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A BEURLING ALGEBRA IS SEMISIMPLE: AN ELEMENTARY PROOF

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The Beurling algebra $L^1(G,\omega)$ on a locally compact Abelian group G with a measurable weight ω is shown to be semisimple. This gives an elementary proof of a result that is implicit in the work of M.C. White (1991), where the arguments are based on amenable (not necessarily Abelian) groups.

Let G be a locally compact Abelian group with Haar measure λ. A *weight* on G is a meaurable function $\omega : G \longrightarrow (0,\infty)$ such that $\omega(s+t) \leqslant \omega(s)\omega(t)$ $(s,t \in G)$. Then the *Beurling algebra* $L^1(G,\omega)$ consists of all complex-valued measurable functions f on G such that $f\omega \in L^1(G)$. It is a commutative Banach algebra with convolution product and with the norm $\|f\|_\omega := \int_G |f(s)|\omega(s)d\lambda(s)$. The authors faced the problem of the semisimplicity of $L^1(G,\omega)$ in the investigation of the unique uniform norm property in Banach algebras ([**1**]). It is shown in [**5**] that if G is amenable, then there exists a continuous, positive, ω-bounded character on G. Then Lemma 2 (below) quickly implies that $L^1(G,\omega)$ is semisimple for an Abelian G. Since the theory of amenable groups is not (yet) a standard part of Harmonic Analysis, and certainly not a part of Abelian Harmonic Analysis, we present an elementary proof of this basic result within the context of Abelian groups.

THEOREM 1. *The Beurling algebra $L^1(G,\omega)$ is semisimple.*

LEMMA 2. *$L^1(G,\omega)$ is either semisimple or radical.*

PROOF: Assume that $L^1(G,\omega)$ is not radical. So its Gelfand space $\Delta\big(L^1(G,\omega)\big)$ is non-empty. Let $\varphi \in \Delta\big(L^1(G,\omega)\big)$. Then there exists a function $\alpha \in L^\infty(G,1/\omega)$, the Banach space dual of $L^1(G,\omega)$, such that

$$\varphi(f) = \int_G f(s)\alpha(s)d\lambda(s)$$

for all $f \in L^1(G,\omega)$. By the standard argument in the case of $L^1(G)$, one can show that α is a continuous function, $0 < |\alpha(s)| \leqslant \omega(s)$ $(s \in G)$ and $\alpha(s+t) = \alpha(s)\alpha(t)$ $(s,t \in G)$.

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For each $\theta \in \widehat{G}$, define α_θ by

$$\alpha_\theta(g) = \int_G g(s)\alpha(s)\theta(s)d\lambda(s), \; g \in L^1(G,\omega).$$

Then $\alpha_\theta \in \Delta\big(L^1(G,\omega)\big)$. Now let $f \in \operatorname{rad} L^1(G,\omega)$, the radical of $L^1(G,\omega)$. Then $\alpha_\theta(f) = \widehat{f}(\alpha_\theta) = \widehat{f\alpha}(\theta) = 0 \;\; (\theta \in \widehat{G})$. Since $f \in L^1(G,\omega)$, we have $f\alpha \in L^1(G)$. Since $L^1(G)$ is semisimple and $\widehat{f\alpha}(\theta) = 0 \;\; (\theta \in \widehat{G})$, we have $f\alpha \equiv 0$ almost everhwhere on G. But $\alpha(s) \neq 0$ for any $s \in G$; and hence $f \equiv 0$ almost everywhere on G. This proves that $L^1(G,\omega)$ is semisimple. ∎

Lemma 3. *Let G_1 be a locally compact Abelian group such that $L^1(G_1,\omega)$ is semisimple for every weight ω on G_1. Let G_2 be a locally compact Abelian group such that $L^1(G_2,\omega)$ is semisimple for every weight ω on G_2. Let $G = G_1 \oplus G_2$ be the direct sum. Then $L^1(G,\omega)$ is semisimple for every weight ω on G.*

Proof: Let ω be a weight on G. By Lemma 2, it is enough to prove that $L^1(G,\omega)$ is not radical. Let U_1 and U_2 be symmetric neighbourhoods of the identities in G_1 and G_2 respectively such that their closures are compact. Define $f = \chi_{U_1 \times U_2}$, the characteristic function of $U_1 \times U_2$. Then f is a non-zero element of $L^1(G,\omega)$. It is clear that $f^n = \chi^n_{U_1}\chi^n_{U_2}$ for all $n \in \mathcal{N}$. It is enough to show that $\lim\limits_{n\to\infty} \|f^n\|_\omega^{1/n} > 0$. So define

$$\begin{aligned} \omega_1(s) = \omega(s,0) \;\; (s \in G_1) \quad &\text{and} \quad \omega_2(s) = \omega(0,s) \;\; (s \in G_2); \\ m = \inf\{\omega_1(s) : s \in U_1\} \quad &\text{and} \quad M = \sup\{\omega_2(s) : s \in U_2\}. \end{aligned}$$

It is clear that ω_i is a weight on G_i $(i = 1,2)$. Then by [**2**, Proposition 2.1], $m > 0$ and $M < \infty$. Also note that for any $n \in \mathcal{N}$, $\omega_2(s) \leqslant M^n$ for all $s \in U_2 + \cdots + U_2$ (n-times) and

$$\begin{aligned} \|f^n\|_\omega &= \int_G \big|f^n(s,t)\big|\omega(s,t)d\lambda_1(s)d\lambda_2(t) \\ &= \int_{G_1}\int_{G_2} \big|\chi^n_{U_1}(s)\big|\,\big|\chi^n_{U_2}(t)\big|\omega(s,t)d\lambda_1(s)d\lambda_2(t) \\ &\geqslant \int_{G_1}\int_{G_2} \big|\chi^n_{U_1}(s)\big|\,\big|\chi^n_{U_2}(t)\big|\frac{\omega_1(s)}{\omega_2(-t)}d\lambda_1(s)d\lambda_2(t) \\ &= \int_{G_1} \big|\chi^n_{U_1}(s)\big|\omega_1(s)d\lambda_1(s)\int_{G_2}\big|\chi^n_{U_2}(t)\big|\frac{1}{\omega_2(-t)}d\lambda_2(t) \\ &\geqslant \|\chi^n_{U_1}\|_{\omega_1}\frac{1}{M^n}\int_{G_2}\big|\chi^n_{U_2}(t)\big|d\lambda_2(t) \\ &= \frac{1}{M^n}\|\chi^n_{U_1}\|_{\omega_1}\|\chi^n_{U_2}\|_1, \end{aligned}$$

where $\|\cdot\|_1$ denotes the L^1-norm and λ_i denotes the Haar measure on G_i for $i = 1,2$. Then $\lim\limits_{n\to\infty}\|f^n\|_\omega^{1/n} \geqslant (1/M)\lim\limits_{n\to\infty}\|\chi^n_{U_1}\|_{\omega_1}^{1/n}\lim\limits_{n\to\infty}\|\chi^n_{U_2}\|_1^{1/n} > 0$. This proves that $L^1(G,\omega)$ is semisimple. ∎

Proof of Theorem 1: Note that if G is a compact Abelian group, then $L^1(G,\omega) = L^1(G)$ for any weight ω on G; so it is semisimple. By [**3**, p. 113], $L^1(\mathcal{R},\omega)$ is semisimple for any weight ω on $\mathcal{R}$; so Lemma 3 implies that $L^1(\mathcal{R}^n,\omega)$ is semisimple for any weight ω on $\mathcal{R}^n$, where $n \geqslant 1$. Hence, again by Lemma 3, $L^1(\mathcal{R}^n \oplus H,\omega)$ is semisimple for any weight ω on $\mathcal{R}^n \oplus H$, where $n \geqslant 0$ and H is a compact Abelian group.

Now let G be an arbitrary locally compact Abelian group and let ω be a weight on G. By [**4**, Theorem 2.4.1], there exists an open subgroup G_1 of G such that $G_1 = \mathcal{R}^n \oplus H$, where $n \geqslant 0$ and H is a compact Abelian group. By above argument $L^1(G_1,\omega_{|G_1})$ is semisimple. But the later is a closed subalgebra of $L^1(G,\omega)$. Hence $L^1(G,\omega)$ is not radical. Thus it is semisimple due to Lemma 2. ∎

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