A TRANSFORMATION CONNECTING PRODUCTS OF GENERALISED BASIC HYPERGEOMETRIC FUNCTIONS

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1. Introduction. Darling [2] in 1932 gave two types (equations 11 and 18) of transformations connecting generalised hypergeometric functions. The first was studied by Bailey [1] and extended by Sears [4] to a transformation connecting products of basic hypergeometric functions of the type ${}_{r+1}\phi_r \times {}_{r+1}\phi_r$. In a number of papers [6, 7, 8] the author has extended these results to both unilateral and bilateral series with bases q and $q^{1/2}$. The second type of transformation by Darling for a product ${}_1F_0 \times {}_3F_2$ was extended by Bailey [1] to a transformation between ${}_1F_0 \times {}_{r+1}F_r$. In the same paper Bailey mentioned the transformation of a ${}_1\phi_0 \times {}_3\phi_2$ without proof.

In the present paper, I have extended the second type of transformation by investigating a transformation between products ${}_1\phi_0 \times {}_{r+1}\phi_r$. Thus it has been shown that this transformation not only includes the results of Bailey and Darling but also that of Sears.

2. Notations. Let $[a]_n = (1-a)(1-aq) \ldots (1-aq^{n-1})$; $[a]_0 = 1$, then the generalised basic hypergeometric function ${}_r\phi_s\left[\begin{matrix}(\alpha_r) ; z \\ (\beta_s)\end{matrix}\right]$ is defined as

$$
{}_r\phi_s\left[\begin{matrix}(\alpha_r); & z \\ (\beta_s) & \end{matrix}\right] \equiv {}_r\phi_s\left[\begin{matrix}q^{(\alpha_r)}; & z \\ q^{(\beta_s)} & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[q^{(\alpha_r)}]_n z^n}{[q]_n [q^{(\beta_s)}]_n} \; ; |z| < 1, \; |q| < 1 \; ,
$$

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where $(\alpha_{N,M})$ means the sequence of parameters $\alpha_N, \alpha_{N+1}, \ldots, \alpha_M$. But when $N = 1$ we shall write (α_M) instead of writing $(\alpha_{1,M})$. Following Hahn [3] the basic difference operator $\mathfrak{D}f(z)$ is defined as

$$\mathfrak{D}f(z) = \frac{f(z) - f(qz)}{z(1-q)} = \frac{1-q^{\theta}}{z(1-q)} f(z),$$

where $\theta = z\frac{d}{dz}$. Following Jackson [4], we define

$$[\theta] f(z) = \frac{1-q^{\theta}}{1-q} f(z) = \frac{f(z) - f(qz)}{1-q}.$$

But when we write $[\theta+\alpha]$ or $[\theta+2\beta]$ we mean $\frac{1-q^{\theta+\alpha}}{1-q}$ or $\frac{1-q^{\theta+2\beta}}{1-q}$ respectively. Lastly to avoid the use of the summation symbol in certain formulae, the following notation will be used. Let $g(a_1, a_2, \ldots, a_M)$ be a function of the M parameters specified and let $N \leq M$. The sum

$$\sum_{i=1}^{N} g(a_i, a_2, a_3, \ldots, a_{i-1}, a_1, a_{i+1}, \ldots, a_M)$$

will be written as

$$g(a_1, a_2, \ldots, a_M) + \text{idem}\,(a_1, a_2, \ldots, a_M).$$

3. In the first instance we deduce a convenient form of the q-difference equation satisfied by the generalised basic hypergeometric function ${}_{r+1}\phi_r[(\alpha_{r+1});(\beta_r);z]$. To do so, we start from the fact that ${}_{r+1}\phi_r[(\alpha_{r+1});(\beta_r);z]$ satisfies the q-difference equation (Jackson [4]).

(3.1) $\{z[\theta+\alpha_1][\theta+\alpha_2]\ldots[\theta+\alpha_{r+1}] - [\theta][\theta+\beta_1-1]\ldots[\theta+\beta_r-1]\}\,\phi(z) = 0,$

and the r solutions of the difference equation are

$$z^{1-\beta_m}\,{}_{r+1}\phi_r\left[\begin{matrix}(\alpha_{r+1})-\beta_m+1\,;\,z\\ 2-\beta_m,\,1-\beta_m+(\beta_r)'\end{matrix}\right] : m = 1, 2, \ldots, r,$$

where $(\beta_r)' - \beta_m$ means the sequence of $r-1$ parameters $\beta_1-\beta_m$, $\beta_2-\beta_m, \ldots, \beta_{m-1}-\beta_m, \beta_{m+1}-\beta_m, \ldots, \beta_r-\beta_m$, i.e. the parameter $\beta_m-\beta_m$ being dropped out. But the equation (3.1), after some

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simplification, can be put in the form

$$(3.2) \qquad \sum_{t=0}^{r+1} (-)^t q^{r-t} [s_t(\beta)+q\{s_{t-1}(\beta)-zq^{t-1}s_t(\alpha)\}]\phi(xq^t) = 0,$$

where $S_t(\beta)$ and $S_t(\alpha)$ means the sum of all possible combinations of products of t factors taken at a time from the sequence of numbers $q^{\beta_1}, q^{\beta_2}, \ldots, q^{\beta_r}$ and $q^{\alpha_1}, q^{\alpha_2}, \ldots, q^{\alpha_{r+1}}$, respectively.
It is obviously assumed that

$$S_0(\alpha) = S_0(\beta) = 1 \text{ and } S_{-1}(\alpha) = S_{-1}(\beta) = S_{r+1}(\beta) = 0.$$

Furthermore, we can easily show by induction that

$$(3.3) \qquad \phi(xq^n) = \sum_{t=0}^{n} (-)^t \left[{n \atop t}\right] (1-q)^t z^t q^{(1/2)t(t-1)} \mathcal{T}^t \phi(z),$$

by using the simple relation

$$\left[{x+1 \atop k}\right] = \left[{x \atop k-1}\right] + q^k \left[{x \atop k}\right],$$

where

$$\left[{x \atop k}\right] = [q]_x / [q]_{x-k} [q]_k .$$

Substituting from (3.3) in (3.2) the difference equation can be written in the desired form

$$(3.4) \qquad \sum_{p=0}^{r+1} P_p(z) \mathcal{T}^p \phi(z) = 0$$

where the coefficients $P_p(z)$ are given by the relation

(3.5)

$$P_{r-p}(z) = (-)^{p+1} (1-q)^{r-p} z^{r-p} q^{1/2(r-p)(r-p-1)}$$
$$\times \sum_{t=0}^{p+1} (-)^t [s_{r+1-t}(\beta) + q\{s_{r-t}(\beta) - zq^{r-t} s_{r-t+1}(\alpha)\}]\left[{r+1-t \atop r-p}\right] q^{t-1},$$

for $-1 \le p \le r$.

4. Next, to deduce the main transformation let us consider the

contour integral

$$(4.1) \quad \int \frac{[sq^{(\alpha_{r+1})}]_n}{[s]_{n-m+1}[sq^{(\beta r)-1}]_{n+1}} ds; \; q = e^{-t}, \; t > 0, \; m = 0, 1, \ldots, r,$$

taken round the semi-circle $|s| = R$ to the right of the imaginary axis. The imaginary axis is indented, if necessary, to ensure that the sequence of increasing poles lie to the right of it.

This integral evidently tends to zero as $R \to \infty$. Equating to zero the sum of the residues at the poles, we obtain the following $R + 1$ identities*

$$(4.2) \quad A^*(z)\mathfrak{T}^t A(z) = \frac{[q-(\beta_{2,r})]\beta_1^{r-1}}{[\beta_1-(\beta_r)']q^{r-1}} B^*(\beta_1;z)\mathfrak{T}^t B(\beta_1;z) + \text{idem}(\beta_1;\beta_2,\ldots,\beta_r)$$

$$t = 0, 1, \ldots, r-1,$$

and

$$(4.3) \quad A^*(z)\mathfrak{T}^r A(z) = \frac{[q-(\beta_{2,r})]\beta_1^{r-1}}{[\beta_1-(\beta_r)']q^{r-1}} B^*(\beta_1;z)\mathfrak{T}^r(\beta_1;z)$$

$$+ \text{idem}\,(\beta_1;\beta_2,\ldots,\beta_r) + \frac{a}{z^r(1-z)} .$$

In case, when $m = r$, the integrand is $[\frac{1}{s} + O(\frac{1}{s^2})]$ and thus the integral tends to $2\pi i$ as $R \to \infty$. The term $a/z^r(1-z)$ arises from the series $\sum_{n=0}^{\infty} z^{n-r}$, a is a constant which can be determined, but its value is immaterial. When $n \leq m-1$, $[s]_{n-m+1}$ must be replaced by

$$(-)^{n-m+1} q^{1/2(n-m+1)(n-m+2) + s(m-n-1)}/[q/s]_{m-n-1} .$$

*The relation (4.2) for the value $m = 0$ is the one already given by Sears [5:6.2].

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In the equations (4.2) and (4.3) we have assumed

$$A(z) = {}_{r+1}\phi_r \left[\begin{matrix} q^{(\alpha_{r+1})} ; z \\ q^{(\beta_r)} \end{matrix} \right]$$

$$A^*(z) = {}_{r+1}\phi_r \left[\begin{matrix} q^{1-(\alpha_{r+1})} ; zq^{\Sigma(\alpha_{r+1})-\Sigma(\beta_r)+r-1} \\ q^{2-(\beta_r)} \end{matrix} \right] ,$$

$$B(\beta_t;z) = {}_{r+1}\phi_r \left[\begin{matrix} q^{1+(\alpha_{r+1})-\beta_t} ; z \\ q^{2-\beta_t}, q^{1+(\beta_r)'-\beta_t} \end{matrix} \right] z^{1-\beta_t}, \quad z^{\beta_t-1} \quad t = 1, 2, \dots, r,$$

and

$$B^*(\beta_t;z) = {}_{r+1}\phi_r \left[\begin{matrix} q^{\beta_t-(\alpha_{r+1})} ; zq^{\Sigma(\alpha_{r+1})-\Sigma(\beta_r)+r-1} \\ q^{\beta_t}, q^{1+\beta_t-(\beta_r)'} \end{matrix} \right]$$

$$t = 1, 2, \dots, r.$$

Now if we suppose

$$\Delta(z) = \begin{vmatrix} B(\beta_1;z) & B(\beta_2;z) \dots\dots & B(\beta_r;z) \\ \mathfrak{I}B(\beta_1;z) & \mathfrak{I}B(\beta_2;z) \dots\dots & \mathfrak{I}B(\beta_r;z) \\ \mathfrak{I}^2B(\beta_1;z) & \mathfrak{I}^2B(\beta_2;z) \dots\dots & \mathfrak{I}^2B(\beta_r;z) \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \mathfrak{I}^{r-1}B(\beta_1;z) & \mathfrak{I}^{r-1}B(\beta_2;z) \dots & \mathfrak{I}^{r-1}B(\beta_r;z) \end{vmatrix} .$$

We deduce from the equations (4.2) and (4.3) that

(4.4) $$A^*(z)\,\Delta_1(z) = \frac{a}{z^r(1-z)}\,\Delta ,$$

where
$$\Delta_1(z) = \begin{vmatrix} A(z) & B(\beta_1;z) & B(\beta_2;z) \dots & B(\beta_r;z) \\ \mathfrak{I}A(z) & \mathfrak{I}B(\beta_1;z) & \mathfrak{I}B(\beta_2;z) \dots & \mathfrak{I}B(\beta_r;z) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \mathfrak{I}^rA(z) & \mathfrak{I}^rB(\beta_1;z) & \mathfrak{I}^rB(\beta_2;z) \dots & \mathfrak{I}^rB(\beta_r;z) \end{vmatrix} ;$$

again since

$$(4.5)\quad \mathcal{T}\Delta_1(z) = \begin{vmatrix} \mathcal{T}^{r+1}A(z) & \mathcal{T}^{r+1}B(\beta_1;z) & \mathcal{T}^{r+1}B(\beta_2;z)\ldots\mathcal{T}^{r+1}B(\beta_r;z) \\ \mathcal{T}^{r-1}A(z) & \mathcal{T}^{r-1}B(\beta_1;z) & \mathcal{T}^{r-1}B(\beta_2;z)\ldots\mathcal{T}^{r-1}B(\beta_r;z) \\ \mathcal{T}^{r-2}A(z) & \mathcal{T}^{r-2}B(\beta_1;z) & \mathcal{T}^{r-2}B(\beta_2;z)\ldots\mathcal{T}^{r-2}B(\beta_r;z) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ A(qz) & B(\beta_1;qz) & B(\beta_2;qz) \quad B(\beta_r;qz) \end{vmatrix}$$

and the functions $A(z)$, $B(\beta_t:z)$: $t = 1, 2, \ldots, r$ satisfy the q-difference equation (3.4), we obtain

$$(4.6)\qquad P_{r+1}(z)\mathcal{T}\Delta_1(z) + \Delta_1(z)\sum_{t=0}^{r}(1-q)^t z^t P_{r-t}(z) = 0 \quad .$$

Substituting the value of the coefficients $P_p(z) : t = 0, 1, 2, \ldots, r+1$ from (3.5), we get, on some simplification

$$(4.7)\qquad z(1-q)q^{(1/2)r(r+1)}[s_r(\beta) - zq^r s_{r+1}(\alpha)]\mathcal{T}\Delta_1(z) -$$

$$\Bigg[\sum_{t=0}^{r}[s_{r-t+1}(\beta) + q\{s_{r-t}(\beta) - zq^{r-t}s_{r-t+1}(\alpha)\}](-)^{r+t}q^{t-1}\times$$

$${}_1\phi_0[q^{-r-1+t};\,;q^{r+1-t}] + q^{(1/2)r(r+1)+1}[s_r(\beta) - zq^r s_{r+1}(\alpha)] -$$

$$(1-z)q^{r+1}\Bigg]\Delta_1(z) = 0.$$

But since ${}_1\phi_0[q^{-1-r+t};\,;q^{r+1-t}] = 0$, $t = 0, 1, \ldots, r$, the equation (4.7) simplifies to

$$(4.8)\qquad z(1-q)q^{(1/2)r(r+1)}[s_r(\beta) - zq^r s_{r+1}(\alpha)]\mathcal{T}\Delta_1(z) -$$

$$\Big[q^{(1/2)r(r+1)}[s_r(\beta) - zq^r s_{r+1}(\alpha)] - [1-z]q^r\Big]\Delta_1(z) = 0.$$

One can without any difficulty solve the q-difference equation (4.8) and find its solution as

$$(4.9)\quad \Delta_1(z) = Kz^{-\Sigma(\beta_r)-(1/2)r(r-1)}{}_1\phi_0[q^{\Sigma(\alpha_{r+1})-\Sigma(\beta_r)+r};\,;z],$$

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where K is a constant independent of z.

Substituting the value of $\Delta_1(z)$ from (4.9) in (4.4) we get

$$(4.10)\quad \Delta(z) = Mz^{-\Sigma(\beta_r) - (1/2)r(r-3)} A^*(z)\ {}_1\phi_0[q^{\Sigma(\alpha_r+1) - \Sigma(\beta_r)+r-1}\ ;\ ;\ qz].$$

where M is some other constant.

To find the value of the constant M, we equate the coefficient of $z^{-\Sigma(\beta_r) - (1/2)r(r-3)}$ on both the sides of (4.10) we obtain

$$(4.11)\quad M = \frac{1}{(1-q)^r}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ (1-q^{1-\beta_1}) & (1-q^{1-\beta_2}) & \cdots & (1-q^{1-\beta_r}) \\ (1-q^{1-\beta_1})(1-q^{-\beta_1}) & (1-q^{1-\beta_2})(1-q^{-\beta_2}) & \cdots & (1-q^{1-\beta_r})(1-q^{-\beta_r}) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (1-q^{1-\beta_1})(1-q^{-\beta_1})\ldots(1-q^{-\beta_1-r+2}) & (1-q^{1-\beta_2})(1-q^{-\beta_2})\ldots(1-q^{-\beta_2-r+2}) & \cdots & (1-q^{1-\beta_r})(1-q^{-\beta_r})\ldots(1-q^{-\beta_r-r+2}) \end{vmatrix}.$$

The transformation (4.10) along with (4.11) is the desired transformation. The result reduces to the one given by Bailey [1] as $q \to 1$. And if one follows the technique of Darling [2] one can get the transformation of Sears [3:6.2] also.

In conclusion, it is worthwhile to remark that we have considered, only for the sake of simplicity of the proof, the product ${}_1\phi_0 \times {}_{r+1}\phi_r$. In fact, from the transformation for the product ${}_1\phi_0 \times {}_{r+1}\phi_r$, transformations involving the products ${}_1\phi_0 \times {}_M\phi_N$ could be deduced by confluence of suitable number of parameters.

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