define $g := \sup_j g_j$ which satisfies $G_{\alpha,\beta}(g) \geq 1$ on $\bigcup_{j=1}^\infty E_j$. We can get

$$\|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^q = \left\{\int_0^\infty \left(\int_{\mathbb{R}^n} |g(t,x)|^p dx\right)^{q/p} dt\right\} \leq \sum_{j=1}^\infty \left\{\int_0^\infty \left(\int_{\mathbb{R}^n} |g_j(t,x)|^p dx\right)^{q/p} dt\right\}.$$

This implies that

$$C_{p,q}^{(\alpha,\beta)}\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty C_{p,q}^{(\alpha,\beta)}(E_j).$$

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Case 2: $p < q$, i.e., $p \wedge q = p$. For $\{g_j\}$ such that $g_j \geq 0$ and $G_{\alpha,\beta}(g_j) \geq 1$ on E_j, let $g = \sup_j g_j$. For $(t,x) \in \bigcup_{j=1}^{\infty} E_j$, there exists a j_0 such that $(t,x) \in E_{j_0}$. Then $g(t,x) \geq g_{j_0}(t,x) \geq 0$, i.e., $g \geq 0$ on $\bigcup_{j=1}^{\infty} E_j$. We get $G_{\alpha,\beta}(g)(t,x) \geq G_{\alpha,\beta}(g_{j_0})(t,x) \geq 0$ for $(t,x) \in \bigcup_{j=1}^{\infty} E_j$. This indicates that

$$\begin{aligned}
\|g\|^p_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} &\leq \left\{ \int_0^\infty \left(\sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |g_j(t,x)|^p \, dx \right)^{q/p} dt \right\}^{p/q} \\
&\leq \left\| \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |g_j(t,x)|^p \, dx \right\|_{L_t^{q/p}} \\
&\leq \sum_{j=1}^{\infty} \left\| \int_{\mathbb{R}^n} |g_j(t,x)|^p \, dx \right\|_{L_t^{q/p}} = \sum_{j=1}^{\infty} \|g_j\|^p_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)},
\end{aligned}$$

which gives

$$C^{(\alpha,\beta)}_{p,q}\left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} C^{(\alpha,\beta)}_{p,q}(E_j).$$

(iv) Because $g_{x_0}(x,t) = g(t, x + x_0)$, it is easy to see that $\|g_{x_0}\|_{L^q_t L^p_x} = \|g\|_{L^q_t L^p_x}$ and

$$C^{(\alpha,\beta)}_{p,q}(K + (0,x_0)) = C^{(\alpha,\beta)}_{p,q}(K). \qquad \blacksquare$$

At the end of this subsection, we investigate the inner and outer capacitary properties of $C^{(\alpha,\beta)}_{p,q}(\cdot)$. We first prove that the capacity $C^{(\alpha,\beta)}_{p,q}(\cdot)$ is an outer capacity.

Proposition 3.3 *For any $E \subset \mathbb{R}^{n+1}_+$,*

$$C^{(\alpha,\beta)}_{p,q}(E) = \inf\left\{ C^{(\alpha,\beta)}_{p,q}(O) : \ O \supset E \ \& \ O \text{ open} \right\}.$$

Proof Without loss of generality, we assume that $C^{(\alpha,\beta)}_{p,q}(E) < \infty$. By (ii) of Theorem 3.2,

$$C^{(\alpha,\beta)}_{p,q}(E) \leq \inf\left\{ C^{(\alpha,\beta)}_{p,q}(O) : \ O \supset E \ \& \ O \text{ open} \right\}.$$

For $\varepsilon \in (0,1)$, there exists a measurable, nonnegative function f such that $G_{\alpha,\beta} f \geq 1$ on E and $\|f\|^{p\wedge q}_{L^q_t L^p_x} \leq C^{(\alpha,\beta)}_{p,q}(E) + \varepsilon$. Since $G_{\alpha,\beta} f$ is lower semicontinuous, then the set $O_\varepsilon := \{(x,t) \in \mathbb{R}^{n+1}_+ : \ G_{\alpha,\beta} f(x,t) > 1 - \varepsilon\}$ is an open set. On the other hand, $E \subset O_\varepsilon$. This implies that

$$\begin{aligned}
C^{(\alpha,\beta)}_{p,q}(O_\varepsilon) &\leq \frac{1}{(1-\varepsilon)^{p\wedge q}} \left(\int_0^\infty \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{q/p} dt \right)^{(p\wedge q)/q} \\
&< \frac{1}{(1-\varepsilon)^{p\wedge q}} (C^{(\alpha,\beta)}_{p,q}(E) + \varepsilon).
\end{aligned}$$

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The arbitrariness of ε indicates that

$$C^{(\alpha,\beta)}_{p,q}(E) \geq \inf\left\{C^{(\alpha,\beta)}_{p,q}(O):\ O \supset E \ \&\ O \text{ open}\right\}. \qquad \blacksquare$$

The following result can be deduced from Proposition 2.1 as an immediate corollary.

Corollary 3.4 *If K_j is a decreasing sequence of compact sets, then*

$$C^{(\alpha,\beta)}_{p,q}(\cap_{j=1}^{\infty} K_j) = \lim_{j\to\infty} C^{(\alpha,\beta)}_{p,q}(K_j).$$

Proposition 3.5 *Let $1 < p < \infty$ and $1 < q < \infty$. If $\{E_j\}_{j=1}^{\infty}$ is an increasing sequence of arbitrary subsets of $\mathbb{R}^n$, then*

$$C^{(\alpha,\beta)}_{p,q}(\cup_{j=1}^{\infty} E_j) = \lim_{j\to\infty} C^{(\alpha,\beta)}_{p,q}(E_j).$$

Proof Since $\{E_j\}_{j=1}^{\infty}$ is increasing, then

$$C^{(\alpha,\beta)}_{p,q}(\cup_{j=1}^{\infty} E_j) \geq \lim_{j\to\infty} C^{(\alpha,\beta)}_{p,q}(E_j).$$

Conversely, without loss of generality, we assume that $\lim_{j\to\infty} C^{(\alpha,\beta)}_{p,q}(E_j)$ is finite. For each j, let f_{E_j} be the unique function such that $f_{E_j} \geq 1$ on E_j and $\|f_{E_j}\|^{p\wedge q}_{L^q_t L^p_x} = C^{(\alpha,\beta)}_{p,q}(E_j)$. Then, for $i < j$, it holds that $G_{\alpha,\beta} f_{E_j} \geq 1$ on E_i and further, $G_{\alpha,\beta}((f_{E_i} + f_{E_j})/2) \geq 1$ on E_i, which means that

$$\left(\int_0^{\infty}\left(\int_{\mathbb{R}^n}((f_{E_i} + f_{E_j})(x,t)/2)^p\,dx\right)^{q/p} dt\right)^{(p\wedge q)/q} \geq C^{(\alpha,\beta)}_{p,q}(E_i).$$

Since the space $L^q_t L^p_x(\mathbb{R}^{n+1}_+)$ is a uniformly convex Banach space for $1 < p < \infty$ and $1 < q < \infty$ (cf. [7, Theorem 1, p. 317]), by [3, Corollary 1.3.3], the sequence $\{f_{E_j}\}_{j=1}^{\infty}$ converges strongly to a function f satisfying

$$\|f\|^{p\wedge q}_{L^q_t L^p_x} = \lim_{j\to\infty} C^{(\alpha,\beta)}_{p,q}(E_j).$$

Similar to [3, Proposition 2.3.12], we can prove that $G_{\alpha,\beta} f \geq 1$ on $\cup_{j=1}^{\infty} E_j$, except possibly on a countable union of sets with $C^{(\alpha,\beta)}_{p,q}(\cdot)$ zero. Hence,

$$\lim_{j\to\infty} C^{(\alpha,\beta)}_{p,q}(E_j) \geq \|f\|^{p\wedge q}_{L^q_t L^p_x} \geq C^{(\alpha,\beta)}_{p,q}(\cup_{j=1}^{\infty} E_j). \qquad \blacksquare$$

As a corollary of Proposition 3.5, we can get the following result.

Corollary 3.6 *Assume that $1 < p < \infty$ and $1 < q < \infty$. Let O be an open subset of $\mathbb{R}^{n+1}_+$. Then*

$$C^{(\alpha,\beta)}_{p,q}(O) = \sup\left\{C^{(\alpha,\beta)}_{p,q}(K):\ \textit{compact } K \subset O\right\}.$$

3.3 $L^q_t L^p_x$-capacity of fractional parabolic balls in $\mathbb{R}^{1+n}_+$

For $(t_0, x_0, r) \in (0,\infty)\times\mathbb{R}^n\times(0,\infty)$, the fractional parabolic ball is defined as

$$B_r^{(\alpha,\beta)}(t_0,x_0) = \left\{(t,x)\in\mathbb{R}^{1+n}_+,\ r^\alpha < t-t_0 < 2r^\alpha, |x-x_0| < r^\beta/2\right\}.$$

Theorem 3.7 Assume that $(\alpha,\beta,p,q)\in(n,\infty)\times(0,1)\times[1,\infty)\times(1,\infty)$ *satisfying* $n\beta/p+\alpha/q-\alpha>0$. *Then, as* $r_0\to 0$ *and* $(t_0,x_0)\in\mathbb{R}^{1+n}_+$,

$$C_{p,q}^{(\alpha,\beta)}(B_{r_0}^{(\alpha,\beta)}(t_0,x_0)) \simeq r_0^{(p\wedge q)(n\beta/p+\alpha/q-\alpha)}.$$

Proof We first assume that $(t_0,x_0)=(0,0)$. Let $g\ge 0$ satisfying $G_{\alpha,\beta}(g)(t,x)\ge 1_{B_{r_0}^{(\alpha,\beta)}(0,0)}(t,x)$. If $(t,x)\in B_{r_0}^{(\alpha,\beta)}(0,0)$, then $|t|<2r_0^\alpha$ and $|x|<r_0^\beta/2$. We can see that $(t,x)\in B_{r_0}^{(\alpha,\beta)}(0,0)$ if and only if $(s,y)\in B_1^{(\alpha,\beta)}(0,0)$, where $s=t/r_0^\alpha$ and $y=x/r_0^\beta$. Then, by (2.5), we utilize the change of variables $r_0^{-\alpha}u=v$ & $r_0^{-\beta}z=w$ to deduce that

$$\begin{aligned} G_{\alpha,\beta}(g)(t,x) &= \int_0^{r_0^\alpha s}\int_{\mathbb{R}^n} G_{r_0^\alpha s-u}(r_0^\beta y - z)F(u,z)\,du\,dz \\ &= \int_0^{r_0^\alpha s}\int_{\mathbb{R}^n} \frac{|r_0^\alpha s-u|^\beta}{(|r_0^\alpha s-u|^{\beta/\alpha}+|r_0^\beta y-z|)^{n+\alpha}} g(u,z)\,du\,dz \\ &= \int_0^{s}\int_{\mathbb{R}^n} \frac{r_0^\alpha|s-v|^\beta}{(|s-v|^{\beta/\alpha}+|y-w|)^{n+\alpha}} g(r_0^\alpha v, r_0^\beta w)\,dv\,dw. \end{aligned}$$

Denote $r_0^\alpha g(r_0^\alpha t, r_0^\beta x)$ by $g^*(t,x)$. The above computation gives

$$G_{\alpha,\beta}(g)(t,x) = G_{\alpha,\beta}(g^*)(s,y).$$

This means that $G_{\alpha,\beta}(g)(t,x)\ge 1$ for $(t,x)\in B_{r_0}^{(\alpha,\beta)}(0,0)$ if and only if $G_{\alpha,\beta}(g^*)(s,y)\ge 1$ for $(s,y)\in B_1^{(\alpha,\beta)}(0,0)$. It is easy to get

$$C_{p,q}^{(\alpha,\beta)}(B_1^{(\alpha,\beta)}(0,0)) \le \|g^*\|^{p\wedge q}_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)} = \|r_0^\alpha g(r_0^\alpha\cdot, r_0^\beta\cdot)\|^{p\wedge q}_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}.$$

Below a change of variables gives

$$\begin{aligned} \|r_0^\alpha g(r_0^\alpha\cdot, r_0^\beta\cdot)\|^{p\wedge q}_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)} &= \left\{\int_0^\infty\left(\int_{\mathbb{R}^n}|r_0^\alpha g(r_0^\alpha t, r_0^\beta x)|^p\,dx\right)^{q/p} dt\right\}^{(p\wedge q)/q} \\ &= r^{(\alpha-\beta n/p-\alpha/q)(p\wedge q)}\|g\|^{p\wedge q}_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}, \end{aligned}$$

which indicates that

$$C_{p,q}^{(\alpha,\beta)}(B_1^{(\alpha,\beta)}(0,0)) \le \|g^*\|^{p\wedge q}_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)} \le r^{(\alpha-\beta n/p-\alpha/q)(p\wedge q)}\|g\|^{p\wedge q}_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}.$$

By the arbitrariness of g, we know

$$C_{p,q}^{(\alpha,\beta)}(B_1^{(\alpha,\beta)}(0,0)) \le r^{(\alpha-\beta n/p-\alpha/q)(p\wedge q)} C_{p,q}^{(\alpha,\beta)}(B_{r_0}^{(\alpha,\beta)}(0,0)).$$

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Changing the order of $B_{r_0}^{(\alpha,\beta)}(0,0)$ and $B_1^{(\alpha,\beta)}(0,0)$, similarly, it holds

$$C_{p,q}^{(\alpha,\beta)}(B_{r_0}^{(\alpha,\beta)}(0,0)) \le r^{-(\alpha-\beta n/p-\alpha/q)(p\wedge q)} C_{p,q}^{(\alpha,\beta)}(B_1^{(\alpha,\beta)}(0,0)).$$

This shows that

$$C_{p,q}^{(\alpha,\beta)}(B_{r_0}^{(\alpha,\beta)}(0,0)) \simeq r_0^{(p\wedge q)(n\beta/p+\alpha/q-\alpha)}.$$

Now, for arbitrary $(t_0,x_0) \in \mathbb{R}_+^{1+n}$, let $g \ge 0$ and $G_{\alpha,\beta}(g) \ge 1_{B_{r_0}^{(\alpha,\beta)}(t_0,x_0)}$. Then, for $1 \le p < \infty$ and $1 < q < \infty$, there exist $\widetilde{p}$ and $\widetilde{q}$ such that

$$\begin{cases} 1 \le p < \widetilde{p} < \infty, \\ 1 < q < \widetilde{q} < \infty, \\ (1/q - 1/\widetilde{q}) + (\beta n/\alpha)(1/p - 1/\widetilde{p}) = 1. \end{cases}$$

By Theorem 2.6, $\|G_{\alpha,\beta}(g)\|_{L_t^{\widetilde{q}} L_x^{\widetilde{p}}(\mathbb{R}_+^{1+n})} \lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}$. It is obvious that

$$\begin{aligned} \|G_{\alpha,\beta}(g)\|_{L_t^{\widetilde{q}} L_x^{\widetilde{p}}(\mathbb{R}_+^{1+n})} &\ge \left\{ \int_{r_0^\alpha+t_0}^{2r_0^\alpha+t_0} \left(\int_{B(x_0,r_0^\beta/2)} |G_{\alpha,\beta}(g)(t,x)|^{\widetilde{p}} dx \right)^{\widetilde{q}/\widetilde{p}} dt \right\}^{1/\widetilde{q}} \\ &\ge \left\{ \int_{r_0^\alpha+t_0}^{2r_0^\alpha+t_0} \left(\int_{B(x_0,r_0^\beta/2)} 1 dx \right)^{\widetilde{q}/\widetilde{p}} dt \right\}^{1/\widetilde{q}} \gtrsim r_0^{n\beta/\widetilde{p}+\alpha/\widetilde{q}}, \end{aligned}$$

which implies $r_0^{(n\beta/\widetilde{p}+\alpha/\widetilde{q})(p\wedge q)} \lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p\wedge q}$. Since $(1/q - 1/\widetilde{q}) + (\beta n/\alpha)(1/p - 1/\widetilde{p}) = 1$, the arbitrariness of g shows that

$$r_0^{(\alpha/q+n\beta/p-\alpha)(p\wedge q)} \lesssim C_{p,q}^{(\alpha,\beta)}(B_{r_0}^{(\alpha,\beta)}(t_0,x_0)).$$

Below, we investigate the upper bound of $C_{p,q}^{(\alpha,\beta)}(B_{r_0}^{(\alpha,\beta)}(t_0,x_0))$. Consider the ball

$$B_{r_0,\eta}^{(\alpha,\beta)}(t_0,x_0) = \left\{ (t,x) \in \mathbb{R}_+^{1+n} : |t-t_0| < (\eta r_0)^\alpha \ \& \ |x-x_0| < r_0^\beta \right\},$$

where η is a sufficiently large constant. Since $(t,x) \in B_{r_0,\eta}^{(\alpha,\beta)}(t_0,x_0)$ ensures that we have, for $r_0 > 0$ small enough,

$$\begin{aligned} G_{\alpha,\beta} 1_{B_{r_0,\eta}^{(\alpha,\beta)}(t_0,x_0)}(t,x) &= \int_0^t \int_{\mathbb{R}^n} G_{t-s}(x-y) 1_{B_{r_0,\eta}^{(\alpha,\beta)}(t_0,x_0)}(s,y)\, dy ds \\ &= \int_{(0,t)\cap\{s:|s-t_0|<(\eta r_0^\alpha)\}} \int_{|y-x_0|<r_0^\beta} G_{t-s}(x-y)\, dy ds \\ &\ge \int_{(0,t)\cap\{s:|s-t_0|<(\eta r_0^\alpha)\}\cap\{s:t-s>(\eta^\alpha-1)r_0^\alpha/2\}} \int_{|y-x_0|<r_0^\beta} G_{t-s}(x-y)\, dy ds. \end{aligned}$$

Because $t-s > (\frac{\eta^\alpha-1}{2}) r_0^\alpha$ and $|y-x_0| < r_0^\beta$, then $|y-x_0| < (\frac{\eta^\alpha-1}{2})^{-\beta/\alpha}(t-s)^{\beta/\alpha}$ and

$$G_{t-s}(x-y) \gtrsim \frac{(t-s)^\beta}{(|x-y|+|t-s|^{\beta/\alpha})^{n+\alpha}} \gtrsim (\frac{\eta^\alpha-1}{2})^{\beta(n/\alpha+1)}(t-s)^{-n\beta/\alpha}.$$

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If $|t-t_0|<r_0^\alpha$,

$$G_{\alpha,\beta}1_{B^{(\alpha,\beta)}_{r_0,\eta}(t_0,x_0)}(t,x)\geq\int_{(0,t)\cap\{s:|s-t_0|<(\eta r_0^\alpha)\}\cap\{s:t-s>(\eta^\alpha-1)r_0^\alpha/2\}}\int_{|y-x_0|<r_0^\beta}|t-s|^{-n\beta/\alpha}dyds$$
$$=\int_{(0,t)\cap\{s:|s-t_0|<(\eta r_0^\alpha)\}\cap\{s:t-s>(\eta^\alpha-1)r_0^\alpha/2\}}r_0^{\beta n}|t-s|^{-n\beta/\alpha}ds.$$

It is easy to see that if $\eta>>1$ and

$$\begin{cases}0<s<t,\\ t_0-(\eta r_0)^\alpha<s<t_0+(\eta r_0)^\alpha,\\ s>t+\frac{\eta^\alpha-1}{2}r_0^\alpha \text{ or } s<\frac{\eta^\alpha-1}{2}r_0^\alpha-t,\end{cases}$$

then

$$\begin{cases}0<s<t,\\ t_0-(\eta r_0)^\alpha<s<t_0+(\eta r_0)^\alpha,\\ s<\frac{\eta^\alpha-1}{2}r_0^\alpha-t.\end{cases}$$

On the other hand, it can be deduced from $r_0^\alpha<t-t_0<2r_0^\alpha$ and $t_0\lesssim r_0^\alpha$ that $t\simeq t_0+r_0^\alpha\leq 2r_0^\alpha$. If $t_0\lesssim r_0^\alpha$, then $t_0-(\eta r_0)^\alpha<0$, which implies that $0<s<t$ and $s<\frac{\eta^\alpha-1}{2}r_0^\alpha-t$. For η large enough, we can see that $\frac{\eta^\alpha-1}{2}r_0^\alpha-t\geq\frac{\eta^\alpha-1}{2}r_0^\alpha-2r_0^\alpha\geq r_0^\alpha\geq t$. Hence, we obtain

$$G_{\alpha,\beta}\left(1_{B^{(\alpha,\beta)}_{r_0,\eta}(t_0,x_0)}\right)(t,x)\gtrsim\int_0^t r_0^{\beta n}|t-s|^{-n\beta/\alpha}ds\gtrsim r_0^{\beta n}(r_0^\alpha+t_0)^{(1-\beta n/\alpha)}\gtrsim r_0^\alpha.$$

Equivalently, there exists a constant c independent of r_0 such that $G_{\alpha,\beta}\left(cr_0^{-\alpha}1_{B^{(\alpha,\beta)}_{r_0,\eta}(t_0,x_0)}\right)(t,x)\geq 1$ for $(t,x)\in B^{(\alpha,\beta)}_{r_0,\eta}(t_0,x_0)$. Therefore,

$$\begin{aligned}C^{(\alpha,\beta)}_{p,q}(B^{(\alpha,\beta)}_{r_0}(t_0,x_0))&\leq\left\|cr_0^{-\alpha}1_{B^{(\alpha,\beta)}_{r_0,\eta}(t_0,x_0)}\right\|^{p\wedge q}_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\\&\lesssim r_0^{-\alpha(p\wedge q)}\left\{\int_{|t-t_0|<(\eta r_0)^\alpha}\left(\int_{|x-x_0|<r_0^\beta}1dx\right)^{q/p}dt\right\}^{(p\wedge q)/q}\\&\lesssim r_0^{-\alpha(p\wedge q)}r_0^{\beta n(p\wedge q)/p}(\eta r_0)^{\alpha(p\wedge q)/q}\\&\lesssim r_0^{(p\wedge q)(\beta n/p+\alpha/q-\alpha)}.\end{aligned}$$

■

3.4 A strong-type inequality related to $L^q_tL^p_x$-capacity

Theorem 3.8 *Given $1\leq p,q<\infty$, $\alpha>n$, and $\beta\in(0,1)$, for any $g\in L^q_tL^p_x(\mathbb{R}^{1+n}_+)$, we have*

$$\int_0^\infty\lambda^{p\wedge q}C^{(\alpha,\beta)}_{p,q}(E_\lambda)\frac{d\lambda}{\lambda}\lesssim\|g\|^{p\wedge q}_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)},$$

where $E_\lambda:=\left\{(t,x)\in\mathbb{R}^{1+n}_+:\ G_{\alpha,\beta}(g)(t,x)>\lambda\right\}$.

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Proof Without loss of generality, we assume that $\|g\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)} < \infty$. For fixed $g \in L^q_tL^p_x(\mathbb{R}^{1+n}_+)$ and any $K = K_t \times K_x \subset \mathbb{R}^{1+n}_+$, define

$$\Phi_g(K) := \frac{\|g\,|_K\,\|^{p\wedge q}_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}}{\|g\|^{p\wedge q}_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}},$$

where

$$\begin{aligned}\|g\,|_K\,\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)} &:= \left\{\int_{K_t}\left(\int_{K_x}|g(t,x)|^p dx\right)^{q/p} dt\right\}^{1/q}\\ &= \left\{\int_0^\infty\left(\int_{\mathbb{R}^n}|g(t,x)|^p\chi_{K_x}(x)dx\right)^{q/p}\chi_{K_t}(t)dt\right\}^{1/q}.\end{aligned}$$

For any disjoint sets A and B, it follows from the identity $\chi_{A\cup B}(t,x) = \chi_{A_t}(t)\chi_{A_x}(x) + \chi_{B_t}(t)\chi_{B_x}(x)$ that

$$\begin{aligned}&\|g\,|_{A\cup B}\,\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\\ &\le \left\{\int_{A_t}\left(\int_{A_x}|g(t,x)|^p dx\right)^{q/p} dt\right\}^{1/q} + \left\{\int_{B_t}\left(\int_{B_x}|g(t,x)|^p dx\right)^{q/p} dt\right\}^{1/q}\\ &\le \|g\,|_A\,\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)} + \|g\,|_B\,\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)},\end{aligned}$$

which means $\Phi_F(A\cup B) \le \Phi_F(A) + \Phi_F(B)$. Below, we prove the reverse inequality in two cases.

Case 1: $p < q$. For this case, $q/p > 1$. Then we obtain

$$\begin{aligned}&\|g\,|_{A\cup B}\,\|^{p\wedge q}_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}\\ &= \left\{\int_{(A\cup B)_t}\left(\int_{(A\cup B)_x}|g(t,x)|^p dx\right)^{q/p} dt\right\}^{p/q}\\ &= \left\{\int_0^\infty\left(\int_{\mathbb{R}^n}|g(t,x)|^p\chi_{A\cup B}(t,x)dx\right)^{q/p} dt\right\}^{p/q}\\ &= \left\{\int_0^\infty\left(\int_{\mathbb{R}^n}|g(t,x)|^p\Big(\chi_A(t,x)+\chi_B(t,x)\Big)dx\right)^{q/p} dt\right\}^{p/q}\\ &= \left\{\int_0^\infty\left(\int_{\mathbb{R}^n}|g(t,x)|^p\chi_A(x,t)dx + \int_{\mathbb{R}^n}|g(t,x)|^p\chi_B(t,x)dx\right)^{q/p} dt\right\}^{p/q},\end{aligned}$$

which, together with the inequality: $(a+b)^{q/p} \ge a^{q/p} + b^{q/p}, a, b > 0,$ implies that

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$$
\begin{aligned}
&\| g |_{A\cup B} \|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} \\
&\gtrsim \left\{ \int_0^\infty \left(\int_{\mathbb{R}^n} |g(t,x)|^p \chi_A(t,x)dx \right)^{q/p} dt + \int_0^\infty \left(\int_{\mathbb{R}^n} |g(t,x)|^p \chi_B(t,x)dx \right)^{q/p} dt \right\}^{p/q} \\
&\gtrsim \left\{ \int_0^\infty \left(\int_{\mathbb{R}^n} |g(t,x)|^p \chi_A(t,x)dx \right)^{q/p} dt \right\}^{p/q} + \left\{ \int_0^\infty \left(\int_{\mathbb{R}^n} |g(t,x)|^p \chi_B(t,x)dx \right)^{q/p} dt \right\}^{p/q} \\
&= \| g |_A \|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} + \| g |_B \|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)}.
\end{aligned}
$$

Case 2: $p \geq q$. For this case, $p \wedge q = q$. Then

$$
\begin{aligned}
\| g |_{A\cup B} \|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} &= \int_{(A\cup B)_t} \left(\int_{(A\cup B)_x} |g(t,x)|^p dx \right)^{q/p} dt \\
&= \int_0^\infty \left(\int_{\mathbb{R}^n} |g(t,x)|^p \Big(\chi_A(t,x) + \chi_B(t,x) \Big) dx \right)^{q/p} dt \\
&\gtrsim \int_0^\infty \left(\int_{\mathbb{R}^n} |g(t,x)|^p \chi_A(t,x)dx \right)^{q/p} dt \\
&\quad + \int_0^\infty \left(\int_{\mathbb{R}^n} |g(t,x)|^p \chi_B(t,x)dx \right)^{q/p} dt \\
&= \| g |_A \|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} + \| F |_A \|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)}.
\end{aligned}
$$

Finally, we prove that

$$
\Phi_g(A \cup B) \simeq \Phi_g(A) + \Phi_g(B).
$$

By [14, Corollary 2.3, p. 187], there exists a measure ψ on $\mathbb{R}^{1+n}_+$ such that $\Phi \leq \psi$ and $\psi(\mathbb{R}^{1+n}_+) \leq c(n)$, where $c(n)$ is a constant only depending on n. For $E_\lambda \backslash E_{a\lambda}$ with $a \in (1,2)$, noting that $E_\infty = \varnothing$, we have

$$
\begin{aligned}
\int_0^\infty \Phi_F(E_\lambda \backslash E_{a\lambda}) \frac{d\lambda}{\lambda} &\leq \int_0^\infty \psi(E_\lambda \backslash E_{a\lambda}) \frac{d\lambda}{\lambda} \\
&= \int_0^\infty \left(\int_\lambda^{a\lambda} d\psi(E_s) \right) \frac{d\lambda}{\lambda} \\
&= \int_0^\infty \left(\int_{s/a}^s \frac{d\lambda}{\lambda} \right) d\psi(E_s) \\
&\leq \psi(\mathbb{R}^{1+n}_+) \log a \\
&\leq c(n) \log a.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_0^\infty \| g |_{E_\lambda \backslash E_{a\lambda}} \|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} \frac{d\lambda}{\lambda} &= \int_0^\infty \| g \|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} \Phi_F(E_\lambda \backslash E_{a\lambda}) \frac{d\lambda}{\lambda} \\
&\leq c(n) \log a \| g \|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)}.
\end{aligned}
$$

By (1.3), $u(t,x) := G_{\alpha,\beta}(g)(t,x)$ is a solution to the space–time fractional dissipative equation:

$$\begin{cases} \partial_t^\beta u(t,x) + (-\Delta)^{\alpha/2} u(t,x) = f(t,x), & \forall\ (t,x) \in \mathbb{R}_+^{1+n}, \\ u(0,x) = 0, & \forall\ x \in \mathbb{R}^n. \end{cases}$$

Set a function $\widetilde{g}$ as

$$\widetilde{g}(t,x) := \begin{cases} 0, & (t,x) \in E_{a\lambda}, \\ \dfrac{g(t,x)}{(a-1)\lambda}, & (t,x) \in E_\lambda \backslash E_{a\lambda}, \\ 0, & (t,x) \in \mathbb{R}_+^{1+n} \backslash E_\lambda. \end{cases}$$

Define $\widetilde{f}(t,x) = I_t^{1-\beta} \widetilde{g}(t,x)$. Then a direct computation indicates that the following function

$$\widetilde{u}(t,x) := \begin{cases} 1, & (t,x) \in E_{a\lambda}, \\ \dfrac{u(t,x)-\lambda}{(a-1)\lambda}, & (t,x) \in E_\lambda \backslash E_{a\lambda}, \\ 0, & (t,x) \in \mathbb{R}_+^{1+n} \backslash E_\lambda \end{cases}$$

is a solution to the equation

$$\begin{cases} \partial_t^\beta \widetilde{u}(t,x) + (-\Delta)^{\alpha/2} \widetilde{u}(t,x) = \widetilde{f}(t,x), & \forall\ (t,x) \in \mathbb{R}_+^{1+n}, \\ \widetilde{u}(0,x) = 0, & \forall\ x \in \mathbb{R}^n. \end{cases}$$

Based on the definition of $C_{p,q}^{(\alpha,\beta)}(\cdot)$, we obtain

$$\begin{aligned} \int_0^\infty \lambda^{p\wedge q} C_{p,q}^{(\alpha,\beta)}(E_{a\lambda}) \frac{d\lambda}{\lambda} &\le \int_0^\infty \lambda^{p\wedge q} \|\widetilde{g}\|_{L_t^q L_x^p(E_\lambda)}^{p\wedge q} \frac{d\lambda}{\lambda} \\ &\le \int_0^\infty \frac{\|g\,|_{E_\lambda \backslash E_{a\lambda}}\|_{L_t^q L_x^p(E_\lambda)}^{p\wedge q}}{(a-1)^{p\wedge q}} \frac{d\lambda}{\lambda} \\ &\le \frac{c(n)\log a}{(a-1)^{p\wedge q}} \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p\wedge q}, \end{aligned}$$

which, via a change of variables, implies that

$$\int_0^\infty \lambda^{p\wedge q} C_{p,q}^{(\alpha,\beta)}(E_\lambda) \frac{d\lambda}{\lambda} \lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p\wedge q}. \qquad \blacksquare$$

The following weak-type estimate is an immediate corollary of Theorem 3.8.

Corollary 3.9 *Let $1 \le p, q < \infty$, $\alpha > n$, and $\beta \in (0,1)$. For all $\lambda > 0$, it holds*

$$\lambda^{p\wedge q} C_{p,q}^{(\alpha,\beta)}(E_\lambda) \lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p\wedge q}.$$

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As an application of Theorem 3.8, we deduce the embedding from $L_t^q L_x^p(\mathbb{R}_+^{1+n})$ to $L^{(r,s)}(\mathbb{R}_+^{1+n},\mu)$, which is the Lorentz space of all functions $u(\cdot,\cdot)$ satisfying

$$\|u\|_{L^{(r,s)}(\mathbb{R}_+^{1+n},\mu)} := \left(\int_0^\infty \mu\left(\left\{(t,x)\in\mathbb{R}_+^{1+n} : |u(t,x)|>\lambda\right\}\right)^{s/r} d\lambda^s\right)^{1/s} < \infty,$$

where $r,s\in(0,\infty)$ and μ is a nonnegative Borel measure on $\mathbb{R}_+^{1+n}$.

Theorem 3.10 *Assume that $1\le p,q<\infty$, $\alpha>n$, and $\beta\in(0,1)$. Let μ be a nonnegative Borel measure on $\mathbb{R}_+^{1+n}$. Then*

$$\|G_{\alpha,\beta}(g)\|_{L^{(p\vee q,p\wedge q)}(\mathbb{R}_+^{1+n},\mu)} \lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \tag{3.3}$$

holds for all $g\in L_t^q L_x^p(\mathbb{R}_+^{1+n})$ if and only if

$$(\mu(K))^{p\wedge q} \lesssim \left(C_{p,q}^{(\alpha,\beta)}(K)\right)^{p\vee q} \tag{3.4}$$

holds for all compact sets $K\subset\mathbb{R}_+^{1+n}$.

Proof Assume that (3.4) holds. Then it follows from Theorem 3.8 that

$$\begin{aligned}
&\|G_{\alpha,\beta}(g)\|_{L^{(p\vee q,p\wedge q)}(\mathbb{R}_+^{1+n},\mu)}\\
&\lesssim \left(\int_0^\infty \lambda^{p\wedge q-1} C_{p,q}^{(\alpha,\beta)}\left(\left\{(t,x)\in\mathbb{R}_+^{1+n} : |G_{\alpha,\beta}(g)(t,x)|>\lambda\right\}\right)d\lambda\right)^{1/(p\wedge q)}\\
&\lesssim \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}.
\end{aligned}$$

Conversely, suppose that (3.3) holds for all $g\in L_t^q L_x^p(\mathbb{R}_+^{1+n})$. Fix a compact set $K\subset\mathbb{R}_+^{1+n}$. By the definition of $C_{p,q}^{(\alpha,\beta)}(\cdot)$, for any $\varepsilon>0$, there exists a function $g\in L_t^q L_x^p(\mathbb{R}_+^{1+n})$ such that

$$\begin{cases} G_{\alpha,\beta}(g)\ge 1_K,\\ \|g\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p\wedge q}+\varepsilon < C_{p,q}^{(\alpha,\beta)}(K).\end{cases}$$

Hence, $K\subset\left\{(t,x)\in\mathbb{R}_+^{1+n} : |G_{\alpha,\beta}(g)(t,x)|>1\right\}$. We get

$$\begin{aligned}
&\left(\int_0^\infty \mu\left(\left\{(t,x)\in\mathbb{R}_+^{1+n} : |G_{\alpha,\beta}(g)(t,x)|>\lambda\right\}\right)^{(p\wedge q)/(p\vee q)} d\lambda^{p\wedge q}\right)^{1/(p\wedge q)}\\
&\ge \left(\int_0^1 \mu\left(\left\{(t,x)\in\mathbb{R}_+^{1+n} : |G_{\alpha,\beta}(g)(t,x)|>\lambda\right\}\right)^{(p\wedge q)/(p\vee q)} d\lambda^{p\wedge q}\right)^{1/(p\wedge q)}\\
&\ge \left(\int_0^1 \mu\left(\left\{(t,x)\in\mathbb{R}_+^{1+n} : |G_{\alpha,\beta}(g)(t,x)|>1\right\}\right)^{(p\wedge q)/(p\vee q)} d\lambda^{p\wedge q}\right)^{1/(p\wedge q)}\\
&\ge (\mu(K))^{1/(p\vee q)},
\end{aligned}$$

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which implies that

$$(\mu(K))^{(p\wedge q)/(p\vee q)} \leq \|G_{\alpha,\beta}(g)\|^{p\wedge q}_{L^{(p\vee q,p\wedge q)}(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|g\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} \lesssim C^{(\alpha,\beta)}_{p,q}(K) - \varepsilon.$$

Then (3.4) follows from the arbitrariness of ε. This completes the proof of Theorem 3.10. ■

4 Hausdorff dimension of the blow-up set

In this section, we estimate the size of the blow-up set of the so-called fractional time–space dissipative potential $G_{\alpha,\beta}$: for $0 \leq g \in L^q_t L^p_x(\mathbb{R}^{1+n}_+)$,

$$\mathcal{B}[G_{\alpha,\beta}(g), p, q] := \Big\{(t,x) \in \mathbb{R}^{1+n}_+ : \ G_{\alpha,\beta}(g)(t,x) = \infty\Big\}.$$

Let $\phi : [0,\infty) \mapsto [0,\infty]$ be an increasing function with $\phi(0) = 0$. For any compact subset $K \subset \mathbb{R}^{1+n}_+$, let

$$H^{\phi,\alpha,\beta}_{\varepsilon}(K) := \inf\left\{\sum_{j=1}^{\infty} \phi(r_j) : \ K \subseteq \cup_{j=1}^{\infty} B^{(\alpha,\beta)}_{r_j}(t_j, x_j) \text{ with } r_j \in (0,\varepsilon)\right\}$$

be the $L^{\alpha,\beta}$-based (ϕ,ε)-Hausdorff capacity of K. Specially, for $\varepsilon = \infty$,

$$H^{\phi,\alpha,\beta}_{\infty}(K) := \inf\left\{\sum_{j=1}^{\infty} \phi(r_j) : \ K \subseteq \cup_{j=1}^{\infty} B^{(\alpha,\beta)}_{r_j}(t_j, x_j) \text{ with } r_j > 0\right\}.$$

Then the (ϕ,ε)-Hausdorff measure of K is defined by

$$H^{\phi,\alpha,\beta}(K) := \lim_{\varepsilon\to 0} H^{\phi,\alpha,\beta}_{\varepsilon}(K).$$

If $\phi(r) = r^d$ for all $r \in (0,\infty)$, then

$$\begin{cases} H^{\phi,\alpha,\beta}_{\varepsilon}(K) := H^{d,\alpha,\beta}_{\varepsilon}(K), \\ H^{\phi,\alpha,\beta}_{\infty}(K) := H^{d,\alpha,\beta}_{\infty}(K), \\ H^{\phi,\alpha,\beta}(K) := H^{d,\alpha,\beta}(K). \end{cases}$$

The Hausdorff dimension of K is defined as

$$\dim^{(\alpha,\beta)}_H(K) := \inf\Big\{d : \ H^{d,\alpha,\beta}(K) = 0\Big\}.$$

Theorem 4.1 *$\mathscr{L}^1(A)$ and $\mathscr{L}^n(B)$ stand for the one-dimensional and n-dimensional Lebesgue measures of bounded Borel sets $A \subset \mathbb{R}_+$ and $B \subset \mathbb{R}^n$, respectively. If $\alpha > n$, $\beta \in (0,1)$,*

$$\begin{cases} 1 \leq p < \widetilde{p} < \infty, \\ 1 < q < \widetilde{q} < \infty, \\ (1/q - 1/\widetilde{q}) + n\beta(1/p - 1/\widetilde{p})/\alpha = 1, \end{cases}$$

then there exists a $\delta \in (0,1)$ such that

$$(\mathscr{L}^1(A))^{(p\wedge q)/\widetilde{q}}(\mathscr{L}^n(B))^{(p\wedge q)/\widetilde{p}} \lesssim C^{(\alpha,\beta)}_{p,q}(A\times B) \lesssim H^{(p\wedge q)(n\beta/p+\alpha/q-\alpha),\alpha,\beta}_{\delta}(A\times B).$$

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Proof For any $0 \le F$ such that $G_{\alpha,\beta}(g) \ge 1_{A\times B}$, we can get

$$\|G_{\alpha,\beta}(g)\|_{L^{\tilde{q}}((0,\infty)),L^{\tilde{p}}} = \left\{ \int_0^\infty \left(\int_{\mathbb{R}^n} |G_{\alpha,\beta}(g)(t,x)|^{\tilde{p}} dx \right)^{\tilde{q}/\tilde{p}} dt \right\}^{1/\tilde{q}} \gtrsim (\mathscr{L}^1(A))^{1/\tilde{q}} (\mathscr{L}^n(B))^{1/\tilde{p}}.$$

On the other hand, it follows from Theorem 2.6 that

$$\|G_{\alpha,\beta}(g)\|_{L^{\tilde{q}}((0,\infty)),L^{\tilde{p}}} \lesssim \|g\|_{L^q((0,\infty),L^p)},$$

which implies

$$(\mathscr{L}^1(A))^{(p\wedge q)/\tilde{q}} (\mathscr{L}^n(B))^{(p\wedge q)/\tilde{p}} \lesssim \|g\|^{p\wedge q}_{L^q((0,\infty),L^p)}. \tag{4.1}$$

Taking the infimum on the right-hand side of (4.1), we obtain

$$(\mathscr{L}^1(A))^{(p\wedge q)/\tilde{q}} (\mathscr{L}^n(B))^{(p\wedge q)/\tilde{p}} \lesssim C^{(\alpha,\beta)}_{p,q}(A \times B).$$

Let $\left\{B^{(\alpha,\beta)}_{r_j}(t_j,x_j)\right\}$ be a covering of $A \times B$ with $r_j \in (0,\delta)$. Then, by Theorem 3.7, we get

$$\begin{aligned} C^{(\alpha,\beta)}_{p,q}(A \times B) &\le C^{(\alpha,\beta)}_{p,q}\left(\bigcup_{j=1}^\infty B^{(\alpha,\beta)}_{r_j}(t_j,x_j) \right) \\ &\le \sum_{j=1}^\infty C^{(\alpha,\beta)}_{p,q}\left(B^{(\alpha,\beta)}_{r_j}(t_j,x_j) \right) \\ &\lesssim \sum_{j=1}^\infty r_j^{(p\wedge q)(n\beta/p+\alpha/q-\alpha)}, \end{aligned}$$

which gives

$$C^{(\alpha,\beta)}_{p,q}(A \times B) \lesssim H_\delta^{(p\wedge q)(n\beta/p+\alpha/q-\alpha),\alpha,\beta}(A \times B).$$

This completes the proof of Theorem 4.1. ∎

Theorem 4.2 *Let K be a compact subset of $\mathbb{R}^{1+n}_+$. If $\alpha > n$, $\beta \in (0,1)$,*

$$\begin{cases} 1 \le p < \infty, \\ 1 < q < \infty, \\ \theta := (p \wedge q)(n\beta/p + \alpha/q - \alpha) > 0, \end{cases}$$

there exists a $\delta \in (0,1)$ such that $C^{(\alpha,\beta)}_{p,q}(K) \lesssim H^{\theta,\alpha,\beta}_\delta(K)$ and hence

$$\dim_H^{(\alpha,\beta)}\left(\mathcal{B}[G_{\alpha,\beta}(g),p,q]\right) \le n\beta - \alpha(p\wedge q - 1).$$

Proof The first assertion follows from Theorem 3.7. Now, we estimate the Hausdorff dimension of the blow-up set. Let $g \ge 0 \in L^q_t L^p_x(\mathbb{R}^{1+n}_+)$. For any $(t,x) \in \mathcal{B}[G_{\alpha,\beta}(g),p,q]$, $G_{\alpha,\beta}(g)(t,x) = \infty$, which means that for $j \in \mathbb{N}$, $G_{\alpha,\beta}(g)(t,x) > 2^j$ on $\mathcal{B}[G_{\alpha,\beta}(g),p,q]$. Set $g_j = 2^{-j}g$, $j \in \mathbb{N}$. Hence, $G_{\alpha,\beta}(g_j) > 1$ on $\mathcal{B}[G_{\alpha,\beta}(g),p,q]$

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and $\|g_j\|_{L^q_t L^p_x} = 2^{-j}\|g\|_{L^q_t L^p_x}$. Then, by the definition of $C^{(\alpha,\beta)}_{p,q}(\cdot)$, it holds

$$C^{(\alpha,\beta)}_{p,q}\left(\mathcal{B}[G_{\alpha,\beta}(g), p, q]\right) = 0.$$

Let K be any compact subset of $\mathcal{B}[G_{\alpha,\beta}(g), p, q]$. By (ii) of Theorem 3.2, we obtain $C^{(\alpha,\beta)}_{p,q}(K) = 0$ since

$$C^{(\alpha,\beta)}_{p,q}(K) \le C^{(\alpha,\beta)}_{p,q}(\mathcal{B}[G_{\alpha,\beta}(g), p, q]).$$

We need the following Frostman-type lemma: if $\phi : [0,\infty) \mapsto [0,\infty]$ increases and $\phi(0) = 0$, then for any given compact $K \subset \mathbb{R}^{1+n}_+$, there exists a measure $\mu \in \mathcal{M}^+(K)$ such that

$$\begin{cases} \mu(B^{(\alpha,\beta)}_r(t,x)) \lesssim \phi(r), \\ \mu(K) \simeq H^{\phi,\alpha,\beta}_\infty(K). \end{cases}$$

Because K is a compact subset, we assume that K is contained in a ball $B^{(\alpha,\beta)}_R(t_0,x_0)$. Let $0 \le G \le L^q_t L^p_x(\mathbb{R}^{1+n}_+)$ such that $G_{\alpha,\beta}(G) \ge 1$ on K. By the Fubini theorem, we get

$$\begin{aligned} \mu(K) &= \iint_K 1 d\mu(t,x) \\ &\le \iint_K G_{\alpha,\beta}(G)(t,x) d\mu(t,x) \\ &= \iint_K \left\{ \int_0^t \int_{\mathbb{R}^n} G_{t-s}(x-y) G(s,y) ds dy \right\} d\mu(t,x) \\ &= \iint_{\mathbb{R}^{1+n}_+} G(s,y) \left\{ \iint_{K\cap((s,\infty)\times\mathbb{R}^n)} G_{t-s}(x-y) d\mu(t,x) \right\} dy ds. \end{aligned}$$

Applying Proposition 2.1, we obtain

$$\mu(K) \lesssim \iint_{\mathbb{R}^{1+n}_+} G(s,y) \left\{ \iint_{K\cap((s,\infty)\times\mathbb{R}^n)} \frac{|t-s|^\beta}{(|t-s|^{\beta/\alpha} + |x-y|)^{n+\alpha}} d\mu(t,x) \right\} dy ds := M_1 + M_2,$$

where

$$\begin{cases} M_1 := \iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)} G(s,y) \left\{ \iint_{K\cap((s,\infty)\times\mathbb{R}^n)} \frac{|t-s|^\beta}{(|t-s|^{\beta/\alpha} + |x-y|)^{n+\alpha}} d\mu(t,x) \right\} dy ds, \\ M_2 := \iint_{\mathbb{R}^{1+n}_+ \setminus B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)} G(s,y) \left\{ \iint_{K\cap((s,\infty)\times\mathbb{R}^n)} \frac{|t-s|^\beta}{(|t-s|^{\beta/\alpha} + |x-y|)^{n+\alpha}} d\mu(t,x) \right\} dy ds. \end{cases}$$

Now, we estimate the term M_2. Write $\mathbb{R}^{1+n}_+ \setminus B^{(\alpha,\beta)}_{2R_0}(t_0,x_0) \subset S_1 \cup S_2 \cup S_3$, where

$$\begin{cases} S_1 := \left\{ (s,y) \in \mathbb{R}^{1+n}_+,\ |y-x_0| > (2R_0)^\beta \ \&\ |s-t_0| > (2R_0)^\alpha \right\}, \\ S_2 := \left\{ (s,y) \in \mathbb{R}^{1+n}_+,\ |y-x_0| > (2R_0)^\beta \ \&\ |s-t_0| \le (2R_0)^\alpha \right\}, \\ S_3 := \left\{ (s,y) \in \mathbb{R}^{1+n}_+,\ |y-x_0| \le (2R_0)^\beta \ \&\ |s-t_0| > (2R_0)^\alpha \right\}. \end{cases}$$

Hence, $M_2 \lesssim M_{2,1} + M_{2,2} + M_{2,3}$, where

$$M_{2,i} := \iint_{S_i} G(s,y) \left\{ \iint_{K\cap((s,\infty)\times\mathbb{R}^n)} \frac{|t-s|^\beta}{(|t-s|^{\beta/\alpha}+|x-y|)^{n+\alpha}} d\mu(t,x) \right\} dyds, \quad i=1,2,3.$$

For $M_{2,1}$, if $(s,y)\in S_1$ and $(t,x)\in K\cap((s,\infty)\times\mathbb{R}^n)\subset K\subset B_R^{(\alpha,\beta)}(t_0,x_0)$, then $|y-x_0|\simeq|y-x|$ and $|t-s|\simeq|s-t_0|$. We have

$$\iint_{K\cap((s,\infty)\times\mathbb{R}^n)} \frac{|t-s|^\beta}{(|t-s|^{\beta/\alpha}+|x-y|)^{n+\alpha}} d\mu(t,x) \lesssim \frac{|t_0-s|^\beta}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{n+\alpha}}\mu(K).$$

On the other hand, a direct computation gives

$$\int_{|y-x_0|>(2R_0)^\beta} \frac{|t_0-s|^{p'\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)p'}} dy \lesssim |s-t_0|^{n\beta(1-p')/\alpha},$$

which, together with the Hölder inequality, implies that

$$\begin{aligned}
M_{2,1} &\lesssim \mu(K) \iint_{S_1} G(s,y) \frac{|t_0-s|^\beta}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{n+\alpha}} dyds \\
&\lesssim \mu(K)\|G\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})} \left\{ \int_{|s-t_0|>(2R_0)^\alpha} \left(\int_{|y-x_0|>(2R_0)^\beta} \frac{|t_0-s|^{p'\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)p'}} dy \right)^{q'/p'} ds \right\}^{1/q'} \\
&\lesssim \mu(K)\|G\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})} \left\{ \int_{|s-t_0|>(2R_0)^\alpha} |s-t_0|^{nq'\beta(1-p')/p'\alpha} ds \right\}^{1/q'} \\
&\lesssim \mu(K)\|G\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})} R_0^{\alpha-\alpha/q-n\beta/p}.
\end{aligned}$$

Similarly, for S_3, we can see that $|t-s|\simeq|s-t_0|$ and $|t_0-s|^{\beta/\alpha}\gtrsim|y-x_0|$. Then

$$\frac{|t-s|^\beta}{(|t-s|^{\beta/\alpha}+|x-y|)^{(n+\alpha)}} \lesssim |t_0-s|^{\beta-\beta(n+\alpha)/\alpha} \lesssim \frac{|t_0-s|^\beta}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{n+\alpha}}.$$

We can follow the procedure of $M_{2,1}$ to get

$$\begin{aligned}
M_{2,3} &\lesssim \iint_{S_3} \mu(K)G(s,y) \frac{|t_0-s|^\beta}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{n+\alpha}} dyds \\
&\lesssim \mu(K)\|G\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})} \left\{ \int_{|s-t_0|>(2R_0)^\alpha} \left(\int_{\mathbb{R}^n} \frac{|t_0-s|^{p'\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)p'}} dy \right)^{q'/p'} ds \right\}^{1/q'} \\
&\lesssim \mu(K)\|G\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})} R_0^{\alpha-\alpha/q-n\beta/p}.
\end{aligned}$$

Then we estimate $M_{2,2}$. For $(s,y)\in S_2$, $|y-x_0|>|s-t_0|^{\beta/\alpha}$, $|y-x_0|\simeq|x-y|$, and $|s-t|\lesssim R_0^\alpha$. Then

$$\frac{|t-s|^{\beta}}{(|t-s|^{\beta/\alpha}+|x-y|)^{(n+\alpha)}} \lesssim \frac{|t-s|^{\beta}}{(|t-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)}} \lesssim \frac{R_0^{\alpha\beta}}{|y-x_0|^{n+\alpha}} \lesssim \frac{R_0^{\alpha\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)}}.$$

We can get

$$M_{2,2} \lesssim \mu(K)\iint_{S_2} G(s,y)\frac{R_0^{\alpha\beta}}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{n+\alpha}}dyds$$
$$\lesssim \mu(K)R_0^{\alpha\beta}\|G\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})}\left\{\int_{|s-t_0|<(2R_0)^\alpha}\left(\int_{|y-x_0|>(2R_0)^\beta}\frac{dy}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)p'}}\right)^{q'/p'}ds\right\}^{1/q'}.$$

The change of variables gives

$$\int_{|y-x_0|>(2R_0)^\beta}\frac{1}{(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{(n+\alpha)p'}}dy \lesssim (|t_0-s|^{\beta/\alpha}+|x_0-y|)^{-(n+\alpha)p'+n},$$

which, together with the change of variable: $u=|t_0-s|/R_0^\alpha$, indicates that

$$M_{2,2} \lesssim \mu(K)R_0^{\alpha\beta}\|G\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})}\left(\int_{|s-t_0|<(2R_0)^\alpha}(|t_0-s|^{\beta/\alpha}+|x_0-y|)^{-q'(n+\alpha)+nq'/p'}ds\right)^{1/q'}$$
$$\lesssim \mu(K)R_0^{\alpha\beta}R_0^{-(n+\alpha)\beta+n\beta/p'+\alpha/q'}\|G\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})}\left(\int_0^1(1+u)^{-(n+\alpha)q'+nq'/p}du\right)^{1/q'}$$
$$\lesssim \mu(K)\|G\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})}R_0^{\alpha-\alpha/q-n\beta/p}.$$

The estimates for $M_{2,i}$, $i=1,2,3$, imply that $M_2 \lesssim \mu(K)\|G\|_{L_t^qL_x^p(\mathbb{R}_+^{1+n})}R_0^{\alpha-\alpha/q-n\beta/p}$.

Now, we estimate M_1. If $(s,y)\in B_{2R_0}^{(\alpha,\beta)}(t_0,x_0)$ and $(t,x)\in B_{2R_0}^{(\alpha,\beta)}(t_0,x_0)$, then $|t-s|<(4R_0)^\alpha$ and $|y-x|\le(4R_0)^\beta$, which indicate that $(t,x)\in B_{4R_0}^{(\alpha,\beta)}(s,y)$. We can get

$$\iint_{K\cap((s,\infty)\times\mathbb{R}^n)}\frac{|t-s|^\beta d\mu(t,x)}{(|t-s|^{\beta/\alpha}+|x-y|)^{n+\alpha}}$$
$$\lesssim \iint_{B_{2R_0}^{(\alpha,\beta)}(t_0,x_0)}\frac{|t-s|^\beta d\mu(t,x)}{(|t-s|^{\beta/\alpha}+|x-y|)^{n+\alpha}}$$
$$\lesssim \iint_{B_{4R_0}^{(\alpha,\beta)}(s,y)}\frac{|t-s|^\beta d\mu(t,x)}{(|t-s|^{\beta/\alpha}+|x-y|)^{n+\alpha}}$$
$$\lesssim \sum_{j=2}^{\infty}\iint_{B_{2^{-j}R_0}^{(\alpha,\beta)}(s,y)\backslash B_{2^{-j-1}R_0}^{(\alpha,\beta)}(s,y)}\frac{|t-s|^\beta d\mu(t,x)}{(|t-s|^{\beta/\alpha}+|x-y|)^{n+\alpha}}.$$

Let $(t,x)\in B_{2^{-j}R_0}^{(\alpha,\beta)}(s,y)\backslash B_{2^{-j-1}R_0}^{(\alpha,\beta)}(s,y)$. If $|t-s|\simeq(2^{-j_0}R_0)^\alpha$ and $|x-x_0|<(2^{-j_0}R_0)^\beta$, then

$$\frac{|t-s|^\beta}{(|t-s|^{\beta/\alpha}+|x-y|)^{n+\alpha}} \lesssim \frac{1}{|t-s|^{\beta n/\alpha}} \simeq \frac{1}{(2^{-j_0}R_0)^{\beta n}}.$$

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If $|t-s|<(2^{-j_0}R_0)^\alpha$ and $|x-x_0|\simeq(2^{-j_0}R_0)^\beta$, then

$$\frac{|t-s|^\beta}{(|t-s|^{\beta/\alpha}+|x-y|)^{n+\alpha}}\lesssim\frac{1}{|x-y|^n}\simeq\frac{1}{(2^{-j_0}R_0)^{\beta n}}.$$

We have

$$\begin{aligned}
M_1&\lesssim\iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)}G(s,y)\left\{\sum_{j=0}^{\infty}\frac{\mu(B^{(\alpha,\beta)}_{2^{-j}R_0}(s,y))}{(2^{-j_0}R_0)^{\beta n}}\right\}dyds\\
&\lesssim\iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)}G(s,y)\left\{\sum_{j=0}^{\infty}\int_{2^{-j_0}R_0}^{2^{-j_0+1}R_0}\mu(B^{(\alpha,\beta)}_{r}(s,y))\frac{dr}{r^{\beta n+1}}\right\}dyds\\
&\lesssim\iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)}G(s,y)\left\{\int_0^{2R_0}\mu(B^{(\alpha,\beta)}_{r}(s,y))\frac{dr}{r^{\beta n+1}}\right\}dyds\\
&\lesssim\int_0^{2R_0}\left\{\iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)}G(s,y)\mu(B^{(\alpha,\beta)}_{r}(s,y))dyds\right\}\frac{dr}{r^{\beta n+1}}.
\end{aligned}$$

Case 1: $p\le q$. Applying Hölder's inequality, we get

$$\begin{aligned}
(4.2)\quad &\|G\|_{L^p_tL^p_x(B^{(\alpha,\beta)}_{2R_0}(t_0,x_0))}\\
&\le\left\{\int_{|s-t_0|<(2R_0)^\alpha}\left(\int_{|y-x_0|<(2R_0)^\beta}|G(s,y)|^pdy\right)ds\right\}^{1/p}\\
&\lesssim\left\{\int_{|s-t_0|<(2R_0)^\alpha}\left(\int_{|y-x_0|<(2R_0)^\beta}|G(s,y)|^pdy\right)^{q/p}ds\right\}^{1/q}\left(\int_{|s-t_0|<(2R_0)^\alpha}1ds\right)^{1/p-1/q}\\
&\lesssim\|G\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}R_0^{\alpha(1/p-1/q)}.
\end{aligned}$$

On the other hand, it follows from the condition $\mu(B^{(\alpha,\beta)}_r(s,y))\lesssim\phi(r)$ that

$$(4.3)\qquad\left(\mu(B^{(\alpha,\beta)}_r(s,y))\right)^{p/(p-1)}\lesssim(\phi(r))^{1/(p-1)}\mu(B^{(\alpha,\beta)}_r(s,y)).$$

The estimates (4.2) and (4.3) imply that

$$\begin{aligned}
M_1&\lesssim\int_0^{2R_0}\|G\|_{L^p_tL^p_x(B^{(\alpha,\beta)}_{2R_0}(t_0,x_0))}\left\{\iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)}\left(\mu(B^{(\alpha,\beta)}_r(s,y))\right)^{p/(p-1)}dyds\right\}^{1-1/p}\frac{dr}{r^{\beta n+1}}\\
&\lesssim\|G\|_{L^q_tL^p_x(\mathbb{R}^{1+n}_+)}R_0^{\alpha(1/p-1/q)}\int_0^{2R_0}\left\{\iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)}\mu(B^{(\alpha,\beta)}_r(s,y))dyds\right\}^{1-1/p}\frac{(\phi(r))^{1/p}dr}{r^{\beta n+1}}.
\end{aligned}$$

Since $\mu\in\mathcal{M}^+(K)$, $\mu(B^{(\alpha,\beta)}_r(s,y))\le\mu(K)$. For any $0<r<2R_0$ and $(t,x)\in B^{(\alpha,\beta)}_r(s,y)$, it holds

$$\begin{aligned}
\iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)}\mu(B^{(\alpha,\beta)}_r(s,y))dyds&=\iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)}\left(\iint_{B^{(\alpha,\beta)}_r(s,y)}1d\mu(t,x)\right)dyds\\
&\lesssim\iint_{\mathbb{R}^{1+n}_+}\left(\iint_{B^{(\alpha,\beta)}_r}1dyds\right)d\mu(t,x)\lesssim r^{\alpha+\beta n}\mu(K).
\end{aligned}$$

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Hence,

$$M_1 \lesssim \|G\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} R_0^{\alpha(1/p-1/q)} \int_0^{2R_0} \left(r^{\alpha+\beta n}\mu(K)\right)^{1-1/p} \frac{(\phi(r))^{1/p} dr}{r^{\beta n+1}}$$
$$\lesssim \|G\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} R_0^{\alpha(1/p-1/q)} (\mu(K))^{1-1/p} \int_0^{2R_0} (\phi(r))^{1/p}\, r^{\alpha(1-1/p)-\beta n/p-1} dr.$$

Case 2: $p > q$. Similar to Case 1, we have

$$M_1 \lesssim \int_0^{2R_0} \left\{ \iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)} G(s,y)\mu(B^{(\alpha,\beta)}_r(s,y))dyds \right\} \frac{dr}{r^{1+\beta n}}$$
$$\lesssim \|G\|_{L^q_t L^q_x(B^{(\alpha,\beta)}_{2R_0}(t_0,x_0))} \int_0^{2R_0} \left(\iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)} \left(\mu(B^{(\alpha,\beta)}_r(s,y))\right)^{q/(q-1)} dyds \right)^{1-1/q} \frac{dr}{r^{\beta n+1}}$$
$$\lesssim R_0^{n(1/q-1/p)} \|G\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} \int_0^{2R_0} \left(\iint_{B^{(\alpha,\beta)}_{2R_0}(t_0,x_0)} \mu(B^{(\alpha,\beta)}_r(s,y))dyds \right)^{1-1/q} \frac{(\phi(r))^{1/q} dr}{r^{\beta n+1}}$$
$$\lesssim R_0^{n(1/q-1/p)} \|G\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} (\mu(K))^{1-1/q} \int_0^{2R_0} (\phi(r))^{1/q} r^{\alpha(1-1/q)\beta n/q-1} dr.$$

Hence, there exists a constant c_0 depending on p, q, α, β, R_0 such that

$$\mu(K) \le c_0 \|G\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} \left\{ \mu(K) + (\mu(K))^{1-1/(p\wedge q)} \int_0^{2R_0} (\phi(r))^{1/(p\wedge q)} r^{-1+\alpha-(\alpha+\beta n)/(p\wedge q)} dr \right\}.$$

Take $\phi(r) := r^\eta$, $\eta > n\beta - \alpha(p \wedge q - 1)$, such that

$$\int_0^{2R_0} (\phi(r))^{1/(p\wedge q)} r^{-1+\alpha-(\alpha+\beta n)/(p\wedge q)} dr < \infty.$$

This gives

$$\frac{(\mu(K))^{1/(p\wedge q)}}{1 + (\mu(K))^{1/(p\wedge q)}} \le c_0 \|G\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)},$$

and therefore

$$\frac{(\mu(K))^{1/(p\wedge q)}}{1 + (\mu(K))^{1/(p\wedge q)}} \le c_0 C^{(\alpha,\beta)}_{p,q}(K) = 0,$$

which implies that $\mu(K) = 0$ and $H^{\phi,\alpha,\beta}_\infty(K) = 0$. Hence, $H^{\phi,\alpha,\beta}(K) = 0$. By the arbitrariness of $K \subset \mathcal{B}(G_{\alpha,\beta}(g), p, q)$,

$$H^{\phi,\alpha,\beta}(\mathcal{B}(G_{\alpha,\beta}(g), p, q)) = 0,$$

which indicates that

$$\dim_H^{(\alpha,\beta)} \left(\mathcal{B}[G_{\alpha,\beta}(g), p, q]\right) \le n\beta - \alpha(p \wedge q - 1). \qquad \blacksquare$$

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