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Decay of Mean Values of Multiplicative Functions

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Abstract. For given multiplicative function f, with $|f(n)| \le 1$ for all n, we are interested in how fast its mean value $(1/x)\sum_{n\le x} f(n)$ converges. Halász showed that this depends on the minimum M (over $y \in \mathbb{R}$) of $\sum_{p\le x}\big(1 - \mathrm{Re}(f(p)p^{-iy})\big)/p$, and subsequent authors gave the upper bound $\ll (1+M)e^{-M}$. For many applications it is necessary to have explicit constants in this and various related bounds, and we provide these via our own variant of the Halász-Montgomery lemma (in fact the constant we give is best possible up to a factor of 10). We also develop a new type of hybrid bound in terms of the location of the absolute value of y that minimizes the sum above. As one application we give bounds for the least representatives of the cosets of the k-th powers mod p.

1 Introduction

Given a multiplicative function f with $|f(n)| \le 1$ for all n, define

$$\Theta(f,x) := \prod_{p\le x}\Big(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\Big)\Big(1 - \frac{1}{p}\Big).$$

We are concerned with understanding the mean value of f up to x, that is, $\frac{1}{x}\sum_{n\le x} f(n)$. For real-valued f it turns out that

$$\frac{1}{x}\sum_{n\le x} f(n) \to \Theta(f,\infty) \quad \text{as } x \to \infty. \tag{1.1a}$$

In 1944 Wintner [19] proved this when $\Theta(f,\infty) \ne 0$, which is equivalent to the hypothesis that $\sum_p \big(1 - f(p)\big)\big/ p$ converges. In 1967, Wirsing [20] settled the harder remaining case when $\Theta(f,\infty) = 0$, thereby establishing an old conjecture of Erdős and Wintner that every multiplicative function f with $-1 \le f(n) \le 1$ has a mean value.

On the other hand not all complex valued multiplicative functions have a mean value tending to a limit; for example, the function $f(n) = n^{i\alpha}$, with $\alpha \in \mathbb{R}\setminus\{0\}$, since $\frac{1}{x}\sum_{n\le x} n^{i\alpha} \sim x^{i\alpha}/(1+i\alpha)$. In the early seventies, Gábor Halász [8, 9] brilliantly realized that the correct question to ask is whether $\sum_p \big(1 - \mathrm{Re}\big(f(p)p^{-i\alpha}\big)\big)\big/ p$ diverges for all real numbers α. His fundamental result states:

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(I) If $\sum_p \big(1 - \mathrm{Re}\big(f(p)/p^{i\alpha}\big)\big) / p$ diverges for all α then $\frac{1}{x}\sum_{n\le x} f(n) \to 0$ as $x \to \infty$.

(II) If there exists α for which $\sum_p \big(1 - \mathrm{Re}\big(f(p)/p^{i\alpha}\big)\big) / p$ converges then

$$\frac{1}{x}\sum_{n\le x} f(n) \sim \frac{x^{i\alpha}}{1+i\alpha}\Theta(f_\alpha, x) \tag{1.1b}$$

where $f_\alpha(n) := f(n)/n^{i\alpha}$. Now $|\Theta(f_\alpha, x)| \to |\Theta(f_\alpha, \infty)|$ as $x \to \infty$ so we can rewrite the above as

$$\frac{1}{x}\sum_{n\le x} f(n) \sim \frac{x^{i\alpha}}{1+i\alpha}|\Theta(f_\alpha, \infty)|e^{ir(x)}$$

where $r(x) = \arg\Theta(f_\alpha, x)$ (which varies very slowly, for example $r(x^2) = r(x)+o(1)$). Also note that if $\sum_p |1 - f(p)/p^{i\alpha}|/p$ converges then (II) holds and $\Theta(f_\alpha, x) \to \Theta(f_\alpha, \infty)$ as $x \to \infty$.

In case (I), Halász also quantified how rapidly the limit is attained. His method was modified and refined by Montgomery [15], and Tenenbaum [18, p. 343] recently deduced the following, easily applicable, version of the result: Throughout define

$$M(x, T) := \min_{|y|\le 2T} \sum_{p\le x} \frac{1 - \mathrm{Re}\big(f(p)p^{-iy}\big)}{p}. \tag{1.2}$$

***Theorem* (Halász–Montgomery–Tenenbaum)** *Let f be a multiplicative function with $|f(n)| \le 1$ for all n. Let $x \ge 3$ and $T \ge 1$ be real numbers, and let $M = M(x, T)$. Then*

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \ll (1+M)e^{-M} + O\Big(\frac{1}{\sqrt{T}}\Big).$$

Here and throughout the constants implied by "$\ll$" and "$O(\cdot)$" are absolute unless otherwise indicated and, in particular, independent of the function f.

Our first theorem leads to an *explicit* refinement of this result, replacing the "$\ll$" by a constant. For any complex number s with $\mathrm{Re}(s) > 0$, let

$$F(s) = F(s; x) := \prod_{p\le x}\Big(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\Big),$$

and define

$$L = L(x, T) := \frac{1}{\log x}\Big(\max_{|y|\le 2T} |F(1+iy)|\Big). \tag{1.3}$$

Theorem 1 *Let f be a multiplicative function with $|f(n)| \le 1$ for all n. Let $x \ge 3$ and $T \ge 1$ be real numbers. Then*

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \le L\Big(\log\frac{e^\gamma}{L} + \frac{12}{7}\Big) + O\Big(\frac{1}{T} + \frac{\log\log x}{\log x}\Big).$$

We can deduce from this the promised *explicit* refinement of the Halász–Montgomery–Tenenbaum result.

Corollary 1 *Let f be a multiplicative function with $|f(n)| \le 1$ for all n. Let $x \ge 3$ and $T \ge 1$ be real numbers, and let $M = M(x, T)$. If f is completely multiplicative then*

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \le \Big(M + \frac{12}{7}\Big)\, e^{\gamma - M} + O\Big(\frac{1}{T} + \frac{\log\log x}{\log x}\Big).$$

If f is multiplicative then

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \le \prod_p \Big(1 + \frac{2}{p(p-1)}\Big)\Big(M + \frac{4}{7}\Big)\, e^{\gamma - M} + O\Big(\frac{1}{T} + \frac{\log\log x}{\log x}\Big).$$

As we will discuss after Theorem 5, Corollary 1 (and so Theorem 1) is essentially "best possible" (up to a factor 10) in that for any given sufficiently large m_0, we can construct f and x so that $M = M(x, \infty) = m_0 + O(1)$ and $|\sum_{n\le x} f(n)| \ge (M + 12/7)e^{\gamma - M}x/10$.

If the maximum in (1.3) (or, the minimum in (1.2)) occurs for $y = y_0$ then one might expect that $f(n)$ looks roughly like n^{iy_0}, so that the mean value of $f(n)$ should be around size $|x^{iy_0}/(1 + iy_0)| \asymp 1/(1 + |y_0|)$. Our next result confirms this expectation.

Theorem 2a *Let f be a multiplicative function with $|f(n)| \le 1$ for all n. Suppose that the maximum in (1.3) with $T = \log x$ is attained at $y = y_0$. Then*

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \ll \frac{1}{1 + |y_0|} + \frac{(\log\log x)^{1+2(1-\frac{2}{\pi})}}{(\log x)^{1-\frac{2}{\pi}}}.$$

The constant $4/\pi$ which appears here and frequently in the rest of the introduction, does so because it is the average of $|1 - z|$ for z on the unit circle.

Taking $f(n) = n^{iy_0}$ we see that Theorem 2a is best possible in terms of y_0. However, in this case $M = 0$, so we might guess that one can obtain a hybrid bound of Corollary 1 and Theorem 2a, of the shape $\ll (M + 1)e^{-M}/(1 + |y_0|)$.

Theorem 2b *Let f be a multiplicative function with $|f(n)| \le 1$ for all n. Suppose that the maximum in (1.3) with $T = \log x$ is attained at $y = y_0$, and let $M = M(x, \log x)$. If f is completely multiplicative then*

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \le \Big(M + \frac{12}{7}\Big)\frac{e^{\gamma - M}}{\sqrt{1 + y_0^2}} + O\Big(\frac{\log\log x}{(\log x)^{2-\sqrt{3}}}\Big).$$

If f is multiplicative then

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \le \prod_p \Big(1 + \frac{2}{p(p-1)}\Big)\Big(M + \frac{4}{7}\Big)\frac{e^{\gamma - M}}{\sqrt{1 + y_0^2}} + O\Big(\frac{\log\log x}{(\log x)^{2-\sqrt{3}}}\Big).$$

The key constituents of the main term are the $1/\sqrt{1+y_0^2}$ which corresponds to the best approximation of $f(n)$ by a function of the form $n^{i\alpha}$, and the $(M+1)e^{-M}$ which corresponds to how much $f(n)$ differs from $n^{i\alpha}$. Note that the size of the right hand side of (1.1b) is $|x^{i\alpha}\Theta(f_\alpha, x)/(1+i\alpha)| \asymp e^{-M}/(1+|y_0|)$, which implies that there is little room to reduce the bound in Theorem 2b. In fact for any given α and sufficiently large m_0 we can determine f, such that $M = m_0 + O(1)$, $y_0 = \alpha$ and the bound in Theorem 2b is too big by a factor of at most 10.

The maximum in (1.3) and the minimum in (1.2) can be unwieldly to determine, so it is desirable to get similar decay estimates in terms of $\sum_{p\le x}\big(1 - \operatorname{Re} f(p)\big)\big/p$ (or equivalently $|F(1)|$). However, in light of case (II) above, this is only possible if we have some additional information on f, since the $\sum_p\big(1 - \operatorname{Re} f(p)\big)\big/p$ may diverge while the absolute value of the mean value may converge. One can avoid case (II) altogether by insisting that all $f(p)$ lie in some closed convex subset D of the unit disc $\mathbb{U}$ (this is a natural restriction for many applications, such as when f is a Dirichlet character of a given order), as in Halász [8, 9], R. Hall and G. Tenenbaum [13], and Hall [12]. The result of Hall is the most general, perhaps qualitatively definitive. To describe it we require some information on the geometry of D:

Throughout we let D be a closed, convex subset of $\mathbb{U}$ with $1 \in D$, and define $\nu = \nu(D) = \max_{\delta\in D}(1 - \operatorname{Re}\delta)$. For $\alpha \in [0,1]$ define

$$\bar{h}(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \max_{\delta\in D} \operatorname{Re}(1-\delta)(\alpha - e^{-i\theta})\, d\theta, \tag{1.4}$$

which is a continuous, increasing, convex function of α. Note that $\bar{h}(0) = \lambda(D)/2\pi$, where $\lambda(D)$ is the length of the boundary of D. Define $\kappa = \kappa(D)$ to be the largest value of $\alpha \in [0,1]$ such that $\bar{h}(\alpha) \le 1$, which exists since $\bar{h}(0) \le 1$. When $0 \in D$, Hall showed that $\kappa(D) = 0$ only when $D = \mathbb{U}$, and $\kappa(D) = 1$ only when $D = [0,1]$. He also proved

Lemma 1.1 *For any closed, convex subset of $\mathbb{U}$ with $1 \in D$ we have*

$$\kappa(D) \ge \min\Big(1, \frac{1-\bar{h}(0)}{\bar{h}(1)-\bar{h}(0)}\Big) \ge \min\Big(1, \frac{1}{\nu(D)}\Big(1 - \frac{\lambda(D)}{2\pi}\Big)\Big).$$

Moreover $\kappa(D)\nu(D) \le 1$ for all D, with equality holding if and only if $D = [0,1]$.

***Theorem* (Hall)** *Let D be a closed, convex subset of $\mathbb{U}$ with $1 \in D$, and define $\kappa(D)$ as above. Let f be a multiplicative function with $|f(n)| \le 1$ and $f(p) \in D$ for all primes p. Then*

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \ll_D \exp\Big(-\kappa(D)\sum_{p\le x}\frac{1-\operatorname{Re} f(p)}{p}\Big). \tag{1.5}$$

Hall states this result under the additional constraint that $0 \in D$, but this is unnecessary. Hall proved that the constant $\kappa(D)$ in (1.5) is optimal for *every D*, in that it cannot be replaced by any larger value. For completely multiplicative functions, we have obtained the following more explicit version of Hall's theorem. Let

$$C(D) = -\kappa\nu\gamma + \min_{\epsilon=\pm 1}\int_0^{2\pi} \frac{\min\big(0, 1-\kappa-\max_{\delta\in D}\mathrm{Re}\,\delta(e^{\epsilon ix}-\kappa)\big)}{x}\,dx. \tag{1.6}$$

Theorem 3 *Let D be a closed, convex subset of $\mathbb{U}$ with $1 \in D$, and define $\kappa = \kappa(D)$, $\nu = \nu(D)$ and $C(D)$ as above. Let f be a multiplicative function with $f(p) \in D$ for all primes p, and put $y = \exp\big((\log x)^{\frac{2}{3}}\big)$. If $\kappa\nu < 1$ then*

$$\begin{aligned}\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \le |\Theta(f,y)|\Big(\frac{2-\kappa\nu}{1-\kappa\nu}\Big)\\ \times \exp\Big(-\kappa\sum_{y<p\le x}\frac{1-\mathrm{Re}\,f(p)}{p} - C(D) + \gamma(1-\kappa\nu)\Big)\\ + O\bigg(\frac{1}{(\log x)^{\frac{1}{3}}}\exp\Big(\sum_{p\le x}\frac{|1-f(p)|}{p}\Big)\bigg).\end{aligned}$$

If $D = [0,1]$, that is $\kappa\nu = 1$, then

$$\frac{1}{x}\sum_{n\le x} f(n) \le e^{\gamma}|\Theta(f,x)| + O\Big(\frac{1}{\log x}\Big).$$

The error term in the first part of Theorem 3 can be bounded using

$$\sum_{p\le x}\frac{|1-f(p)|}{p} \le \Big(2\log\log x\sum_{p\le x}\frac{1-\mathrm{Re}\,f(p)}{p}\Big)^{\frac{1}{2}} + O(1).$$

A version of the second statement in Theorem 3 was first proved by Hall [12]. Theorem 3 is essentially "best possible" (up to the constant of multiplication), for *every* such D, as noted in [12] and [13].

The first statement of Theorem 3 gives an explicit, quantitative and useful version of Hall's theorem, so long as $\sum_{p\le x}\big(1-\mathrm{Re}\,f(p)\big)/p \ll_D \log\log x$. However if this fails then Hall's original theorem shows that $\sum_{n\le x} f(n) \ll_D x/(\log x)^{B_D}$ for some constant $B_D > 0$, so we have the following corollary:

Corollary 2 *Retain the notation and variables of Theorem 3. Let f be a multiplicative function with $f(p) \in D$ for all p. If $\kappa\nu < 1$ then*

$$\begin{aligned}\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \le |\Theta(f,y)|\Big(\frac{2-\kappa\nu}{1-\kappa\nu}\Big)\\ \times \exp\Big(-\kappa\sum_{y<p\le x}\frac{1-\mathrm{Re}\,f(p)}{p} - C(D) + \gamma(1-\kappa\nu)\Big)\\ + O_D\Big(\frac{1}{(\log x)^{B_D}}\Big),\end{aligned}$$

where $B_D := \big((1+\kappa/3) - \sqrt{1+2\kappa/3}\big)/\kappa$.

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Lastly, we apply our ideas to study how averages of multiplicative functions vary. One would like to be able to say that

$$\frac{1}{x}\sum_{n\le x} f(n) - \frac{w}{x}\sum_{n\le x/w} f(n) \ll \Big(\frac{\log 2w}{\log x}\Big)^{\beta}, \tag{1.7}$$

for all $1 \le w \le x$, with as large an exponent β as possible (thus showing that averages of multiplicative functions vary slowly). However (1.7) is not true in general, as the ubiquitous example $f(n) = n^{i\alpha}$ reveals. On the other hand P. D. T. A. Elliott [3] proved that the *absolute value* of the means of multiplicative functions does vary slowly. He showed that

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| - \frac{w}{x}\Big|\sum_{n\le x/w} f(n)\Big| \ll \Big(\frac{\log 2w}{\log x}\Big)^{\frac{1}{19}},$$

for all multiplicative functions f with $|f(n)| \le 1$, and all $1 \le w \le x$. By applying Theorem 1, and the ideas underlying it, we have obtained the following result which leads to an improvement on Elliott's theorem.

Theorem 4 *Let f be a multiplicative function with $|f(n)| \le 1$ for all n. For any $x \ge 3$ there exists a real number y_1 such that for all $1 \le w \le x/10$, we have*

$$\Big|\frac{1}{x}\sum_{n\le x} f(n)n^{-iy_1} - \frac{w}{x}\sum_{n\le x/w} f(n)n^{-iy_1}\Big| \\ \ll \Big(\frac{\log 2w}{\log x}\Big)^{1-\frac{2}{\pi}}\log\Big(\frac{\log x}{\log 2w}\Big) + \frac{(\log\log x)^{1+2(1-\frac{2}{\pi})}}{(\log x)^{1-\frac{2}{\pi}}}.$$

If the maximum in (1.3) with $T = \log x$ is attained at $y = y_0$ then we can take $y_1 = y_0$ if $|y_0| < (\log x)/2$, and $y_1 = 0$ otherwise.

Note that $1 - \frac{2}{\pi} = 0.36338\cdots$ and $2 - \sqrt{3} = 0.267949\cdots$.

Corollary 3 *Let f be a multiplicative function with $|f(n)| \le 1$ for all n. Then for $1 \le w \le x/10$, we have*

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| - \frac{w}{x}\Big|\sum_{n\le x/w} f(n)\Big| \ll \Big(\frac{\log 2w}{\log x}\Big)^{1-\frac{2}{\pi}}\log\Big(\frac{\log x}{\log 2w}\Big) + \frac{\log\log x}{(\log x)^{2-\sqrt{3}}}.$$

In the special case that $f(n)$ is non-negative we can improve the $1-2/\pi$ in Corollary 3 to $1-1/\pi$, see [7]. There we apply this idea of slowly varying averages to refine the upper bound of $e^{\gamma}\Theta(f,x)$ in Theorem 3.

Another application of estimates such as Corollary 3, as Hildebrand [14] observed, is to extend slightly the range of validity of Burgess' famous character sum estimate. For characters χ with cubefree conductor q, one gets $\sum_{n\le N}\chi(n) = o(N)$ for $N > q^{1/4-o(1)}$ rather than $N > q^{1/4+o(1)}$.

We shall give a third application, of a kind first observed by Elliott [4], and improving results of Davenport and Erdős [2]. For integer $k \geq 2$ define η_k to be the infimum of those real numbers η such that for all primes p there exists a representative of each coset of the k-th powers mod p, which is $\ll_k p^\eta$. We expect $\eta_k = 0$ for all k, but the best result to date, due to Elliott, is that $\eta_k \leq 1/4 - c/k^{19}$ for some constant $c > 0$. Using our Corollary 3 in Elliott's argument we may replace 19 here with any constant $> 1/(1 - 2/\pi) = 2.752\cdots$. We also work out bounds for η_k explicitly for $k = 2$ and $k = 3$, and modify the argument of Davenport and Erdős to get $\eta_k \leq 1/4 - e^\gamma/(4k^2) + O(k^{-3})$ for prime values of k. These results are collected together in Corollary 4 below.

The problem of estimating η_k may be reduced to the following (difficult) optimization problem. Given $k \geq 2$ consider the class of all completely multiplicative functions f which take values on the k-th roots of unity (like a character of order k), such that for a given large x, and for each k-th root of unity ξ, there are $\sim x/k$ integers $n \leq x$ with $f(n) = \xi$ (to be precise, we mean that for some given function $\epsilon = \epsilon(x) \to 0$ as $x \to \infty$, there are between $(1 - \epsilon)x/k$ and $(1 + \epsilon)x/k$ integers $n \leq x$ with $f(n) = \xi$). Define $\tau_k(x)$ to be the smallest real number τ such that for every k-th root of unity ξ, there is an integer $n \leq x^\tau$ with $f(n) = \xi$; and then $\tau_k := \limsup_{x\to\infty} \tau_k(x)$. Bounds for τ_k give bounds for η_k since using Burgess' theorem we have $\eta_k \leq \tau_k/4$. We determine below τ_2 and τ_3, but the value of τ_k for $k \geq 4$ remains an open question.

Corollary 4 *For all $k \geq 2$ we have*

$$\eta_k \leq \frac{1}{4} - \frac{c}{(k \log k)^{1/(1-2/\pi)}},$$

for some $c > 0$. If k is prime then $\eta_k \leq 1/4 - e^\gamma/4k^2 + O(1/k^3)$. Further $\tau_2 = e^{-1/2} = 0.60653\cdots$, and $\tau_3 = 0.765423\cdots$, so that $\eta_2 \leq 1/(4\sqrt{e})$ and $\eta_3 \leq 0.191355\cdots$.

Our proofs of Theorems 1, 2a,b, and 4 are based on the following key proposition (and its variant, Proposition 3.3, below), which we establish by a variation of Halász' method. Proposition 1 below is a variant of Montgomery's lemma (see [15], and also Montgomery and R. C. Vaughan [17]) which is one of the main ingredients in the proof of Hall's theorem.

Proposition 1 *Let f, x, T, and F be as in Theorem 1. Then*

$$\frac{1}{x}\Big|\sum_{n\leq x} f(n)\Big| \leq \frac{2}{\log x}\int_0^1 \Big(\frac{1-x^{-2\alpha}}{2\alpha}\Big)\Big(\max_{|y|\leq T} |F(1+\alpha+iy)|\Big)\, d\alpha + O\Big(\frac{1}{T} + \frac{\log\log x}{\log x}\Big).$$

To prove Theorem 3, we adopt a different strategy, turning to integral equations. Let $\chi\colon [0,\infty) \to \mathbb{U}$ be a measurable function, with $\chi(t) = 1$ for $t \leq 1$. We let $\sigma(u)$ denote the solution to

$$u\sigma(u) = (\sigma * \chi)(u) = \int_0^u \sigma(t)\chi(u-t)\,dt, \qquad \text{with initial condition } \sigma(u) = 1 \text{ for } 0 \leq u \leq 1. \tag{1.8}$$

We showed in [6] that (1.8) has a unique solution, and this solution is continuous. Further let $I_0(u;\chi) = 1$, and for $n \geq 1$ define

$$(1.9a) \qquad I_n(u;\chi) = \int_{\substack{t_1,\dots,t_n \geq 0\\ t_1+\cdots+t_n \leq u}} \frac{1-\chi(t_1)}{t_1}\frac{1-\chi(t_2)}{t_2}\cdots\frac{1-\chi(t_n)}{t_n}\, dt_1\cdots dt_n.$$

Then we showed that

$$(1.9b) \qquad \sigma(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} I_n(u;\chi).$$

The relevance of the class of integral equations (1.8) to the study of multiplicative functions was already observed by Wirsing [20]. We illustrate this connection by means of the following Proposition, proved in [6] (Proposition 1 there).

Proposition 2 *Let f be a multiplicative function with $|f(n)| \leq 1$ for all n and $f(n) = 1$ for $n \leq y$. Let $\vartheta(x) = \sum_{p\leq x} \log p$ and define*

$$\chi(u) = \chi_f(u) = \frac{1}{\vartheta(y^u)} \sum_{p\leq y^u} f(p)\log p.$$

Then $\chi(t)$ is a measurable function taking values in the unit disc and with $\chi(t) = 1$ for $t \leq 1$. Let $\sigma(u)$ be the corresponding unique solution to (1.8). Then

$$\frac{1}{y^u}\sum_{n\leq y^u} f(n) = \sigma(u) + O\Big(\frac{u}{\log y}\Big).$$

Proposition 2 allows us to handle mean values of multiplicative functions which are known to be 1 on the small primes. We borrow another result from [6] (see Proposition 4.5 there) which allows us to remove the impact of the small primes on the multiplicative functions to be explored.

Proposition 3 *Let f be a multiplicative function with $|f(n)| \leq 1$ for all n. For any $2 \leq y \leq x$, let g be the completely multiplicative function with $g(p) = 1$ if $p \leq y$, and $g(p) = f(p)$ otherwise. Then*

$$\frac{1}{x}\sum_{n\leq x} f(n) = \Theta(f,y)\frac{1}{x}\sum_{m\leq x} g(m) + O\bigg(\frac{\log y}{\log x}\exp\Big(\sum_{p\leq x}\frac{|1-f(p)|}{p}\Big)\bigg).$$

We prove Theorem 3 by establishing a decay estimate, Theorem 5, for solutions of (1.8) when $\chi(t)$ is constrained to lie in D for all t. Then using Propositions 2 and 3 we unwind this result to deduce Theorem 3. It should be noted that it is *unnecessary* to work with integral equations to prove these results, and that one can proceed directly. However we find it easier to understand these proofs when formulated in this way. Moreover we discovered these proofs, which are rather different from those of Halász and Montgomery, in the context of integral equations, and it would seem disingenuous to disguise their origins.

Theorem 5 *Let D be a closed, convex subset of $\mathbb{U}$ with $1 \in D$, and define $\kappa = \kappa(D), \nu = \nu(D)$ and $C(D)$ as above. Let $\chi\colon [0,\infty) \to D$ be a measurable function with $\chi(t) = 1$ for $t \leq 1$, and let σ denote the corresponding solution to (1.8). Put*

$$M_0 = M_0(u;\chi) = \int_0^u \frac{1 - \operatorname{Re}\chi(v)}{v}\,dv.$$

Then, if $\kappa\nu < 1$,

$$|\sigma(u)| \leq \Big(\frac{2-\kappa\nu}{1-\kappa\nu}\Big)\exp\big(-\kappa M_0 - C(D) + \gamma(1-\kappa\nu)\big) - \Big(\frac{\kappa\nu}{1-\kappa\nu}\Big)\exp\Big(-\frac{M_0}{\nu} - \frac{C(D)}{\kappa\nu}\Big).$$

If $\kappa\nu = 1$ (so that $D = [0,1]$) then $|\sigma(u)| \leq e^{\gamma - M_0}$.

When studying mean values of multiplicative functions we have seen how the example $f(n) = n^{i\alpha}$ led Halász to consider convex regions D that are not dense on the unit circle. Given that we now have $\chi(t) = 1$ for $0 \leq t \leq 1$, it is perhaps unclear whether such restrictions are necessary when considering (1.8). In fact they are, and in Section 10a we shall see that if $\chi(t) = e^{i\alpha t}$ for all $t > 1$ then $\limsup |\sigma(u)| \gg_\alpha 1$.

By Proposition 2, we know that statements about multiplicative functions can be interpreted to give information on solutions to (1.8). For example, the remark after the statement of Theorem 3 translates to saying that Theorem 5 is "best possible" for *every D*, up to the constant of multiplication, via [12] and [13]. Moreover we can state integral equations versions of Corollary 1 and Theorem 4.

Corollary 1′ *If χ and σ are as in Theorem 5 then $|\sigma(u)| \leq (M + 12/7)e^{\gamma - M}$ where*

$$M = M(u) := \min_{y\in\mathbb{R}} \int_0^u \frac{1 - \operatorname{Re}\chi(v)e^{-ivy}}{v}\,dv.$$

In fact this is "best possible", up to a factor 10, in the sense that for any sufficiently large m_0 we can find χ and σ as in Theorem 5 with $M = m_0 + O(1)$ and $|\sigma(u)| \geq (M + 12/7)e^{\gamma - M}/10$; see Section 10b for our construction. This implies the same of Corollary 1 and hence of Theorem 1, by Proposition 2.

The analogue of Theorem 4 shows that $|\sigma(u)|$ obeys a strong Lipschitz-type estimate.

Theorem 4′ *Let $\chi\colon [0,\infty) \to \mathbb{U}$ be a measurable function with $\chi(t) = 1$ for $t \leq 1$, and let σ denote the corresponding solution to (1.8). Then for all $1 \leq v \leq u$,*

$$\big|\, |\sigma(u)| - |\sigma(v)| \,\big| \ll \Big(\frac{u-v}{u}\Big)^{1-\frac{2}{\pi}} \log\frac{u}{u-v}.$$

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We illustrate Theorem 5, and thus Theorem 3, by working out several examples. In each of our examples we will have $D = \bar{D}$, which allows us to restate Theorem 5 as $|\sigma(u)| \le c' e^{-\kappa M_0} < c e^{-\kappa M_0}$ where

$$\begin{aligned} c' &:= c \exp\Bigl(-2\pi \int_0^\pi \frac{\min\bigl(0, 1-\kappa - \max_{\delta\in D} \operatorname{Re} \delta(e^{i\theta} - \kappa)\bigr)}{\theta(2\pi - \theta)} \, d\theta \Bigr) \\ &< c := \Bigl(\frac{2-\kappa\nu}{1-\kappa\nu} \Bigr) e^{\gamma}. \end{aligned}$$

Example 1 *D is the convex hull of the m-th roots of unity.* For $m = 2$ we have $D = [-1, 1]$, $\nu = 2$, $\kappa = 0.32867416320\cdots$ and $c' = 6.701842225\cdots < c = 6.978982\cdots$. For larger m we can determine a formula for $\bar{h}(\alpha)$; for example, for odd $m \ge 3$, define $\delta_j = \theta_j - \pi(2j-1)/m$ where $\sin\theta_j/(\cos\theta_j - \alpha) = \tan\bigl(\pi(2j-1)/m\bigr)$, for $1 \le j \le (m+1)/2$. Then

$$\bar{h}(\alpha) = \alpha + \frac{1}{\pi}\Bigl(\sin\frac{\pi}{m}\Bigl(1 + 2\sum_{j=1}^{(m-1)/2} \cos\delta_j\Bigr) - \alpha\Bigl(\delta_1 + \sum_{j=1}^{(m-1)/2} (\delta_{j+1} - \delta_j)\cos\frac{2\pi j}{m}\Bigr)\Bigr)$$

An analogous formula holds for even m. We computed κ and c (not c') for various m:

m	3	4	5	6	7	8	9	10
κ	.167216	.098589	.063565	.044673	.032971	.025359	.020086	.016305
c	4.15845	3.99959	3.79356	3.73689	3.68124	3.65731	3.63435	3.62219

The c and κ values for D, the convex hull of the m-th roots of unity.

One can show that, as $m \to \infty$, we have $\kappa = \pi^2/6m^2 + O(1/m^4)$ and $c = 2e^{\gamma} + O(1/m^2)$. Therefore, following the proof of Theorem 2 of [6] we have that if x is sufficiently large and p is a prime $\equiv 1 \pmod m$, then there are at least $\{\pi_m + o(1)\}x$ integers $\le x$ which are m-th power residues (mod p), where

$$\pi_m \ge \exp\bigl(-\exp(\{3/\pi^4 + o(1)\}m^4 \log m)\bigr).$$

It is shown in [6] that $\pi_m \le \exp(-\{1 + o(1)\}m\log m)$, and that $\pi_2 = .1715\cdots$, the only m for which the best possible value has been determined.

Example 2 *D is the disc going through 1 with radius $r \le 1$.* Note that $\kappa = 0$ if $r = 1$. We have the (relatively) simple formula,

$$\bar{h}(\alpha) = r\Bigl(\alpha + \frac{1}{\pi}\int_{\theta_0}^{\pi} |e^{i\theta} - \alpha| \, d\theta\Bigr),$$

so that $\kappa = 1$ if $r \le \pi/(\pi + 4) = .43990084\cdots$. For various radii r, we computed κ and c:

r	$.4399\cdots$	.45	.5	.6	.7	.8	.9	.95
κ	1	.968330	.822168	.580480	.390142	.236024	.108183	.051957
c	16.5986	15.6413	11.7966	7.65099	5.70586	4.64287	3.99284	3.75723

The c and κ values for D, the disc of radius r, with center $1 - r$.

One can show that, as r gets close to 1, that is $r = 1 - \delta$ where $\delta \to 0^+$, then $\kappa = \delta + 3\delta^2/4 + O(\delta^3)$ and $c = 2e^\gamma\big(1 + \delta + O(\delta^2)\big)$.

Example 3 *D is the sector of the circle bounded by the lines from* 1 *to* $e^{\pm i\varphi}$. In other words, D is the convex hull of the point $\{1\}$ together with the arc from $e^{i\varphi}$ to $e^{-i\varphi}$ on the unit circle. Select $\theta_0 < \theta_1$ so that $\tan(\varphi/2) = \sin\theta_0/(\cos\theta_0 - \alpha)$ and $\tan\varphi = \sin\theta_1/(\cos\theta_1 - \alpha)$, and thus, with $I := \big(\theta_0 + (\theta_1 - \theta_0)\cos\varphi\big)$, we have

$$\bar h(\alpha) = \alpha + \frac{1}{\pi}\left(\sin\theta_0 + \sin(\theta_1 - \varphi) - \sin(\theta_0 - \varphi) - \alpha I + \int_{\theta_1}^{\pi} |e^{i\theta} - \alpha|\, d\theta\right).$$

Notice that if $\varphi = \pi$ then $D = [-1, 1]$ so, as above, $\kappa = \kappa^* := .328674163\cdots$ and $c = 6.978982\cdots$ We computed the following values:

φ	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	$9\pi/10$	$.99\pi$
κ	.006293	.014597	.046181	.140280	.188459	.235961	.317918	.328674
c	3.58485	3.61571	3.74339	4.01647	4.25671	4.63956	5.15381	6.67192

The c and κ values for D, the cone with lines from 1 to $e^{\pm i\varphi}$.

One can show that as $\varphi \to 0$ we have $\kappa \sim \varphi^3/24\pi$. Moreover if $\varphi \to \pi$ then we have $\kappa^* - \kappa \sim \eta(\pi - \varphi)$, for some absolute constant $\eta > 0$.

2 Preliminaries

We begin with the following lemma, weaker versions of which may be found in the works of Halász [8], Halberstam and Richert [10], and Montgomery and Vaughan [17].

Lemma 2.1 *Let f be a multiplicative function with $|f(n)| \le 1$ for all n. Put $S(x) = \sum_{n\le x} f(n)$. Then for $x \ge 3$,*

$$|S(x)| \le \frac{x}{\log x}\int_2^x \frac{|S(y)|}{y^2}\, dy + O\Big(\frac{x}{\log x}\Big). \tag{2.1}$$

Further, if $1 \le w \le x$, *then*

$$\left|\frac{S(x)}{x} - \frac{S(x/w)}{x/w}\right| \le \frac{1}{\log x}\int_{2w}^{x}\left|\frac{S(y)}{y} - \frac{S(y/w)}{y/w}\right|\frac{dy}{y} + O\Big(\frac{\log 2w}{\log x}\Big). \tag{2.2}$$

Proof First note that

$$S(x)\log x - \sum_{n\le x} f(n)\log n = \sum_{n\le x} f(n)\log\frac{x}{n} = O\Big(\sum_{n\le x}\log\frac{x}{n}\Big) = O(x).$$

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Further

$$\sum_{n\le x} f(n)\log n = \sum_{n\le x} f(n)\sum_{p^k|n}\log p = \sum_{p^k\le x}\log p \sum_{m\le x/p^k} f(mp^k).$$

Since

$$\sum_{m\le x/p^k} f(mp^k) = f(p^k)\sum_{m\le x/p^k} f(m) + O\Big(\sum_{\substack{m\le x/p^k\\ p|m}} 1\Big) = f(p^k)S\Big(\frac{x}{p^k}\Big) + O\Big(\frac{x}{p^{k+1}}\Big),$$

it follows that

$$S(x)\log x = \sum_{d\le x} f(d)\Lambda(d)S\Big(\frac{x}{d}\Big) + O(x). \tag{2.3}$$

Hence

$$|S(x)|\log x \le \sum_{d\le x}\Lambda(d)\Big|S\Big(\frac{x}{d}\Big)\Big| + O(x). \tag{2.4}$$

Writing $\psi(x) = \sum_{n\le x}\Lambda(n)$, as usual, we see that

$$\begin{aligned}\sum_{d\le x}\Lambda(d)\Big|S\Big(\frac{x}{d}\Big)\Big| &= \sum_{d\le x}\big(\psi(d)-\psi(d-1)\big)\Big|S\Big(\frac{x}{d}\Big)\Big|\\ &= \sum_{d\le x}\psi(d)\Big(\Big|S\Big(\frac{x}{d}\Big)\Big| - \Big|S\Big(\frac{x}{d+1}\Big)\Big|\Big).\end{aligned}$$

We now use the prime number theorem in the form $\psi(d) = d + O\big(d/(\log 2d)^2\big)$, together with the simple observation that $|S(x/d)| - \big|S\big(x/(d+1)\big)\big| \le \sum_{x/(d+1)<n\le x/d} 1$. It follows that

$$\begin{aligned}\sum_{d\le x}\Lambda(d)\Big|S\Big(\frac{x}{d}\Big)\Big| &= \sum_{d\le x} d\Big(\Big|S\Big(\frac{x}{d}\Big)\Big| - \Big|S\Big(\frac{x}{d+1}\Big)\Big|\Big)\\ &\quad + O\Big(\sum_{d\le x}\frac{d}{\log^2(2d)}\sum_{x/(d+1)<n\le x/d} 1\Big).\end{aligned}$$

The main term above is plainly $\sum_{d\le x}|S(x/d)|$, and the remainder term is

$$\begin{aligned}&\ll \sum_{d\le\sqrt{x}}\frac{d}{\log^2(2d)}\frac{x}{d(d+1)} + \frac{1}{\log^2 x}\sum_{\sqrt{x}\le d\le x} d\sum_{x/(d+1)<n\le x/d} 1\\ &\ll x + \frac{1}{\log^2 x}\sum_{n\le\sqrt{x}}\frac{x}{n} \ll x.\end{aligned}$$

Combining these observations and (2.4), we have shown that

$$|S(x)| \log x \leq \sum_{d \leq x} \left| S\Big(\frac{x}{d}\Big) \right| + O(x).$$

Now $|S(x/d)| = \int_d^{d+1} |S(x/t)|\, dt + O(\sum_{x/(d+1)<n\leq x/d} 1)$, and so the right side above is

$$\int_1^{x+1} \left| S\Big(\frac{x}{t}\Big) \right| dt + O(x).$$

By changing variables $y = x/t$ this is

$$x \int_{x/(x+1)}^{x} \frac{|S(y)|}{y^2}\, dy + O(x) = x \int_2^x \frac{|S(y)|}{y^2}\, dy + O(x),$$

proving (2.1).

To show (2.2), we note by (2.3) that

$$\begin{aligned} \log x \Big(\frac{S(x)}{x} - \frac{S(x/w)}{x/w} \Big) &= O(\log 2w) + \frac{1}{x} \sum_{d \leq x} f(d)\Lambda(d) S\Big(\frac{x}{d}\Big) \\ &\qquad - \frac{w}{x} \sum_{d \leq x/w} f(d)\Lambda(d) S\Big(\frac{x}{wd}\Big) \\ &= O(\log 2w) + \sum_{d \leq x/w} f(d)\Lambda(d) \Big(\frac{S(x/d)}{x} - \frac{S(x/wd)}{x/w} \Big). \end{aligned}$$

Hence

$$\left| \frac{S(x)}{x} - \frac{S(x/w)}{x/w} \right| \leq \frac{1}{\log x} \sum_{d \leq x/w} \Lambda(d) \left| \frac{S(x/d)}{x} - \frac{S(x/wd)}{x/w} \right| + O\Big(\frac{\log 2w}{\log x} \Big).$$

We now mimic the partial summation argument used to deduce (2.1) from (2.4). This shows (2.2).

The next lemma provides our alternative way to develop this theory, different from that of Montgomery (see III.4.3 of [18]).

Lemma 2.2 *Let a_n be a sequence of complex numbers such that $\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$. Define $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ which is absolutely convergent in $\mathrm{Re}(s) \geq 1$. For all real numbers $T \geq 1$, and all $0 \leq \alpha \leq 1$ we have*

$$\max_{|y| \leq T} |A(1 + \alpha + iy)| \leq \max_{|y| \leq 2T} |A(1 + iy)| + O\Big(\frac{\alpha}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n} \Big), \tag{2.5}$$

and for any $w \geq 1$,

$$\max_{|y| \leq T} |A(1+\alpha+iy)(1-w^{-\alpha-iy})| \leq \max_{|y| \leq 2T} |A(1+iy)(1-w^{-iy})| + O\Big(\frac{\alpha}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n} \Big). \tag{2.6}$$

Proof We shall only prove (2.6); the proof of (2.5) is similar. Note that the Fourier transform of $k(z) = e^{-\alpha|z|}$ is $\hat{k}(\xi) = \int_{-\infty}^{\infty} e^{-\alpha|z|-i\xi z}\,dz = \frac{2\alpha}{\alpha^2+\xi^2}$ which is always non-negative. The Fourier inversion formula gives for any $z \geq 1$,

$$\begin{aligned} z^{-\alpha} = k(\log z) = k(-\log z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{k}(\xi) z^{-i\xi}\,d\xi \\ &= \frac{1}{\pi}\int_{-T}^{T} \frac{\alpha}{\alpha^2+\xi^2} z^{-i\xi}\,d\xi + O\Big(\frac{\alpha}{T}\Big). \end{aligned}$$

Using this appropriately, we get that for all $n \geq 1$, and $0 \leq \alpha \leq 1$,

$$\frac{1}{n^{\alpha}}(1-w^{-\alpha-iy}) = \frac{1}{\pi}\int_{-T}^{T} \frac{\alpha}{\alpha^2+\xi^2} n^{-i\xi}(1-w^{-iy-i\xi})\,d\xi + O\Big(\frac{\alpha}{T}\Big).$$

Multiplying the above by a_n/n^{1+iy}, and summing over all n, we conclude that

$$\begin{aligned} A(1+\alpha+iy)(1-w^{-\alpha-iy}) = \frac{1}{\pi}\int_{-T}^{T} &\frac{\alpha}{\alpha^2+\xi^2} A(1+iy+i\xi)(1-w^{-iy-i\xi})\,d\xi \\ &+ O\Big(\frac{\alpha}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n}\Big). \end{aligned}$$

If $|y| \leq T$ then $|y+\xi| \leq |y| + |\xi| \leq 2T$, and so we deduce that

$$\begin{aligned} &\max_{|y|\leq T} |A(1+\alpha+iy)(1-w^{-\alpha-iy})| \\ &\qquad \leq \Big(\max_{|y|\leq 2T} |A(1+iy)(1-w^{-iy})|\Big)\frac{1}{\pi}\int_{-T}^{T}\frac{\alpha}{\alpha^2+\xi^2}\,d\xi \\ &\qquad\quad + O\Big(\frac{\alpha}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n}\Big), \end{aligned}$$

and (2.6) follows since

$$\frac{1}{\pi}\int_{-T}^{T}\frac{\alpha}{\alpha^2+\xi^2}\,d\xi \leq \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha}{\alpha^2+\xi^2}\,d\xi = 1.$$

Our next lemma was inspired by Lemma 2 of Montgomery and Vaughan [17], who consider (essentially) the quotient $\big|F\big(1+i(y+\beta)\big)/F(1+iy)\big|$ rather than the product below.

Lemma 2.3 *Let f, x, and F be as in Theorem 1. Then for all real numbers y, and $1/\log x \leq |\beta| \leq \log x$, we have*

$$\big|F(1+iy)F\big(1+i(y+\beta)\big)\big| \ll (\log x)^{\frac{4}{\pi}} \max\Big(\frac{1}{|\beta|}, (\log\log x)^2\Big)^{2(1-\frac{2}{\pi})}.$$

Proof Clearly

$$\big|F(1+iy)F\big(1+i(y+\beta)\big)\big| \ll \exp\Big(\operatorname{Re}\sum_{p\le x}\frac{f(p)p^{-iy}+f(p)p^{-i(y+\beta)}}{p}\Big)$$
$$\ll \exp\Big(\sum_{p\le x}\frac{|1+p^{-i\beta}|}{p}\Big)$$
$$= \exp\Big(\sum_{p\le x}\frac{2|\cos(\frac{|\beta|}{2}\log p)|}{p}\Big). \tag{2.7}$$

By the prime number theorem and partial summation we have for $z \ge w \ge 2$

$$\sum_{w\le p\le z}\frac{1}{p} = \int_w^z \frac{dt}{t\log t} + O\big(\exp(-c\sqrt{\log w})\big),$$

for some constant $c > 0$. Choose $C = 100/c^2$, and put

$$Y = \max\Big(\exp\big(C(\log\log x)^2\big), e^{\frac{1}{|\beta|}}\Big).$$

Put $\delta = 1/\log^3 x$, and divide the interval $[Y, x]$ into $\ll \log^4 x$ subintervals of the type $\big(z, z(1+\delta)\big]$ (with perhaps one shorter interval). For each of these subintervals we have

$$\sum_{z\le p\le z(1+\delta)}\frac{|\cos(\frac{|\beta|}{2}\log p)|}{p} = \Big(\Big|\cos\Big(\frac{|\beta|}{2}\log z\Big)\Big| + O(\delta|\beta|)\Big)\sum_{z\le p\le(1+\delta)z}\frac{1}{p}$$
$$= \Big(\Big|\cos\Big(\frac{|\beta|}{2}\log z\Big)\Big| + O(\delta|\beta|)\Big)\Big(\int_z^{z(1+\delta)}\frac{dt}{t\log t} + O\Big(\frac{1}{\log^{10} x}\Big)\Big)$$
$$= \int_z^{z(1+\delta)}\frac{|\cos(\frac{|\beta|}{2}\log t)|}{t\log t}\,dt + O\Big(\frac{1}{\log^5 x}\Big),$$

where we used $|\beta| \le \log x$. Using this for each of the $\ll \log^4 x$ such subintervals covering $[Y, x]$, we conclude that

$$\sum_{Y\le p\le x}\frac{|\cos(\frac{|\beta|}{2}\log p)|}{p} = \int_Y^x \frac{|\cos(\frac{|\beta|}{2}\log t)|}{t\log t}\,dt + O\Big(\frac{1}{\log x}\Big)$$
$$= \int_{\frac{|\beta|}{2}\log Y}^{\frac{|\beta|}{2}\log x}\frac{|\cos y|}{y}\,dy + O(1).$$

Splitting the integral over y above into intervals of length 2π (with maybe one shorter interval), and noting that $\frac{1}{2\pi}\int_0^{2\pi} |\cos\theta|\, d\theta = \frac{2}{\pi}$, we deduce that

$$\sum_{Y \le p \le x} \frac{|\cos(\frac{|\beta|}{2}\log p)|}{p} \le \frac{2}{\pi}\log\frac{\log x}{\log Y} + O(1).$$

Trivially, we also have

$$\sum_{p \le Y} \frac{|\cos(\frac{|\beta|}{2}\log p)|}{p} \le \sum_{p \le Y}\frac{1}{p} = \log\log Y + O(1).$$

Combining the above two bounds, we get that

$$\sum_{p \le x} \frac{|\cos(\frac{|\beta|}{2}\log p)|}{p} \le \frac{2}{\pi}\log\log x + \Big(1 - \frac{2}{\pi}\Big)\log\log Y + O(1).$$

The lemma follows upon using this in (2.7), and recalling the definition of Y.

We conclude this section by offering a proof of Lemma 1.1.

Proof of Lemma 1.1 For a fixed θ, note that $\max_{\delta\in D}\operatorname{Re}(1-\delta)(\alpha - e^{-i\theta})$ is an increasing function of α. Integrating, we see that $\bar{h}(\alpha)$ is an increasing function. Clearly $\bar{h}$ is continuous, and we now show that it is convex: that is, given $0 \le \alpha < \beta \le 1$, and $t \in [0,1]$, $\bar{h}\big(t\alpha + (1-t)\beta\big) \le t\bar{h}(\alpha) + (1-t)\bar{h}(\beta)$. Indeed, for a fixed θ, we have

$$\begin{aligned}&\max_{\delta\in D}\operatorname{Re}(1-\delta)\big(t(\alpha - e^{-i\theta}) + (1-t)(\beta - e^{-i\theta})\big)\\ &\qquad \le t\max_{\delta\in D}\operatorname{Re}(1-\delta)(\alpha - e^{-i\theta}) + (1-t)\max_{\delta\in D}\operatorname{Re}(1-\delta)(\beta - e^{-i\theta});\end{aligned}$$

so, integrating this, we get that $\bar{h}$ is convex.

Note that $2\pi\bar{h}(0) = \int_0^{2\pi}\max_{\delta\in D}\operatorname{Re}(1-\delta)(-e^{-i\theta})\, d\theta = \int_0^{2\pi}\max_{\delta\in D}\operatorname{Re}\,\delta e^{-i\theta}\, d\theta$. This last expression equals $\lambda(D)$, the perimeter of D, a result known as Crofton's formula (see [1], page 65).

We now show the lower bounds for κ. If $\kappa = 1$ there is nothing to prove; and suppose $\kappa < 1$ so that $\bar{h}(1) > 1$. By convexity we see that

$$\bar{h}\Big(\frac{1-\bar{h}(0)}{\bar{h}(1)-\bar{h}(0)}\Big) \le \frac{1-\bar{h}(0)}{\bar{h}(1)-\bar{h}(0)}\bar{h}(1) + \Big(1 - \frac{1-\bar{h}(0)}{\bar{h}(1)-\bar{h}(0)}\Big)\bar{h}(0) = 1,$$

and so it follows that $\kappa \ge \frac{1-\bar{h}(0)}{\bar{h}(1)-\bar{h}(0)}$. Clearly

$$\bar{h}(\alpha) \le \bar{h}(0) + \frac{1}{2\pi}\int_0^{2\pi}\max_{\delta\in D}\operatorname{Re}(1-\delta)\alpha\, d\theta = \bar{h}(0) + \alpha\nu.$$

Hence we see that $\frac{1-\bar{h}(0)}{\bar{h}(1)-\bar{h}(0)} \geq \big(1-\bar{h}(0)\big)/\nu = \frac{1}{\nu}(1-\frac{\lambda(D)}{2\pi})$.

Lastly it remains to show that $\kappa\nu \leq 1$ with equality only when $D = \kappa\nu$. By definition we have $\bar{h}(\alpha) \geq \max_{\delta\in D} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}(1-\delta)(\alpha - e^{-i\theta})\,d\theta = \alpha\nu$. It follows that $\kappa\nu \leq 1$ always. Moreover if $\kappa\nu = 1$ then $\bar{h}(\kappa) = 1$ and there exists $d \in D$ such that the maximum of $\mathrm{Re}(1-\delta)(\kappa - e^{-i\theta})$ for $\delta \in D$, occurs at $\delta = d$. Therefore if $d+\eta \in D$ then $\mathrm{Re}\,\eta(\kappa - e^{-i\theta}) \geq 0$ for all $\theta \in [0, 2\pi)$. Therefore $\kappa = \nu = 1$ else as θ runs through $[0, 2\pi)$, so does $\arg(\kappa - e^{-i\theta})$, which implies $\mathrm{Re}\,\eta(\kappa - e^{-i\theta}) < 0$ for some θ. Now $\arg(1-e^{-i\theta})$ runs through $(-\pi/2, \pi/2)$ so $\eta \in \mathbb{R}$ else $\mathrm{Re}\,\eta(\kappa - e^{-i\theta}) < 0$ for some θ. Thus $D \subset \mathbb{R}$ and so $D = [0, 1]$ since $\nu = 1$.

3 The Key Proposition

3a The Integral Equations Version

Our tool in analysing (1.8) is the Laplace transform, which, for a measurable function $f\colon [0,\infty) \to \mathbb{C}$ is given by

$$\mathcal{L}(f, s) = \int_0^\infty f(t)e^{-ts}\,dt$$

where s is some complex number. If f is integrable and grows sub-exponentially (that is, for every $\epsilon > 0$, $|f(t)| \ll_\epsilon e^{\epsilon t}$ almost everywhere) then the Laplace transform is well defined for all complex numbers s with $\mathrm{Re}(s) > 0$. Laplace transforms occupy a role in the study of differential equations analogous to Dirichlet series in multiplicative number theory.

Below, χ will be measurable with $\chi(t) = 1$ for $t \leq 1$ and $|\chi(t)| \leq 1$ for all t, and $\sigma(u)$ will denote the corresponding solution to (1.8). Observe that for any two 'nice' functions f and g, $\mathcal{L}(f * g, s) = \mathcal{L}(f, s)\mathcal{L}(g, s)$. From the definition of σ, it follows that

$$\mathcal{L}(v\sigma(v), t+iy) = \mathcal{L}(\sigma, t+iy)\mathcal{L}(\chi, t+iy), \tag{3.1}$$

where $t > 0$ and y are real numbers.

Further, recalling from (1.9a,b) that $\sigma(v) = \sum_{j=0}^{\infty}(-1)^j I_j(v;\chi)/j!$, we have

$$\begin{aligned}\mathcal{L}(\sigma, t+iy) &= \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\mathcal{L}\big(I_j(v;\chi), t+iy\big)\\ &= \frac{1}{t+iy}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\mathcal{L}\Big(\frac{1-\chi(v)}{v}, t+iy\Big)\right)^j\\ &= \frac{1}{t+iy}\exp\left(-\mathcal{L}\Big(\frac{1-\chi(v)}{v}, t+iy\Big)\right). \end{aligned} \tag{3.2}$$

We now give our integral equations version of Proposition 1.

Proposition 3.1 *Fix $u \geq 1$, and define for $t > 0$*

$$M_+(t) = \int_u^\infty \frac{e^{-tv}}{v}\,dv + \min_{y\in\mathbb{R}} \int_0^u \frac{1 - \operatorname{Re}\chi(v)e^{-ivy}}{v} e^{-tv}\,dv.$$

Then

$$|\sigma(u)| \leq \frac{1}{u}\int_0^\infty \Big(\frac{1-e^{-2tu}}{t}\Big)\frac{\exp\big(-M_+(t)\big)}{t}\,dt.$$

Since $M_+(t) \geq \max\big(0, -\log(tu) + O(1)\big)$ we see that the integral in the proposition converges.

Proof Define $\hat\chi(v) = \chi(v)$ if $v \leq u$, and $\hat\chi(v) = 0$ if $v > u$. Let $\hat\sigma$ denote the corresponding solution to (1.8). Note that $\hat\sigma(v) = \sigma(v)$ for $v \leq u$. Thus

$$\begin{aligned}|\sigma(u)| = |\hat\sigma(u)| &\leq \frac{1}{u}\int_0^u |\hat\sigma(v)|\,dv = \frac{1}{u}\int_0^u 2v|\hat\sigma(v)|\int_0^\infty e^{-2tv}\,dt\,dv\\ &= \frac{1}{u}\int_0^\infty \Big(\int_0^u 2v|\hat\sigma(v)|e^{-2tv}\,dv\Big)\,dt. \qquad (3.3)\end{aligned}$$

By Cauchy's inequality

$$\begin{aligned}\Big(\int_0^u 2v|\hat\sigma(v)|e^{-2tv}\,dv\Big)^2 &\leq \Big(4\int_0^u e^{-2tv}\,dv\Big)\Big(\int_0^\infty |v\hat\sigma(v)|^2 e^{-2tv}\,dv\Big)\\ &= 2\frac{1-e^{-2tu}}{t}\int_0^\infty |v\hat\sigma(v)|^2 e^{-2tv}\,dv. \qquad (3.4)\end{aligned}$$

By Plancherel's formula (Fourier transform is an isometry on L^2)

$$\int_0^\infty |v\hat\sigma(v)|^2 e^{-2tv}\,dv = \frac{1}{2\pi}\int_{-\infty}^\infty \big|\mathcal{L}\big(v\hat\sigma(v), t+iy\big)\big|^2\,dy$$

and, using (3.1), this is

$$\begin{aligned}&= \frac{1}{2\pi}\int_{-\infty}^\infty |\mathcal{L}(\hat\sigma, t+iy)|^2 |\mathcal{L}(\hat\chi, t+iy)|^2\,dy\\ &\leq \big(\max_{y\in\mathbb{R}} |\mathcal{L}(\hat\sigma, t+iy)|^2\big)\frac{1}{2\pi}\int_{-\infty}^\infty |\mathcal{L}(\hat\chi, t+iy)|^2\,dy.\end{aligned}$$

Applying Plancherel's formula again, we get

$$\frac{1}{2\pi}\int_{-\infty}^\infty |\mathcal{L}(\hat\chi, t+iy)|^2\,dy = \int_0^\infty |\hat\chi(v)|^2 e^{-2tv}\,dv \leq \int_0^u e^{-2tv}\,dv = \frac{1-e^{-2tu}}{2t}.$$

Hence

$$\int_0^\infty |v\hat{\sigma}(v)|^2 e^{-2tv}\,dv \le \frac{1-e^{-2tu}}{2t} \max_{y\in\mathbb{R}} |\mathcal{L}(\hat{\sigma}, t+iy)|^2. \tag{3.5}$$

By (3.2), we have

$$\mathcal{L}(\hat{\sigma}, t+iy) = \frac{1}{t+iy} \exp\Big(-\mathcal{L}\Big(\frac{1-\hat{\chi}(v)e^{-ivy}}{v}, t\Big) + \mathcal{L}\Big(\frac{1-e^{-ivy}}{v}, t\Big)\Big).$$

Now, we have the identity

$$\operatorname{Re}\mathcal{L}\Big(\frac{1-e^{-ivy}}{v}, t\Big) = \log|1+iy/t|$$

which is easily proved by differentiating both sides with respect to y. Using this we obtain

$$t|\mathcal{L}(\hat{\sigma}, t+iy)| = \exp\Big(-\operatorname{Re}\mathcal{L}\Big(\frac{1-\hat{\chi}(v)e^{-ivy}}{v}, t\Big)\Big), \tag{3.6}$$

from which it follows that

$$\max_{y\in\mathbb{R}} |\mathcal{L}(\hat{\sigma}, t+iy)| = \frac{\exp\big(-M_+(t)\big)}{t}.$$

Inserting this in (3.5), and that into (3.4), and then (3.3), we obtain the proposition.

3b The Multiplicative Functions Version: Proof of Proposition 1

In this subsection, we prove Proposition 1. We follow closely the ideas behind the proof of Proposition 3.1 above.

Note that

$$\begin{aligned}
\int_2^x \frac{|S(y)|}{y^2}\,dy &= \int_2^x \frac{2\log y}{y^2}\Big|\sum_{n\le y} f(n)\Big| \Big(\int_0^1 y^{-2\alpha}\,d\alpha + O(y^{-2})\Big)\,dy \\
&= \int_2^x \frac{2}{y^2}\Big|\sum_{n\le y} f(n)\log n + O\Big(\sum_{n\le y}\log(y/n)\Big)\Big| \Big(\int_0^1 y^{-2\alpha}\,d\alpha\Big)\,dy \\
&\quad + O(1) \\
&= \int_0^1 \Big(\int_2^x \frac{2}{y^{2+2\alpha}}\Big|\sum_{n\le y} f(n)\log n\Big|\,dy\Big)\,d\alpha + O(\log\log x).
\end{aligned} \tag{3.7}$$

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By Cauchy's inequality

$$\int_2^x \Big|\sum_{n\le y} f(n)\log n\Big| \frac{dy}{y^{2+2\alpha}} \le \Big(\int_1^x \frac{dy}{y^{1+2\alpha}}\Big)^{\frac12}\Big(\int_2^x \Big|\sum_{n\le y} f(n)\log n\Big|^2 \frac{dy}{y^{3+2\alpha}}\Big)^{\frac12}$$

$$(3.8)\qquad = \Big(\frac{1-x^{-2\alpha}}{2\alpha}\Big)^{\frac12}\Big(\int_2^x \Big|\sum_{n\le y} f(n)\log n\Big|^2 \frac{dy}{y^{3+2\alpha}}\Big)^{\frac12}.$$

Now define the multiplicative function $\tilde f$ by $\tilde f(p^k) = f(p^k)$ for $p \le x$, and $\tilde f(p^k) = 0$ for $p > x$, so that $F(s) = \sum_{n\ge1} \tilde f(n)/n^s$. Naturally $\tilde f(n) = f(n)$ for $n \le x$, and so

$$\int_2^x \Big|\sum_{n\le y} f(n)\log n\Big|^2 \frac{dy}{y^{3+2\alpha}} \le \int_1^\infty \Big|\sum_{n\le y} \tilde f(n)\log n\Big|^2 \frac{dy}{y^{3+2\alpha}},$$

and with the change of variables $y = e^t$, this is

$$(3.9)\qquad = \int_0^\infty \Big|\sum_{n\le e^t} \tilde f(n)\log n\Big|^2 e^{-2(1+\alpha)t}\,dt.$$

By Plancherel's formula

$$(3.10)\qquad \int_0^\infty \Big|\sum_{n\le e^t} \tilde f(n)\log n\Big|^2 e^{-2(1+\alpha)t}\,dt = \frac{1}{2\pi}\int_{-\infty}^\infty \Big|\frac{F'(1+\alpha+iy)}{1+\alpha+iy}\Big|^2 dy.$$

Lemma 3.2 *Let $T \ge 1$ be a real number. Then*

$$\Big(\frac{1}{2\pi}\int_{-\infty}^\infty \Big|\frac{F'(1+\alpha+iy)}{1+\alpha+iy}\Big|^2 dy\Big)^{\frac12} \le \big(\max_{|y|\le T} |F(1+\alpha+iy)|\big)\Big(\frac{1-x^{-2\alpha}}{2\alpha}\Big)^{\frac12} + O\Big(\frac{m^{\frac32}}{T} + \sqrt{m}\Big)$$

where, for convenience, we have set $m = m(\alpha) = \min(\log x, 1/\alpha)$.

Proof We split the integral to be bounded into two parts: $|y| \le T$, and $|y| > T$. Split the second region further into intervals of the form $kT \le |y| \le (k+1)T$ where $k \ge 1$ is an integer. Thus

$$\int_{|y|>T} \Big|\frac{F'(1+\alpha+iy)}{1+\alpha+iy}\Big|^2 dy \ll \sum_{k=1}^\infty \frac{1}{k^2T^2}\int_{|y|=kT}^{(k+1)T} |F'(1+\alpha+iy)|^2\,dy$$

$$\ll \sum_{k=1}^\infty \frac{1}{k^2T^2}\sum_{n=1}^\infty \frac{|\tilde f(n)|^2\log^2 n}{n^{2+2\alpha}}(T+n),$$

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by appealing to Corollary 3 of Montgomery and Vaughan [16]. Since $\tilde{f}(n) = 0$ if n is divisible by a prime larger than x, this is

$$\ll \frac{1}{T}\sum_{n=1}^{\infty}\frac{\log^2 n}{n^{2+2\alpha}} + \frac{1}{T^2}\sum_{\substack{n=1\\ p|n\Rightarrow p\le x}}^{\infty}\frac{\log^2 n}{n^{1+2\alpha}} \ll \frac{1}{T} + \frac{m^3}{T^2}.$$

Hence we have that

$$\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big|\frac{F'(1+\alpha+iy)}{1+\alpha+iy}\Big|^2\,dy\Big)^{\frac12} = \Big(\frac{1}{2\pi}\int_{-T}^{T}\Big|\frac{F'(1+\alpha+iy)}{1+\alpha+iy}\Big|^2\,dy\Big)^{\frac12} + O\Big(\frac{1}{\sqrt{T}} + \frac{m^{\frac32}}{T}\Big). \tag{3.11}$$

We now turn to the first region $|y| \le T$. Define $g(n)$ to be the completely multiplicative function given on primes p by $g(p) = \tilde{f}(p)$. Put $G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$, and define $H(s)$ by $F(s) = G(s)H(s)$. Note that $H(s)$ is absolutely convergent in $\mathrm{Re}(s) > \frac12$, and that in the region $\mathrm{Re}(s) \ge 1$ we have uniformly $|H(s)|, |H'(s)| \ll 1$. Using $F' = G'H + GH' = F(G'/G) + O(G)$, together with the inequality $(\int |f+g|^2)^{\frac12} \le (\int |f|^2)^{\frac12} + (\int |g|^2)^{\frac12}$ (which is easily deduced from Cauchy's inequality), we see that

$$\Big(\frac{1}{2\pi}\int_{-T}^{T}\Big|\frac{F'(1+\alpha+iy)}{1+\alpha+iy}\Big|^2\,dy\Big)^{\frac12} \le \Big(\frac{1}{2\pi}\int_{-T}^{T}\Big|\frac{(F\frac{G'}{G})(1+\alpha+iy)}{1+\alpha+iy}\Big|^2\,dy\Big)^{\frac12} + O\Big(\Big(\int_{-T}^{T}\Big|\frac{G(1+\alpha+iy)}{1+\alpha+iy}\Big|^2\,dy\Big)^{\frac12}\Big). \tag{3.12}$$

Splitting the interval $[-T, T]$ into subintervals of length 1, we see that the remainder term above is

$$\ll \Big(\sum_{k=-[T]-1}^{[T]}\frac{1}{1+k^2}\int_{k}^{k+1}|G(1+\alpha+iy)|^2\,dy\Big)^{\frac12}$$

$$\ll \Big(\sum_{k=-[T]-1}^{[T]}\frac{1}{1+k^2}\sum_{n=1}^{\infty}\frac{|g(n)|^2}{n^{2+2\alpha}}(1+n)\Big)^{\frac12}$$

by appealing again to Corollary 3 of [16]. Plainly this is

$$\ll \Big(\sum_{\substack{n=1\\ p|n\Rightarrow p\le x}}^{\infty}\frac{1}{n^{1+2\alpha}}\Big)^{\frac12} \ll \sqrt{m}. \tag{3.13}$$

We focus on the main term in the right side of (3.12). Clearly

$$\Big(\frac{1}{2\pi}\int_{-T}^{T}\Big|\frac{(F\frac{G'}{G})(1+\alpha+iy)}{1+\alpha+iy}\Big|^2\,dy\Big)^{\frac12} \le \big(\max_{|y|\le T}|F(1+\alpha+iy)|\big)\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big|\frac{\frac{G'}{G}(1+\alpha+iy)}{1+\alpha+iy}\Big|^2\,dy\Big)^{\frac12}.$$

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Since $-\frac{G'}{G}(s)=\sum_{n=1}^{\infty} g(n)\Lambda(n)n^{-s}$ we get, by Plancherel's formula, that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\frac{G'}{G}(1+\alpha+iy)}{1+\alpha+iy}\right|^2 dy=\int_0^{\infty}\Big|\sum_{n\le e^t} g(n)\Lambda(n)\Big|^2 e^{-2(1+\alpha)t}\,dt.$$

Since $|g(n)|\le 1$ always, we see that $|\sum_{n\le e^t} g(n)\Lambda(n)|\le \psi(e^t)$ for all t. Further, since $g(n)=0$ if n is divisible by a prime larger than x, we see that if $t\ge \log x$, then $|\sum_{n\le e^t} g(n)\Lambda(n)|\ll x+e^{t/2}$. Using these observations together with the prime number theorem we deduce that the above is

$$\begin{aligned}&\le \int_0^{\log x}\Big(e^t+O\Big(\frac{e^t}{(t+1)^2}\Big)\Big)^2 e^{-2(1+\alpha)t}\,dt+O\Big(\int_{\log x}^{\infty}(x^2+e^t)e^{-2(1+\alpha)t}\,dt\Big)\\&=\frac{1-x^{-2\alpha}}{2\alpha}+O(1).\end{aligned}$$

Thus the main term in the right side of (3.12) is

$$\begin{aligned}&\le \max_{|y|\le T}|F(1+\alpha+iy)|\Big(\frac{1-x^{-2\alpha}}{2\alpha}+O(1)\Big)^{\frac12}\\&=\max_{|y|\le T}|F(1+\alpha+iy)|\Big(\Big(\frac{1-x^{-2\alpha}}{2\alpha}\Big)^{\frac12}+O(m^{-\frac12})\Big),\end{aligned}$$

since $(1-x^{-2\alpha})/2\alpha\asymp m(\alpha)$. Combining this with (3.13), and (3.11), we obtain the lemma, since $|F(1+\alpha+iy)|\ll\prod_{p\le y}(1-1/p^{1+\alpha})^{-1}\ll m$.

We use (3.9), (3.10) and Lemma 3.2 to estimate the right side of (3.8). Inserting that estimate into (3.7) we conclude that

$$\begin{aligned}\int_2^x\frac{|S(y)|}{y^2}\,dy&\le 2\int_0^1\big(\max_{|y|\le T}|F(1+\alpha+iy)|\big)\Big(\frac{1-x^{-2\alpha}}{2\alpha}\Big)\,d\alpha\\&\qquad+O\Big(\int_0^1\Big(\frac{m(\alpha)^2}{T}+m(\alpha)\Big)\,d\alpha+\log\log x\Big)\\&=2\int_0^1\big(\max_{|y|\le T}|F(1+\alpha+iy)|\big)\Big(\frac{1-x^{-2\alpha}}{2\alpha}\Big)\,d\alpha\\&\qquad+O\Big(\frac{\log x}{T}+\log\log x\Big).\end{aligned}$$

When used with (2.1) of Lemma 2.1, this yields Proposition 1.

We end this section by giving a variant of Proposition 1 which will be our main tool in the proof of Theorem 4.

Proposition 3.3 *Let f, T, and x be as in Theorem 1. Then for $1\le w\le x$, we have*

$$\begin{aligned}\left|\frac{S(x)}{x}-\frac{S(x/w)}{x/w}\right|&\ll\frac{1}{\log x}\int_0^1 m(\alpha)\big(\max_{|y|\le T}|(1-w^{-\alpha-iy})F(1+\alpha+iy)|\big)\,d\alpha\\&\qquad+O\Big(\frac1T+\frac{\log 2w}{\log x}\log\frac{\log x}{\log 2w}\Big).\end{aligned}$$

Proof Since the proof is very similar to that of Proposition 1, we shall merely sketch it. Arguing as in (3.7), we get that

$$\int_{2w}^{x} \Big| \frac{S(y)}{y} - \frac{S(y/w)}{y/w} \Big| \frac{dy}{y} \ll \int_0^1 \Big(\int_{2w}^{x} \Big| \frac{1}{y} \sum_{n \le y} f(n) \log n - \frac{1}{y/w} \sum_{n \le y/w} f(n) \log n \Big| \frac{dy}{y^{1+2\alpha}} \Big) \, d\alpha + \log 2w \log\Big(\frac{\log x}{\log 2w} \Big).$$

Using Cauchy's inequality as in (3.8), we see that

$$\int_{2w}^{x} \Big| \frac{1}{y} \sum_{n \le y} f(n) \log n - \frac{1}{y/w} \sum_{n \le y/w} f(n) \log n \Big| \frac{dy}{y^{1+2\alpha}} \ll \min m(\alpha)^{\frac{1}{2}} \Big(\int_{2w}^{x} \Big| \frac{1}{y} \sum_{n \le y} f(n) \log n - \frac{1}{y/w} \sum_{n \le y/w} f(n) \log n \Big|^2 \frac{dy}{y^{1+2\alpha}} \Big)^{\frac{1}{2}}.$$

As before, we handle the second factor above by replacing f by $\tilde{f}$, extending the range of integration to $\int_1^\infty$, substituting $y = e^t$, and invoking Plancherel's formula. The only difference from (3.10) is that $F'(1 + \alpha + iy)/(1 + \alpha + iy)$ in the right side there must be replaced by the Fourier transform of $e^{-(1+\alpha)t} \sum_{n \le e^t} \tilde{f}(n) \log n - we^{-(1+\alpha)t} \sum_{n \le e^t/w} \tilde{f}(n) \log n$ which is $-F'(1+\alpha+iy)(1-w^{-\alpha-iy})/(1+\alpha+iy)$. We make this adjustment, and follow the remainder of the proof of Proposition 1.

4 Proofs of Theorem 1 and Corollary 1

Recall the multiplicative function $\tilde{f}(n)$ defined by $\tilde{f}(p^k) = f(p^k)$ for $p \le x$, and $\tilde{f}(p^k) = 0$ for $p > x$. Then $F(s) = \sum_n \tilde{f}(n) n^{-s}$, and since $|\tilde{f}(n)| \le 1$ always, we get that for all $0 < \alpha \le 1$,

$$\max_{y \in \mathbb{R}} |F(1 + \alpha + iy)| \le \zeta(1 + \alpha) = \frac{1}{\alpha} + O(1). \tag{4.1}$$

Taking $a_n = \tilde{f}(n)$ in Lemma 2.2 and noting that $\sum_n |a_n|/n \ll \log x$, we conclude that for $0 \le \alpha \le 1$

$$\max_{|y| \le T} |F(1 + \alpha + iy)| \le \max_{|y| \le 2T} |F(1 + iy)| + O\Big(\frac{\alpha \log x}{T} \Big). \tag{4.2}$$

Note that $L \le \frac{1}{\log x} \prod_{p \le x} (1 - \frac{1}{p})^{-1} = e^\gamma + O(1/\log x)$, by Mertens' theorem. The theorem is trivial if $1 \le L \le e^\gamma + O(1/\log x)$, and also if $L \le 1/\log x$, so we suppose that $1/\log x < L \le 1$. We use Proposition 1, employing the bound (4.2) when $\alpha \le 1/(L\log x)$, and the bound (4.1) when $1/(L\log x) \le \alpha \le 1$. We deduce that

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \le L\int_0^{1/L\log x} \frac{1-x^{-2\alpha}}{\alpha}\,d\alpha + \frac{2}{\log x}\int_{1/L\log x}^{1} \frac{1-x^{-2\alpha}}{2\alpha}\frac{1}{\alpha}\,d\alpha + O\Big(\frac{1}{T} + \frac{\log\log x}{\log x}\Big). \tag{4.3}$$

Making a change of variables $y = 2\alpha\log x$, we see that the first integral above is

$$\begin{aligned}
&\le L\int_0^{2/L} \frac{1-e^{-y}}{y}\,dy \\
&= L\Big(\int_0^1 \frac{1-e^{-y}}{y}\,dy + \int_1^{2/L}\frac{dy}{y} - \int_1^\infty \frac{e^{-y}}{y}\,dy + \int_{2/L}^\infty \frac{e^{-y}}{y}\,dy\Big) \\
&= L\Big(\gamma + \log\frac{2}{L}\Big) + L\int_{2/L}^\infty \frac{e^{-y}}{y}\,dy,
\end{aligned}$$

since $\gamma = \int_0^1 (1-e^{-y})/y\,dy - \int_1^\infty e^{-y}/y\,dy$. Further, the second integral in (4.3) is

$$\frac{1}{\log x}\int_{1/L\log x}^{1} \frac{1-x^{-2\alpha}}{\alpha^2}\,d\alpha = 2\int_{2/L}^\infty \frac{1-e^{-y}}{y^2}\,dy = L - \int_{2/L}^\infty \frac{2e^{-y}}{y^2}\,dy.$$

Combining the above bounds, we see that the right side of (4.3) is

$$\le L\Big(1 + \log 2 + \log\frac{e^\gamma}{L} + \int_{2/L}^\infty \frac{e^{-y}}{y}\Big(1 - \frac{2/L}{y}\Big)\,dy\Big) + O\Big(\frac{1}{T} + \frac{\log\log x}{\log x}\Big). \tag{4.4}$$

Since the maximum of $\big(1-(2/L)/y\big)\big/y$ for $y \ge 2/L$ is attained at $y = 4/L$, we see that the integral term above is $\le L/8\int_{2/L}^\infty e^{-y}\,dy \le 1/8\int_2^\infty e^{-y}\,dy = 1/(8e^2)$ since $L \le 1$, and theorem then follows from (4.4) since $1 + \log 2 + 1/(8e^2) \le 12/7$.

We now deduce Corollary 1. Suppose f is completely multiplicative. Then, by Mertens' theorem,

$$\begin{aligned}
|F(1+iy)| &= \big(e^\gamma\log x + O(1)\big)\prod_{p\le x}\Big|1 - \frac{f(p)}{p^{1+iy}}\Big|^{-1}\Big(1-\frac{1}{p}\Big) \\
&= \big(e^\gamma\log x + O(1)\big)\exp\Big(-\sum_{\substack{p\le x\\ k\ge 1}} \frac{1 - \operatorname{Re} f(p^k)p^{-iky}}{kp^k}\Big),
\end{aligned} \tag{4.5}$$

and so it follows that $L \le e^{\gamma - M} + O(1/\log x)$. Using this bound in Theorem 1, we get the completely multiplicative case of Corollary 1.

If f is only known to be multiplicative then note that

$$\left|1+\frac{f(p)}{p^{1+iy}}+\frac{f(p^2)}{p^{2+2iy}}+\cdots\right|\cdot\left|1-\frac{f(p)}{p^{1+iy}}\right|\leq 1+\frac{2}{p(p-1)},$$

since $|f(p^k)|\leq 1$ for all k. Using this with the observation of the preceding paragraph, we see that $L\leq\prod_p(1+\frac{2}{p(p-1)})e^{\gamma-M}+O(1/\log x)$ in this case. Appealing now to Theorem 1, and noting that $\log\bigl(\prod_p(1+\frac{2}{p(p-1)})\bigr)\geq 8/7$, we deduce this case of Corollary 1.

5 Proof of Theorem 2a

We may suppose that $|y_0|\geq 10$. Applying Theorem 1 with $T=|y_0|/2-1$ we get that

$$\frac{1}{x}\Bigl|\sum_{n\leq x}f(n)\Bigr|\ll\left(\frac{\max_{|y|\leq|y_0|-2}|F(1+iy)|}{\log x}\right)\log\left(\frac{e^{1+\gamma}\log x}{\max_{|y|\leq y_0-2}|F(1+iy)|}\right)$$

$$+\frac{1}{|y_0|+1}+\frac{\log\log x}{\log x}. \tag{5.1}$$

By the definition of y_0, we see that for $|y|\leq|y_0|-2$,

$$|F(1+iy)|\leq\bigl(|F(1+iy)F(1+iy_0)|\bigr)^{\frac{1}{2}},$$

and appealing to Lemma 2.3, this is (with $\log x\gg|\beta|=|y-y_0|\geq 2$)

$$\ll(\log x)^{\frac{2}{\pi}}(\log\log x)^{2(1-\frac{2}{\pi})}.$$

Using this bound in (5.1), we obtain the theorem.

6 Proof of Theorem 4

If $|y_0|\geq(\log x)/2$, then in view of Theorem 3a, the result follows. Thus we may assume that $|y_0|\leq(\log x)/2$. Put $f_0(n)=f(n)n^{-iy_0}$, and define

$$F_0(s)=\prod_{p\leq x}\bigl(1+f_0(p)p^{-s}+f_0(p^2)p^{-2s}+\cdots\bigr)=F(s+iy_0).$$

We note that

$$\bigl(|F(1+iy_0)|=\bigr)\quad|F_0(1)|=\max_{|y|\leq\log x}|F_0(1+iy)|. \tag{6.1}$$

Indeed, the left side of (6.1) is plainly $\leq$ right side; and further the right side is $=\max_{|y|\leq\log x}|F(1+iy+iy_0)|\leq\max_{|y|\leq\log x+|y_0|}|F(1+iy)|\leq|F(1+iy_0)|$, proving (6.1).

We now appeal to Proposition 3.3, with f there replaced by f_0, and F by F_0, and with $T = (\log x)/2$. Thus we see that

$$\Big|\frac{1}{x}\sum_{n\le x} f_0(n) - \frac{w}{x}\sum_{n\le x/w} f_0(n)\Big| \ll \frac{\log 2w}{\log x}\log\Big(\frac{\log x}{\log 2w}\Big)$$

$$(6.2)\qquad + \frac{1}{\log x}\int_0^1 \min\Big(\log x, \frac{1}{\alpha}\Big)\Big(\max_{|y|\le(\log x)/2}|F_0(1+\alpha+iy)(1-w^{-\alpha-iy})|\Big)\, d\alpha.$$

Next, we use Lemma 2.2 with $a_n = f_0(n)$ if n is divisible only by primes $\le x$, and $a_n = 0$ otherwise. Thus $A(s) = F_0(s)$, and $\sum_{n=1}^{\infty}|a_n|/n \ll \log x$. Taking $T = (\log x)/2$, we deduce from (2.6) of Lemma 2.2 that

$$(6.3)\qquad \max_{|y|\le(\log x)/2}|F_0(1+\alpha+iy)(1-w^{-\alpha-iy})| \le \max_{|y|\le \log x}|F_0(1+iy)(1-w^{-iy})| + O(1).$$

If $|y| \le 1/\log x$, then plainly $|F_0(1+iy)(1-w^{-iy})| \ll \log x(|y|\log 2w) \ll \log 2w$. If $\log x \ge |y| > 1/\log x$, then using (6.1) and Lemma 2.3, we get

$$|F_0(1+iy)| \le \Big(|F_0(1)F_0(1+iy)|\Big)^{\frac12} \ll (\log x)^{\frac{2}{\pi}}\max\Big(\frac{1}{|y|}, (\log\log x)^2\Big)^{(1-\frac{2}{\pi})}.$$

Since $|1-w^{-iy}| \ll \min(1, |y|\log 2w)$, we deduce from these remarks and (6.3) that (6.4)

$$\max_{|y|\le(\log x)/2}|F_0(1+\alpha+iy)(1-w^{-\alpha-iy})| \ll (\log x)^{\frac{2}{\pi}}\max\Big(\log 2w, (\log\log x)^2\Big)^{1-\frac{2}{\pi}}.$$

In addition, we have the trivial estimate

$$(6.5)\qquad \max_{|y|\le(\log x)/2}|F_0(1+\alpha+iy)(1-w^{-\alpha-iy})| \ll \zeta(1+\alpha) \ll \frac{1}{\alpha}.$$

We now use (6.2), employing estimate (6.4) when α is less than

$$\max\Big(\log 2w, (\log\log x)^2\Big)^{-(1-\frac{2}{\pi})}(\log x)^{-\frac{2}{\pi}},$$

and estimate (6.5) for larger α. This gives the theorem.

7 Deduction of Corollary 3 and of Theorem 2b

We require the following lemma, which relates the mean value of $f(n)$ to the mean value of $f(n)n^{i\alpha}$.

Lemma 7.1 *Suppose $f(n)$ is a multiplicative function with $|f(n)| \le 1$ for all n. Then for any real number α we have*

$$\sum_{n\le x} f(n)n^{i\alpha} = \frac{x^{i\alpha}}{1+i\alpha}\sum_{n\le x} f(n) + O\bigg(\frac{x}{\log x}\log(e+|\alpha|)\exp\Big(\sum_{p\le x}\frac{|1-f(p)|}{p}\Big)\bigg).$$

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To prove this Lemma, we require a consequence of Theorem 2 of Halberstam and Richert [10]. Suppose h is a non-negative multiplicative function with $h(p^k) \le 2$ for all prime powers p^k. It follows from Theorem 2 of [10] that

$$\sum_{n\le x} h(n) \le \frac{2x}{\log x}\sum_{n\le x}\frac{h(n)}{n}\Big\{1+O\Big(\frac{1}{\log x}\Big)\Big\}. \tag{7.1}$$

Using partial summation we deduce from (7.1) that for $1 \le y \le x^{1/2}$,

$$\sum_{x/y<n\le x}\frac{h(n)}{n} \le \Big\{\frac{1}{\log x} - \log\Big(1-\frac{\log y}{\log x}\Big)\Big\}\sum_{n\le x}\frac{h(n)}{n}\Big\{2+O\Big(\frac{1}{\log x}\Big)\Big\}. \tag{7.2}$$

Proof of Lemma 7.1 Let g denote the multiplicative function defined by $g(p^k) = f(p^k) - f(p^{k-1})$, so that $f(n) = \sum_{d|n} g(d)$. Then

$$\sum_{n\le x} f(n)n^{i\alpha} = \sum_{n\le x} n^{i\alpha}\sum_{d|n} g(d) = \sum_{d\le x} g(d)\, d^{i\alpha}\sum_{n\le x/d} n^{i\alpha}. \tag{7.3}$$

By partial summation it is easy to see that

$$\sum_{n\le z} n^{i\alpha} = \begin{cases} \frac{z^{1+i\alpha}}{1+i\alpha} + O(1+\alpha^2) \\ O(z). \end{cases}$$

We use the first estimate above in (7.3) when $d \le x/(1+\alpha^2)$, and the second estimate when $x/(1+\alpha^2) \le d \le x$. This gives

$$\sum_{n\le x} f(n)n^{i\alpha} = \frac{x^{1+i\alpha}}{1+i\alpha}\sum_{d\le x}\frac{g(d)}{d} + O\Big((1+\alpha^2)\sum_{d\le x/(1+\alpha^2)}|g(d)| + x\sum_{x/(1+\alpha^2)\le d\le x}\frac{|g(d)|}{d}\Big).$$

Applying (7.1) and (7.2) we deduce that

$$\begin{aligned}\sum_{n\le x} f(n)n^{i\alpha} &= \frac{x^{1+i\alpha}}{1+i\alpha}\sum_{d\le x}\frac{g(d)}{d} + O\Big(\frac{x}{\log x}\log(e+|\alpha|)\sum_{d\le x}\frac{|g(d)|}{d}\Big)\\ &= \frac{x^{1+i\alpha}}{1+i\alpha}\sum_{d\le x}\frac{g(d)}{d} + O\Big(\frac{x}{\log x}\log(e+|\alpha|)\exp\Big(\sum_{p\le x}\frac{|1-f(p)|}{p}\Big)\Big).\end{aligned}$$

Using the above estimate twice, once with α replaced by 0, we obtain the lemma.

Proof of Corollary 3 We may suppose that $w \le \sqrt{x}$, else there's nothing to prove. Let y_0 be as in Theorem 4. By the definition of M and by (4.5) we know that for all $|y| \le 2\log x$,

$$\sum_{p\le x}\frac{1-\operatorname{Re} f(p)p^{-iy}}{p} \ge M = \sum_{p\le x}\frac{1-\operatorname{Re} f(p)p^{-iy_0}}{p} + O(1).$$

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Further we have for $|y| \leq 2\log x$

$$\sum_{p\leq x/w} \frac{1-\operatorname{Re} f(p)p^{-iy}}{p} \geq \sum_{p\leq x} \frac{1-\operatorname{Re} f(p)p^{-iy}}{p} - 2\sum_{x/w\leq p\leq x} \frac{1}{p} \geq M + O(1).$$

By Corollary 1 (with $T = \log x$) it follows that

$$\frac{1}{x}\Big|\sum_{n\leq x} f(n)\Big|, \quad \frac{1}{x/w}\Big|\sum_{n\leq x/w} f(n)\Big| \ll Me^{-M} + \frac{\log\log x}{\log x}.$$

From this estimate, Corollary 3 follows if $M \geq (2-\sqrt{3})\log\log x$. We suppose now that $M \leq (2-\sqrt{3})\log\log x$.

For a complex number z in the unit disc, we have $|1-z| = (1+|z|^2 - 2\operatorname{Re} z)^{\frac{1}{2}} \leq (2-2\operatorname{Re} z)^{\frac{1}{2}}$. Hence, by Cauchy's inequality and our bound on M,

$$\begin{aligned}
\sum_{p\leq x} \frac{|1-f(p)p^{-iy_0}|}{p} &\leq \sum_{p\leq x} \frac{\sqrt{2-2\operatorname{Re} f(p)p^{-iy_0}}}{p} \\
&\leq \Big(\sum_{p\leq x} \frac{2}{p}\Big)^{\frac{1}{2}} \Big(\sum_{p\leq x} \frac{1-\operatorname{Re} f(p)p^{-iy_0}}{p}\Big)^{\frac{1}{2}} \\
&\leq \big(2(2-\sqrt{3})\big)^{\frac{1}{2}} \log\log x + O(1) \\
&= (\sqrt{3}-1)\log\log x + O(1). \qquad (7.4)
\end{aligned}$$

Applying Lemma 7.1, we see that

$$\begin{aligned}
\frac{1}{x}\sum_{n\leq x} f(n) &= \frac{x^{iy_0}}{1+iy_0}\sum_{n\leq x} f(n)n^{-iy_0} + O\Big(\frac{\log\log x}{\log x}\exp\Big(\sum_{p\leq x}\frac{|1-f(p)p^{-iy_0}|}{p}\Big)\Big) \\
&= \frac{x^{iy_0}}{1+iy_0}\sum_{n\leq x} f(n)n^{-iy_0} + O\Big(\frac{\log\log x}{(\log x)^{2-\sqrt{3}}}\Big),
\end{aligned}$$

and similarly

$$\frac{w}{x}\sum_{n\leq x/w} f(n) = \frac{(x/w)^{iy_0}}{1+iy_0}\sum_{n\leq x/w} f(n)n^{-iy_0} + O\Big(\frac{\log\log x}{(\log x)^{2-\sqrt{3}}}\Big).$$

Taking absolute values in these relations, and appealing to Theorem 4, we obtain the corollary.

Proof of Theorem 2b Suppose that the maximum in (1.3) with $T = \log x$ is attained at $y = y_0$. If $|y_0| \geq \log x$ the result follows from Theorem 2a. Thus we may assume $|y_0| < \log x$. Let $g(n) = f(n)/n^{iy_0}$ so that the maximum in (1.3) with f replaced by

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g and $T = \frac{1}{2}\log x$ is attained at $y = 0$ (for in the range there, $|y + y_0| \le |y| + |y_0| \le 2\log x$). Write $M = M_f(x, \log x) = M_g(x, \frac{1}{2}\log x)$. We will give the proof now assuming f is completely multiplicative (the proof for all multiplicative f is entirely analogous): By Corollary 1 (with $T = \frac{1}{2}\log x$) we have

$$\frac{1}{x}\Big|\sum_{n\le x} g(n)\Big| \le \Big(M + \frac{12}{7}\Big)e^{\gamma - M} + O\Big(\frac{\log\log x}{\log x}\Big).$$

By Lemma 7.1 with f replaced by g, and α by y_0, we have

$$\sum_{n\le x} f(n) = \frac{x^{iy_0}}{1+iy_0}\sum_{n\le x} g(n) + O\bigg(\frac{x}{\log x}\log\log x \exp\Big(\sum_{p\le x}\frac{|1-g(p)|}{p}\Big)\bigg).$$

Combining these two statements gives, since $\sum_{p\le x}|1-g(p)|/p \le \sqrt{2M\log\log x} + O(1)$ by Cauchy's inequality,

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \le \frac{1}{\sqrt{1+y_0^2}}\Big(M + \frac{12}{7}\Big)e^{\gamma - M} + O\bigg(\frac{\log\log x}{\log x}\exp\Big(\sqrt{2M\log\log x}\Big)\bigg).$$

The result follows from this provided $M \le (2-\sqrt{3})\log\log x$; and it follows from Corollary 1 directly if $M > (2-\sqrt{3})\log\log x$.

8 Proof of Theorem 5

We recall the notations of Section 3a. We first obtain a lower bound for $M_+(t)$ in terms of $M_0 = \int_0^u \frac{1-\operatorname{Re}\chi(v)}{v}\,dv$.

Proposition 8.1 *For all $u \ge 1$ and $t > 0$ we have*

$$M_+(t) \ge \max\Big(0, \kappa M_0 - \kappa\nu\log(tu) + (1-\kappa\nu)\int_{tu}^\infty \frac{e^{-v}}{v}\,dv + C(D)\Big),$$

where $C(D)$ was defined in (1.6).

Proof First note that $M_+(t) \ge 0$ by definition. Also

$$M_+(t) - \kappa M_0 = I - \kappa\int_0^u \frac{1-\operatorname{Re}\chi(v)}{v}(1-e^{-tv})\,dv + \int_u^\infty \frac{e^{-tv}}{v}\,dv$$

where

$$I := \min_{y\in\mathbb{R}} \int_0^u \frac{1-\kappa-\operatorname{Re}\chi(v)(e^{-ivy}-\kappa)}{v}e^{-tv}\,dv. \tag{8.1}$$

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Since $1 - \operatorname{Re}\chi(v) \le \nu$, we get that

$$\begin{aligned}
-\kappa \int_0^u \frac{1-\operatorname{Re}\chi(v)}{v}(1-e^{-tv})\,dv + \int_u^\infty \frac{e^{-tv}}{v}\,dv \\
&\ge -\kappa\nu\Big(\int_0^u \frac{1-e^{-tv}}{v}\,dv - \int_u^\infty \frac{e^{-tv}}{v}\,dv\Big) + (1-\kappa\nu)\int_u^\infty \frac{e^{-tv}}{v}\,dv \\
&= -\kappa\nu\big(\gamma + \log(tu)\big) + (1-\kappa\nu)\int_{tu}^\infty \frac{e^{-v}}{v}\,dv,
\end{aligned}$$

so that

$$M_+(t) \ge \kappa M_0 - \kappa\nu\big(\gamma+\log(tu)\big) + (1-\kappa\nu)\int_{tu}^\infty \frac{e^{-v}}{v}\,dv + I.$$

Therefore we obtain the Proposition by proving

$$I \ge \min_{\epsilon=\pm1}\int_0^{2\pi} \frac{\min\big(0, 1-\kappa-\max_{\delta\in D}\operatorname{Re}\delta(e^{\epsilon ix}-\kappa)\big)}{x}\,dx. \tag{8.2}$$

If the minimum in (8.1) occurs at $y = 0$, then $I = (1-\kappa)\int_0^u \frac{1-\operatorname{Re}\chi(v)}{v}e^{-tv}\,dv \ge 0$, which is stronger than (8.2). So we may suppose that the minimum in (8.1) occurs for some $y \ne 0$. Put $w(\theta) = 1-\kappa-\max_{\delta\in D}\operatorname{Re}\delta(e^{-i\theta}-\kappa)$. Then we see that, with $\epsilon = \operatorname{sgn}(y)$,

$$I \ge \int_0^u \frac{w(vy)}{v}e^{-tv}\,dv = \int_0^{|y|u}\frac{w(\epsilon v)}{v}e^{-tv/|y|}\,dv = \int_0^{|y|u}\frac{e^{-tv/|y|}}{v}\,d\Big(\int_0^v w(\epsilon x)\,dx\Big).$$

Integrating by parts, we conclude that

$$I \ge \frac{e^{-tu}}{|y|u}\int_0^{|y|u} w(\epsilon x)\,dx + \int_0^{|y|u}\Big(\int_0^v w(\epsilon x)\,dx\Big)\Big(\frac{e^{-tv/|y|}}{v^2} + \frac{te^{-tv/|y|}}{v|y|}\Big)\,dv. \tag{8.3}$$

Note that $w(\epsilon x)$ is a 2π-periodic function, and that $\frac{1}{2\pi}\int_0^{2\pi} w(\epsilon x)\,dx = 1-\bar h(\kappa) \ge 0$. Hence putting $w^-(x) = \min\big(0, w(x)\big)$, we get that

$$\int_0^v w(\epsilon x)\,dx \ge \int_{2\pi[\frac{v}{2\pi}]}^v w^-(\epsilon x)\,dx = W_\epsilon(v), \tag{8.4}$$

say. Observe that W_ϵ is a 2π-periodic function, which is always negative, and that W_ϵ is decreasing in $(0, 2\pi)$.

Using (8.4) in (8.3), and since W_ϵ is negative and $e^{-x}(1+x) \le 1$ for all $x \ge 0$, we get that

$$\begin{aligned}
I &\ge \frac{e^{-tu}}{|y|u}W_\epsilon(|y|u) + \int_0^{|y|u}\frac{W_\epsilon(v)}{v^2}e^{-tv/|y|}\Big(1+\frac{tv}{|y|}\Big)\,dv \\
&\ge \frac{W_\epsilon(|y|u)}{|y|u} + \int_0^{|y|u}\frac{W_\epsilon(v)}{v^2}\,dv. \qquad (8.5)
\end{aligned}$$

If $\alpha \geq 2\pi$ then, since $W_\epsilon(v) \geq W_\epsilon(2\pi-)$ $(= \int_0^{2\pi} w^-(\epsilon x)\,dx)$, we get

$$\begin{aligned}\frac{W_\epsilon(\alpha)}{\alpha} + \int_0^\alpha \frac{W_\epsilon(v)}{v^2}\,dv &\geq \frac{W_\epsilon(2\pi-)}{\alpha} + \int_0^{2\pi} \frac{W_\epsilon(v)}{v^2}\,dv + W_\epsilon(2\pi-)\int_{2\pi}^\alpha \frac{dv}{v^2} \\ &= \int_0^{2\pi} \frac{W_\epsilon(v)}{v^2} + \frac{W_\epsilon(2\pi-)}{2\pi}.\end{aligned}$$

If $\alpha < 2\pi$ then, since $W_\epsilon(x)$ is decreasing in $(0, 2\pi)$,

$$\begin{aligned}\frac{W_\epsilon(\alpha)}{\alpha} + \int_0^\alpha \frac{W_\epsilon(v)}{v^2}\,dv &\geq \frac{W_\epsilon(\alpha)}{\alpha} + \int_0^{2\pi} \frac{W_\epsilon(v)}{v^2} - W_\epsilon(\alpha)\int_\alpha^{2\pi} \frac{dv}{v^2} \\ &= \int_0^{2\pi} \frac{W_\epsilon(v)}{v^2}\,dv + \frac{W_\epsilon(\alpha)}{2\pi} \\ &\geq \int_0^{2\pi} \frac{W_\epsilon(v)}{v^2}\,dv + \frac{W_\epsilon(2\pi-)}{2\pi}.\end{aligned}$$

Using these in (8.5), we conclude that

$$I \geq \int_0^{2\pi} \frac{W_\epsilon(v)}{v^2}\,dv + \frac{W_\epsilon(2\pi-)}{2\pi} = \int_0^{2\pi} \frac{w^-(\epsilon x)}{x}\,dx,$$

which, from the definition of w^-, is greater than or equal to the right side of (8.2) for both $\epsilon = \pm 1$. This completes the proof of the proposition.

We now finish the proof of Theorem 5. We first deal with the case $D \neq [0, 1]$, where $\kappa\nu < 1$. We shall input the bounds for $M_+(t)$ in Proposition 8.1 into the t-integral in Proposition 3.1. We split this integral into three parts: when $0 \leq t \leq t_1 := e^{-\gamma}/u$, when $t_1 \leq t \leq t_2 := \exp(\frac{M_0}{\nu} + \frac{C(D)}{\kappa\nu})/u$, and when $t > t_2$.

We first estimate the contribution of the first range of t. Since

$$\int_{tu}^\infty \frac{e^{-v}}{v}\,dv \geq \int_{tu}^1 \frac{dv}{v} - \int_0^1 \frac{1-e^{-v}}{v}\,dv + \int_1^\infty \frac{e^{-v}}{v}\,dv = -\log(tu) - \gamma,$$

and $1 - \kappa\nu \geq 0$, we see that $M_+(t) \geq \kappa M_0 - \log(tu) + C(D) - \gamma(1-\kappa\nu)$, by Proposition 8.1. Hence, with a little calculation,

$$\begin{aligned}\int_0^{t_1} \Big(\frac{1-e^{-2tu}}{t}\Big) \frac{\exp\big(-M_+(t)\big)}{tu}\,dt &\leq \exp\big(-\kappa M_0 - C(D) + \gamma(1-\kappa\nu)\big) \\ &\qquad \times \int_0^{2e^{-\gamma}} \frac{1-e^{-x}}{x}\,dx \\ &\leq \exp\big(-\kappa M_0 - C(D) + \gamma(1-\kappa\nu)\big).\end{aligned}$$

For the middle range of t, we use the bound $M_+(t) \geq \kappa M_0 - \kappa\nu \log(tu) + C(D)$, which holds since $1 - \kappa\nu$, and $\int_{tu}^{\infty} \frac{e^{-v}}{v}\,dv$ are non-negative. Hence

$$\int_{t_1}^{t_2} \Big(\frac{1 - e^{-2tu}}{t}\Big) \frac{\exp\big(-M_+(t)\big)}{tu}\,dt \leq \exp\big(-\kappa M_0 - C(D)\big) \int_{t_1}^{t_2} \frac{(tu)^{\kappa\nu}}{tu}\frac{dt}{t}$$
$$= \frac{\exp\big(-\kappa M_0 - C(D) + \gamma(1-\kappa\nu)\big)}{1-\kappa\nu} - \frac{\exp(-\frac{M_0}{\nu} - \frac{C(D)}{\kappa\nu})}{1-\kappa\nu}.$$

For the last range of t, we use the trivial bound $M_+(t) \geq 0$. This gives that

$$\int_{t_2}^{\infty} \Big(\frac{1 - e^{-2tu}}{t}\Big) \frac{\exp\big(-M_+(t)\big)}{tu}\,dt \leq \int_{t_2}^{\infty} \frac{dt}{t^2u} = \exp\Big(-\frac{M_0}{\nu} - \frac{C(D)}{\kappa\nu}\Big).$$

Combining the above three bounds with Proposition 3.1, we obtain Theorem 5 in the case $\kappa\nu < 1$.

We now consider the case $D = [0, 1]$ where we shall show that $|\sigma(u)| \leq e^{\gamma - M_0}$. Put $\hat{\chi}(t) = \chi(t)$ if $t \leq u$, and $\hat{\chi}(t) = 0$ for $t > u$, and let $\hat{\sigma}$ denote the corresponding solution to (1.8). Note that both $\sigma(v)$ and $\hat{\sigma}(v)$ are non-negative for all v, and that $\hat{\sigma}(v) = \sigma(v)$ for $v \leq u$. Now, using (3.6),

$$\sigma(u) = \frac{1}{u}\int_0^u \sigma(v)\chi(u-v)\,dv \leq \frac{1}{u}\int_0^u \sigma(v)\,dv \leq \frac{1}{u}\int_0^{\infty} \hat{\sigma}(v)\,dv = \frac{1}{u}\lim_{t\to 0} \mathcal{L}(\hat{\sigma}, t)$$
$$= \frac{1}{u}\lim_{t\to 0}\frac{1}{t}\exp\Big(-\mathcal{L}\Big(\frac{1-\hat{\chi}(v)}{v}, t\Big)\Big) = e^{-M_0}\lim_{t\to 0}\frac{1}{tu}\exp\Big(-\int_u^{\infty}\frac{e^{-tv}}{v}\,dv\Big)$$
$$= e^{-M_0}\lim_{y\to 0}\frac{1}{y}\exp\Big(-\int_y^{\infty}\frac{e^{-v}}{v}\,dv\Big) = e^{\gamma - M_0},$$

which proves the theorem in this case.

9 Deduction of Theorem 3

Let $y = \exp\big((\log x)^{\frac{2}{3}}\big)$, and let g be the completely multiplicative function with $g(p) = 1$ for $p \leq y$, and $g(p) = f(p)$ for larger p. Let $\chi(t) = 1$ for $t \leq 1$, and put for $t > 1$

$$\chi(t) = \frac{1}{\vartheta(y^t)} \sum_{p \leq y^t} g(p)\log p.$$

Let σ denote the corresponding solution to (1.8). Note that for $u \geq 1$

$$\int_0^u \frac{1 - \operatorname{Re}\chi(v)}{v}\,dv = \int_1^u \frac{1}{v\vartheta(y^v)} \sum_{y<p\leq y^v} \big(1 - \operatorname{Re} f(p)\big)\log p\,dv$$
$$= \int_1^u \frac{1}{vy^v} \sum_{y<p\leq y^v} \big(1 - \operatorname{Re} f(p)\big)\log p\,dv + O\Big(\frac{1}{\log y}\Big),$$

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upon using the prime number theorem. Interchanging the sum and the integral, the above is

$$
\begin{aligned}
&= \sum_{y<p\le y^u} \big(1-\operatorname{Re} f(p)\big) \log p \int_{\log p/\log y}^{u} \frac{dv}{v y^v} + O\Big(\frac{1}{\log y}\Big) \\
&= \sum_{y<p\le y^u} \big(1-\operatorname{Re} f(p)\big) \log p \Big(\frac{1}{p\log p} + O\Big(\frac{1}{p\log^2 p} + \frac{1}{u y^u \log y}\Big)\Big) \\
&\qquad + O\Big(\frac{1}{\log y}\Big).
\end{aligned}
$$

We conclude that

$$
\int_0^u \frac{1-\operatorname{Re}\chi(v)}{v}\,dv = \sum_{y<p\le y^u} \frac{1-\operatorname{Re} f(p)}{p} + O\Big(\frac{1}{\log y}\Big). \tag{9.1}
$$

Appealing to Propositions 3 and then 2 we obtain that

$$
\begin{aligned}
\frac{1}{x}\sum_{n\le x} f(n) &= \Theta(f,y)\frac{1}{x}\sum_{n\le x} g(n) + O\bigg(\frac{1}{(\log x)^{\frac{1}{3}}}\exp\Big(\sum_{p\le x}\frac{|1-f(p)|}{p}\Big)\bigg) \\
&= \Theta(f,y)\sigma\Big(\frac{\log x}{\log y}\Big) + O\bigg(\frac{1}{(\log x)^{\frac{1}{3}}}\exp\Big(\sum_{p\le x}\frac{|1-f(p)|}{p}\Big)\bigg).
\end{aligned}
$$

Since $f(p) \in D$ for all p and D is convex, thus $\chi(t) \in D$ for all t. Hence using Theorem 5 and (9.1), we conclude that

$$
\begin{aligned}
\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| &\le |\Theta(f,y)|\Big(\frac{2-\kappa\nu}{1-\kappa\nu}\Big) \\
&\qquad \times \exp\bigg(-\kappa\sum_{y\le p\le x}\frac{1-\operatorname{Re} f(p)}{p} - C(D) + \gamma(1-\kappa\nu)\bigg) \\
&\qquad + O\bigg(\frac{1}{(\log x)^{\frac{1}{3}}}\exp\Big(\sum_{p\le x}\frac{|1-f(p)|}{p}\Big)\bigg),
\end{aligned}
$$

which completes the proof of the first part of Theorem 3.

In fact the second part of Theorem 3 follows from Lemma 2.1 for, from (2.1) we have

$$
(\log x + 1)\frac{1}{x}\sum_{n\le x} f(n) \le \sum_{n\le x}\frac{f(n)}{n} + O(1) = e^{\gamma}\log x\,\Theta(f,x) + O(1)
$$

using Mertens' theorem, and the result follows.

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10 Explicit Constructions

10a Examining Proper Subregions D of $\mathbb{U}$ is Necessary

As we remarked after Theorem 5, it is not, *a priori*, clear that one should look at proper subsets D of $\mathbb{U}$ when looking for bounds (of the shape of Theorem 2) on solutions to (1.8). However if we take $\chi(t) = 1$ for $t \le 1$, and $\chi(t) = e^{i\alpha t}$ for $t > 1$ then

$$\mathcal{L}\Big(\frac{1 - \operatorname{Re}\chi(v)e^{-iv\alpha}}{v}, t\Big) = \int_0^1 \Big(\frac{1-\cos v\alpha}{v}\Big) e^{-tv}\,dv \to \int_0^1 \frac{1-\cos v\alpha}{v}\,dv \gg_\alpha 1,$$

as $t \to 0$. Therefore, by (3.6),

$$\int_0^\infty |\sigma(v)|e^{-tv}\,dv \ge |\mathcal{L}(\sigma, t+i\alpha)| \gg_\alpha 1/t,$$

if t is sufficiently small. Now $\int_{b/t}^\infty |\sigma(v)|e^{-tv}\,dv \le \int_{b/t}^\infty e^{-tv}\,dv \le e^{-b}/t$ for any $b > 0$ and so $\int_0^{b/t} |\sigma(v)|e^{-tv}\,dv \gg_\alpha 1/t$ if b is sufficiently large. Taking $N = b/t$ we deduce that if N is sufficiently large then $\int_0^N |\sigma(v)|\,dv \gg_\alpha N$, and so $\limsup |\sigma(u)| \gg_\alpha 1$.

However

$$M_0(u,\chi) = \int_0^u \frac{1-\operatorname{Re}\chi(v)}{v}\,dv = \int_1^u \frac{1-\cos\alpha v}{v}\,dv = \log u + O_\alpha(1),$$

so no estimate of the shape $|\sigma(u)| \ll \exp(-\kappa M_0)$ can hold (with $\kappa > 0$), as in Theorem 5.

10b Corollary 1′ is Best Possible, Up to the Constant

Assume that Corollary 1′ is not best possible, so that if $M = M(u)$ is sufficiently large then $|\sigma(u)| \le \epsilon M e^{-M}$.

Select u sufficiently large, and choose $\chi(t) = 1$ for $t \le 1$, $\chi(t) = i$ for $1 < t \le u/2$, and $\chi(t) = 0$ for $t > u/2$; let σ denote the corresponding solution to (1.8). Next we take $\hat\chi(t) = \chi(t)$ for $t \le u/2$, or $t > u$, and for $u/2 < t < u$ choose $\hat\chi(t)$ to be a unit vector pointing in the direction of $\overline{\sigma(u-t)}$. Let $\hat\sigma$ denote the corresponding solution to (1.8). By definition we have $\hat\sigma(u-t) = \sigma(u-t)$ in the range $u/2 \le t \le u$; and so $\hat\chi(t)\hat\sigma(u-t) = |\sigma(u-t)|$ throughout this range, by our choice of $\hat\chi(t)$. From (1.9) and then this observation we deduce

(10.1)
$$\hat\sigma(u) - \sigma(u) = \int_{u/2}^u \frac{\hat\chi(t)}{t}\hat\sigma(u-t)\,dt = \int_{u/2}^u \frac{|\sigma(u-t)|}{t}\,dt \ge \frac{1}{u}\int_0^{u/2} |\sigma(v)|\,dv.$$

Multiplicative functions such as this have been explored in some detail in the literature: Let α be a complex number with $\operatorname{Re}(\alpha) < 1$, and let ρ_α denote the unique continuous solution to $u\rho_\alpha'(u) = -(1-\alpha)\rho_\alpha(u-1)$, for $u \ge 1$, with the initial condition $\rho_\alpha(u) = 1$ for $u \le 1$ (The Dickman–De Bruijn function is the case $\alpha = 0$.) For

$\alpha \in [0,1]$, Goldston and McCurley [5] gave an asymptotic expansion of ρ_α. Their proof is in fact valid for all complex α with $\mathrm{Re}(\alpha) < 1$, and shows that when α is not an integer

$$\rho_\alpha(u) \sim \frac{e^{\gamma(1-\alpha)}}{\Gamma(\alpha)u^{1-\alpha}},$$

as $u \to \infty$ (Curiously, when α is an integer the behaviour of ρ_α is very different; in fact $\rho_\alpha(u) = 1/u^{u+o(u)}$). We have $\sigma(v) = \rho_i(v)$ for $v \le u/2$, and so in (10.1) we get: $\hat\sigma(u) - \sigma(u) = \{c + o(1)\} \log u/u$ where $c = e^\gamma/|\Gamma(i)| = 3.414868086\cdots$.

Now we note that

$$\begin{aligned} M(u) &= \min_{y\in\mathbb{R}} \int_0^u \frac{1 - \mathrm{Re}\,\chi(v)e^{-ivy}}{v}\,dv \\ &= \min_{y\in\mathbb{R}} \left(\int_0^1 \frac{1-\cos(vy)}{v}\,dv + \int_1^{u/2} \frac{1-\sin(vy)}{v}\,dv + \log 2 \right), \\ &\ge \log u + \min_{y\in\mathbb{R}} \int_0^y \frac{1-\cos t + \sin t}{t}\,dt - \max_{Y\in\mathbb{R}} \int_0^Y \frac{\sin t}{t}\,dt \\ &\ge \log u - 1.851937052\cdots \end{aligned}$$

and similarly $\hat M(u) \ge \log(u/2) - 1.851937052\cdots$. Let

$$c' = e^{1.851937052\cdots} = 6.372150763\cdots.$$

Therefore $Me^{-M} \le \{c' + o(1)\}\log u/u$ and $\hat M e^{-\hat M} \le \{2c' + o(1)\}\log u/u$, so that

$$|\hat\sigma(u)| + |\sigma(u)| \ge \{c + o(1)\}\log u/u \ge \{c/3c' + o(1)\}(\hat M e^{-\hat M} + Me^{-M}).$$

Thus either $|\sigma(u)| \ge (5/28)Me^{-M}$ or $|\hat\sigma(u)| \ge (5/28)\hat M e^{-\hat M}$, which implies the remarks following Corollaries 1 and 1′ since $c/3c' > 5/28 > e^\gamma/10$.

11 Bounds on Least Members of Cosets of the k-th Powers

11a Bounds For τ_k: k Large

Let f be a completely multiplicative function which takes values on the k-th roots of unity. Suppose x is a large integer such that for each k-th root of unity ξ there are between $(1-\epsilon)x/k$ and $(1+\epsilon)x/k$ integers n below x with $f(n) = \xi$, for some given $\epsilon > 0$. It follows that

$$\sum_{n\le x} f(n)^j \le \{\epsilon + o(1)\}x, \tag{11.1}$$

for each $j = 1,\dots,k-1$. Now suppose $1 \le w \le o(x)$ and observe that

$$\sum_{\substack{n\le x/w\\ f(n)=\xi}} 1 = \frac{1}{k}\sum_{j=0}^{k-1} \xi^{-j} \sum_{n\le x/w} f(n)^j = \frac{1}{k}\left[\frac{x}{w}\right] + \frac{1}{k}\sum_{j=1}^{k-1}\xi^{-j}\sum_{n\le x/w} f(n)^j.$$

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Using (11.1) together with Corollary 3 we conclude that

$$\sum_{\substack{n\le x/w\\ f(n)=\xi}} 1 \ge \frac{x}{w}\Big(\frac{1}{k}-\epsilon - C\Big(\frac{\log 2w}{\log x}\Big)^{1-\frac{2}{\pi}}\log\Big(\frac{\log x}{\log 2w}\Big)+o(1)\Big),$$

for some absolute constant C. If $\epsilon \le k/2$ and $w < x^{c/(k\log k)^{1/(1-2/\pi)}}$ for a suitable constant $c > 0$ then the above is positive, so that $\tau_k < 1 - c/(k\log k)^{1/(1-2/\pi)}$, and our desired bound for η_k, the first part of Corollary 4, follows.

In the case that k is prime we may improve our bound for τ_k by modifying the argument of Davenport and Erdős [2]. Let ϵ, f and x be as above, and suppose that ξ is a k-th root of unity such that $f(n) \ne \xi$ for all $n \le X = x^{\tau_k+o(1)}$. Plainly $f(p) = 1$ for all $p \le X^{1/(k-1)} =: y$, otherwise $\xi = f(p)^j = f(p^j)$ for some $1 \le j \le k-1$ contradicting $f(n) \ne \xi$ for all $n \le X$. Suppose $X \le n \le x$ with $f(n) = \xi$. Write $n = rs$ where $p|r \Rightarrow p \le y$, and $p|s \Rightarrow p > y$. Then $\xi = f(n) = f(r)f(s) = f(s)$ and so we must have $s > X$. Hence

$$(1-\epsilon)\frac{x}{k} \le \sum_{\substack{n\le x\\ f(n)=\xi}} 1 \le \sum_{\substack{X\le s\le x\\ p|s\Rightarrow p>y}}\ \sum_{r\le x/s} 1 \le x \sum_{\substack{X\le s\le x\\ p|s\Rightarrow p>y}} \frac{1}{s}. \tag{11.2}$$

The right side above may be estimated using knowledge of the distribution of integers free of small prime factors (see Theorem 3 of Chapter III.6 of [18]). Using this result and partial summation we get that

$$\sum_{\substack{X\le s\le x\\ p|s\Rightarrow p>y}} \frac{1}{s} = \int_{\log X/\log y}^{\log x/\log y} \omega(z)\,dz + o(1),$$

where ω is Buchstab's function defined by $\omega(z) = 1/z$ for $1 \le z \le 2$ and for $z > 2$ it is the unique continuous solution to the differential-difference equation $\big(u\omega(u)\big)' = \omega(u-1)$. As $z \to \infty$ we have $\omega(z) = e^{-\gamma} + O(z^{-z+o(z)})$ (see Theorem 4 of III.6 of [18]) and hence

$$\sum_{\substack{X\le s\le x\\ p|s\Rightarrow p>y}} \frac{1}{s} = \frac{\log(x/X)}{\log y}e^{-\gamma} + O(k^{-k+o(k)}).$$

Using this in (11.2) we conclude that $e^{-\gamma}(k-1)(1-\tau_k)/\tau_k + O(k^{-k+o(k)}) \ge (1-\epsilon)/k$, and our desired bound on τ_k follows.

11b Evaluating τ_2 and τ_3

That $\tau_2 = 1/\sqrt{e}$ is essentially a classical observation of Vinogradov. First we show that $\tau_2 \le 1/\sqrt{e}$. Suppose f is a completely multiplicative function with $f(n) = \pm 1$.

Suppose x is such that both $\{n \le x : f(n) = 1\}$ and $\{n \le x : f(n) = -1\}$ have cardinality $\sim x/2$. Let n_1 be the first time $f(n) = -1$. Plainly we may suppose that $n_1 > \sqrt{x}$. If $n \le x$ has all prime factors below n_1 then $f(n) = 1$. The number of such integers is $\sim x\big(1 - \log(\log x/\log n_1)\big)$ and so we conclude that $n_1 \ge x^{1/\sqrt{e}+o(1)}$ as desired. To see that $\tau_2 \ge 1/\sqrt{e}$, simply consider the function f given by $f(p) = 1$ for all $p \le x^{1/\sqrt{e}}$ and $f(p) = -1$ for $x \ge p > x^{1/\sqrt{e}}$.

We now focus on evaluating τ_3. Define U to be the unique real number such that $U \le 4/3$, and $1/(2U) + e/(2U^3) \ge 1$ (that is, $U \ge 1.30189\cdots$) and satisfying the equation

$$\frac{1}{3} = \log U + \int_{\frac{1}{2U}}^{1-\frac{e}{2U^3}} \log\Big(\frac{e}{2U^3 y}\Big)\frac{dy}{y} + \int_{1-\frac{e}{2U^3}}^{\frac{1}{2}} \log\Big(\frac{1-y}{y}\Big)\frac{dy}{y}.$$

Then $U = 1.3064664\cdots$ and we claim that $\tau_3 = 1/U = 0.765423\cdots$. We remark here that Davenport and Erdős [2] showed that $\tau_3 \le 0.76549\cdots$.

We first show that $\tau_3 \le 1/U$, and then construct an example giving $\tau_3 \ge 1/U$. Suppose f is a completely multiplicative function with $f(n)^3 = 1$ for all integers $n \ge 1$, and that x is large with

$$\#\{n \le x : f(n) = \omega^j\} = x/3 + o(x) \quad \text{for } j = 0, 1, 2 \text{ where } \omega = e^{2\pi i/3}.$$

Let n_1 denote the smallest integer with $f(n_1) \ne 1$, and without loss of generality suppose that $f(n_1) = \omega$. We then need to show that the smallest n_2 with $f(n_2) = \omega^2$ satisfies $n_2 \le x^{1/U+o(1)}$. We may suppose that $n_2 > x^{3/4}$, and since $n_1^2 \ge n_2$, that $n_1 \ge x^{3/8}$.

Let P_1 denote the set of primes below x with $f(p) = \omega$, and P_2 denote the set of primes below x with $f(p) = \omega^2$. Then $P_1 \subset [n_1, x]$, and $P_2 \subset [n_2, x]$. Since $n_2 > x^{3/4}$ and $n_1 > x^{3/8}$ we see that an integer $n \le x$ either has no prime factors from P_1 and P_2, or has exactly one prime factor from P_1 (and none from P_2), or has exactly two prime factors from P_1 (and none from P_2), or has exactly one prime factor from P_2 (and none from P_1). We call A, B, C and D, the sets of integers corresponding to these four cases. Elements in A satisfy $f(n) = 1$, elements in B that $f(n) = \omega$, and elements in C and D satisfy $f(n) = \omega^2$. Thus

$$|A| \sim |B| \sim |C| + |D| \sim x/3. \tag{11.3}$$

Lastly put $\beta_1 = \sum_{p\in P_1} 1/p$ and $\beta_2 = \sum_{p\in P_2} 1/p$.

Note that

$$|D| = \sum_{p\in\mathcal{P}_2} [x/p] \sim \beta_2 x,$$

and that

$$|B| + 2|C| = \sum_{p\in\mathcal{P}_1} [x/p] \sim \beta_1 x.$$

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Combining these with (11.3) we conclude that

$$\beta_2 + o(1) \le 1/3 \le \beta_1 + o(1), \quad \text{and} \quad \beta_1 + 2\beta_2 = 1 + o(1). \tag{11.4}$$

Given a subset P of the primes in $[w, z]$ with $\sum_{p\in P} 1/p = \beta + o(1)$ we see that

$$\sum_{\substack{p<q\in P\\ pq\le z}} \frac{1}{pq}$$

is maximized when P is the set of all primes in $[w, w^{e^\beta}]$. Using this observation for $P_1 \subset [n_1, x] \subset [\sqrt{n_2}, x]$ we see that

$$\Big(\frac{\beta_1}{2} - \frac{1}{6}\Big) \sim \frac{|C|}{x} \sim \sum_{\substack{p<q\in P_1\\ pq\le x}} \frac{1}{ps} \le \sum_{\substack{p<q\in [n_2^{1/2}, n_2^{e^{\beta_1}/2}]\\ pq\le x}} \frac{1}{pq} + o(1) = f(n_2, \beta_1) + o(1),$$

say. If $\beta < \beta'$ then we see that

$$\begin{aligned} f(n_2, \beta') - f(n_2, \beta) &\le \sum_{n_2^{e^\beta/2} \le q \le n_2^{e^{\beta'}/2}} \frac{1}{q} \sum_{n_2^{\frac12} \le p \le x^{\frac12}} \frac{1}{p} \le (\beta' - \beta) \log\Big(\frac{\log x}{\log n_2}\Big) + o(1)\\ &\le \frac{1}{3}(\beta' - \beta) + o(1), \end{aligned}$$

since $n_2 > x^{3/4}$. Thus we see that $(1/6 + o(1) \ge)\ \beta_1/2 - f(n_2, \beta_1)$ is essentially an increasing function of β_1. Since $\beta_2 \le \sum_{n_2 \le p \le x} 1/p = \log(\log x/\log n_2) + o(1)$ we get by (11.4) that $\beta_1 > 1 - 2\log(\log x/\log n_2)$ and hence we conclude that

$$\frac{1}{6} + o(1) \ge \frac{1}{2} - \log\Big(\frac{\log x}{\log n_2}\Big) - f\bigg(n_2, 1 - 2\log\Big(\frac{\log x}{\log n_2}\Big)\bigg). \tag{11.5}$$

Put now $n_2 = x^{1/u}$ so that $1 \le u \le 4/3$. In case $u \le 1.301890916\cdots$ is such that $1/(2u) + e/(2u^3) \ge 1$ then we see that

$$f(n_2, 1 - 2\log u) = \sum_{x^{1/(2u)} \le p \le x^{1/2}} \frac{1}{p} \sum_{p<q\le x/p} \frac{1}{q} = \int_{\frac{1}{2u}}^{\frac12} \log\Big(\frac{1-y}{y}\Big) \frac{dy}{y} + o(1).$$

In this case (11.5) yields that

$$\frac{1}{3} + o(1) \le \log u + \int_{\frac{1}{2u}}^{\frac12} \log\Big(\frac{1-y}{y}\Big) \frac{dy}{y}.$$

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However the right side is an increasing function of u, and its value at $u = 1.302$ is $0.3284\cdots < 1/3$. Thus we must have $4/3 \geq u > 1.301890916\cdots$, in which case $1/(2u) + e/(2u^3) > 1$. Here we see that

$$\begin{aligned} f(n_2, 1-2\log u) &= \sum_{x^{1/(2u)} \geq p < x^{1-e/(2u^3)}} \frac{1}{p} \sum_{p<q\leq x^{e/(2u^3)}} \frac{1}{q} + \sum_{x^{1-e/(2u^3)} \leq p \leq x^{\frac{1}{2}}} \frac{1}{p} \sum_{p<q\leq x/p} \frac{1}{q} \\ &= \int_{\frac{1}{2u}}^{1-\frac{e}{2u^3}} \log\Big(\frac{e}{2u^3 y}\Big)\frac{dy}{y} + \int_{1-\frac{e}{2u^3}}^{\frac{1}{2}} \log\Big(\frac{1-y}{y}\Big)\frac{dy}{y}. \end{aligned}$$

Thus in this case (11.5) yields that

$$\frac{1}{3} + o(1) \leq \log u + \int_{\frac{1}{2u}}^{1-\frac{e}{2u^3}} \log\Big(\frac{e}{2u^3 y}\Big)\frac{dy}{y} + \int_{1-\frac{e}{2u^3}}^{\frac{1}{2}} \log\Big(\frac{1-y}{y}\Big)\frac{dy}{y}.$$

Again the right side is an increasing function of u in this range, and it equals $1/3$ at $U = 1.306466\cdots$, proving that $u \geq U + o(1)$, and hence our desired upper bound for τ_3.

Our proof above indicates the optimal function f attaining this value of τ_3. Take $f(p) = \omega$ for $p \in [x^{\frac{1}{2U}}, x^{\frac{e}{2U^3}}]$, $f(p) = \omega^2$ for $p \in [x^{\frac{1}{U}}, x]$ and $f(p) = 1$ otherwise. Then we check easily from our earlier considerations that the sets $n \leq x$, with $f(n) = 1$, $f(n) = \omega$, or $f(n) = \omega^2$ all have cardinality $\sim x/3$, and the least n with $f(n) = \omega^2$ exceeds $x^{1/U}$. This completes our determination of τ_3.

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