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Calibrated geometry in hyperkähler cones, 3-Sasakian manifolds, and twistor spaces

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Abstract. We systematically study calibrated geometry in hyperkähler cones C^{4n+4}, their 3-Sasakian links M^{4n+3}, and the corresponding twistor spaces Z^{4n+2}, emphasizing the relationships between submanifold geometries in various spaces. Our analysis highlights the role played by a canonical $\mathrm{Sp}(n)\mathrm{U}(1)$-structure γ on the twistor space Z. We observe that $\mathrm{Re}(e^{-i\theta}\gamma)$ is an S^1-family of semi-calibrations and make a detailed study of their associated calibrated geometries. As an application, we obtain new characterizations of complex Lagrangian and complex isotropic cones in hyperkähler cones, generalizing a result of Ejiri–Tsukada. We also generalize a theorem of Storm on submanifolds of twistor spaces that are Lagrangian with respect to both the Kähler–Einstein and nearly Kähler structures.

1 Introduction

Hyperkähler manifolds C, equipped with a Riemannian metric g_C, complex structures (I_1, I_2, I_3), and Kähler forms $(\omega_1, \omega_2, \omega_3)$, are a rich source of calibrated geometries. They feature not only familiar geometries arising from the Calabi–Yau structure – such as complex submanifolds and special Lagrangians – but also less-familiar ones specific to the hyperkähler setting. For example, a submanifold $N^{2k+2} \subset C^{4n+4}$ is *complex isotropic with respect to* I_1 if it is simultaneously

$$I_1\text{-complex}, \ \omega_2\text{-isotropic, and } \omega_3\text{-isotropic.}$$

Complex Lagrangians $N^{2n+2} \subset C^{4n+4}$, those complex isotropic submanifolds of top dimension $2n+2$, are particularly remarkable, as they are at once complex submanifolds with respect to I_1 and special Lagrangian with respect to I_2 and I_3.

This paper seeks to systematically study the various calibrated cones of hyperkähler manifolds C, with a particular focus on complex isotropic cones. For this, it is of course necessary to assume that $(C^{4n+4}, g_C) = (\mathbb{R}^+ \times M^{4n+3}, dr^2 + r^2 g_M)$ is itself a Riemannian cone.

Hyperkähler cones C^{4n+4} are themselves highly special objects: each induces three associated Einstein spaces, called M, Z, and Q, as we briefly recall. The first of these, M^{4n+3}, is just the link of C, which inherits a 3-Sasakian structure. In view of the simple relationship between C and M, 3-Sasakian manifolds exhibit a wide array of semi-

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calibrated geometries. Indeed, each of the calibrated cones in C that we study has a semi-calibrated counterpart in M.

$\dim(\mathrm{C}(L))$	Calibrated cone $\mathrm{C}(L) \subset C$	Semi-calibrated link $L \subset M$	$\dim(L)$
$2k+2$	Complex	CR	$2k+1$
$2n+2$	Special Lagrangian	Special Legendrian	$2n+1$
$2k+2$	Special isotropic	Special isotropic	$2k+1$
$2n+2$	Complex Lagrangian	CR Legendrian	$2n+1$
$2k+2$	Complex isotropic	CR isotropic	$2k+1$
4	Cayley	Associative	3

The entries of this table will be explained in Sections 2 and 3.

Now, since M is 3-Sasakian, it admits three linearly independent Reeb vector fields A_1, A_2, A_3. In fact, for each $v = (v_1, v_2, v_3) \in S^2$, the Reeb field $A_v = \sum v_i A_i$ yields a one-dimensional foliation $\mathcal{F}_v$ on M, the projection $p_v \colon M \to M/\mathcal{F}_v$ is a principal S^1-orbibundle, and the quotient $Z = M/\mathcal{F}_v$ is a $(4n+2)$-orbifold. It is well known that Z naturally admits both a Kähler–Einstein structure $(g_{\mathrm{KE}}, J_{\mathrm{KE}}, \omega_{\mathrm{KE}})$ and a nearly Kähler structure $(g_{\mathrm{NK}}, J_{\mathrm{NK}}, \omega_{\mathrm{NK}})$. Indeed, Z is the twistor space of a quaternionic-Kähler $4n$-orbifold Q of positive scalar curvature.

The four Einstein spaces C, M, Z, Q may be summarized in the following "diamond diagram" in which $\tau \colon Z \to Q$ denotes the twistor S^2-bundle.

$$
\begin{array}{ccc}
M^{4n+3} & \hookrightarrow & C^{4n+4} \\
 & \searrow^{p_v} & \downarrow \\
\downarrow h & & Z^{4n+2} \\
 & \swarrow_{\tau} & \\
Q^{4n} & &
\end{array}
\tag{1.1}
$$

The *flat model* is $(C, M, Z, Q) = (\mathbb{H}^{n+1}, \mathbb{S}^{4n+3}, \mathbb{CP}^{2n+1}, \mathbb{HP}^n)$, in which each $p_v \colon \mathbb{S}^{4n+3} \to \mathbb{CP}^{2n+1}$ is a complex Hopf fibration, and $h \colon \mathbb{S}^{4n+3} \to \mathbb{HP}^n$ is a quaternionic Hopf fibration.

In addition to all of the structure already discussed, we recover an observation of Alexandrov [3] that twistor spaces Z admit a distinguished complex 3-form γ corresponding to an $\mathrm{Sp}(n)\mathrm{U}(1)$-structure. In fact, we give two different proofs of this result, one in Section 4.2 via the 3-Sasakian geometry of M, and the other in Section 5.1 via the quaternionic-Kähler geometry of Q. Furthermore, we establish the new result that $\mathrm{Re}(\gamma)$ is a semi-calibration and we classify those $\mathrm{Re}(\gamma)$-calibrated submanifolds that are ω_{KE}-isotropic. More precisely:

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Theorem 1.1 *Let Z be the $(4n+2)$-dimensional twistor space of a positive quaternionic-Kähler $4n$-orbifold. Then Z admits an $\mathrm{Sp}(n)\mathrm{U}(1)$-structure $\gamma \in \Omega^3(Z;\mathbb{C})$ compatible with the Kähler–Einstein and nearly Kähler structures. Moreover:*

- *The 3-form $\mathrm{Re}(\gamma)$ is a semi-calibration (i.e., has comass one).*
- *If Σ^3 is compact, $\mathrm{Re}(\gamma)$-calibrated, and ω_{KE}-isotropic, then with respect to the Kähler–Einstein metric, Σ is a geodesic circle bundle over a totally complex surface in Q. (See Definition 5.7.) Conversely, any such circle bundle is $\mathrm{Re}(\gamma)$-calibrated and ω_{KE}-isotropic. (See Theorem 5.16.)*

We remark that there is a difference between the cases $n = 1$ and $n \geq 2$, so our proof handles them separately. In Section 4.3, we undertake a detailed study of $\mathrm{Re}(\gamma)$-calibrated 3-folds in Z^{4n+2}. In a certain precise sense, these are generalizations of special Lagrangian 3-folds in nearly Kähler 6-manifolds.

Geometric structures in place, we establish a series of relationships between the various classes of submanifolds in M, Z, and Q; see diagram (1.1). That is, given a submanifold $\Sigma \subset Z$, we ask how various first-order conditions on Σ (e.g., complex and Lagrangian) influence the geometry of a local $p_{(1,0,0)}$-horizontal lift $\widehat{\Sigma} \subset M$ (provided one exists) and its $p_{(1,0,0)}$-circle bundle $p^{-1}_{(1,0,0)}(\Sigma) \subset M$, and vice versa. Similarly, starting with a totally complex $U \subset Q^{4n}$, we study its *τ-horizontal lift* $\widetilde{U} \subset Z$ and its *geodesic circle bundle lift* $\mathcal{L}(U) \subset Z$:

$$\widetilde{U}|_x = \{j \in Z_x : j(T_xU) = T_xU\}, \qquad \mathcal{L}(U)|_x = \{j \in Z_x : j(T_xU) \subset (T_xU)^{\perp}\}.$$

See Section 5.2 for a detailed discussion.

Altogether, the litany of propositions and theorems – proven in Sections 4.4, 5.2, and 6 – comprise a sort of "dictionary" of submanifold geometries. As an example, in Section 5.2, we obtain the following characterization of the compact submanifolds of Z that are Lagrangian with respect to both ω_{KE} and ω_{NK}, generalizing a result of Storm [30] to higher dimensions.

Theorem 1.2 *Recall diagram (1.1).*

1. *If $\Sigma^{2n+1} \subset Z^{4n+2}$ is a compact $(2n+1)$-dimensional submanifold that is both ω_{KE}-Lagrangian and ω_{NK}-Lagrangian, then $\Sigma = \mathcal{L}(U)$ for some totally complex $2n$-fold $U^{2n} \subset Q^{4n}$ (resp. superminimal surface if $n = 1$).*
2. *Conversely, if $U^{2n} \subset Q^{4n}$ is totally complex and $n \geq 2$, or if U is a superminimal surface and $n = 1$, then $\mathcal{L}(U) \subset Z$ is ω_{KE}-Lagrangian and ω_{NK}-Lagrangian.*

As another example, in Section 6, we provide several characterizations of complex isotropic cones in hyperkähler cones C^{4n+4} in terms of submanifold geometries in M, Z, and Q. In particular, we prove the following theorem, generalizing a result of Ejiri and Tsukada [13] on complex isotropic cones of top dimension $2n+2$ in $C = \mathbb{H}^{n+1}$.

Theorem 1.3 *Recall diagram (1.1). Let $L^{2k+1} \subset M^{4n+3}$ be a compact submanifold, where $3 \leq 2k+1 \leq 2n+1$. The following conditions are equivalent:*

1. *The cone $\mathrm{C}(L)$ is complex isotropic with respect to $\cos(\theta)I_2 + \sin(\theta)I_3$ for some $e^{i\theta} \in S^1$.*
2. *The link L is locally of the form $p^{-1}_{(0,\cos(\theta),\sin(\theta))}(\widetilde{U})$ for some totally complex submanifold $U^{2k} \subset Q$ (resp. superminimal surface if $n = 1$) and some $e^{i\theta} \in S^1$.*

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(3) *The link L is locally a $p_{(1,0,0)}$-horizontal lift of $\mathcal{L}(U) \subset Z$ for some totally complex submanifold $U^{2k} \subset Q^{4n}$ (resp. superminimal surface $U^2 \subset Q^4$ if $n = 1$).*

A more detailed statement appears as Theorem 6.1. Moreover, additional characterizations are available for complex isotropic cones $\mathrm{C}(L) \subset C$ of top dimension $2n + 2$ and lowest dimension 4: see Theorems 6.2 and 6.3, respectively.

Intuitively, Theorem 1.3 states that the link $L^{2k+1} \subset M$ of a complex isotropic cone in C^{4n+4} can be manufactured from a totally complex submanifold $U^{2k} \subset Q$ in two ways. By (2), one can first consider its τ-horizontal lift $\widetilde{U} \subset Z$ and then take the resulting $p_{(0,\cos(\theta),\sin(\theta))}$-circle bundle. On the other hand, by (3), one could instead begin with the geodesic circle bundle lift $\mathcal{L}(U) \subset Z$ and then take a $p_{(1,0,0)}$-horizontal lift to M. Thus, in a sense, the operations of "circle bundle lift" and "horizontal lift" commute with one another.

Broadly speaking, Theorems 1.2 and 1.3 illustrate that a great variety of distinct classes of semi-calibrated submanifolds of a hyperkähler cone, 3-Sasakian manifold, or twistor space can only arise as particular constructions built from totally complex submanifolds, which is not at all evident from their definitions. Consequently, such submanifolds are essentially as plentiful as totally complex submanifolds. See Example 5.2 for some explicit totally complex submanifolds.

1.1 Organization and conventions

In Section 2, we discuss several calibrated geometries in hyperkähler manifolds C^{4n+4}, including the complex, special Lagrangian, complex isotropic, special isotropic, and Cayley submanifolds. Then, starting in Section 3, we assume that $C = \mathrm{C}(M)$ is a hyperkähler cone over a 3-Sasakian manifold M^{4n+3}. We spend Section 3.1 reviewing 3-Sasakian geometry, turning to the submanifold theory of M in Sections 3.2 and 3.3. In Section 3.4, we introduce a complex 3-form $\Gamma_1 \in \Omega^3(M;\mathbb{C})$ and prove that it descends via $p_{(1,0,0)}: M \to Z$ to a 3-form $\gamma \in \Omega^3(Z;\mathbb{C})$ on the twistor space.

Section 4 concerns submanifold theory in twistor spaces. After discussing $\mathrm{Sp}(n)\mathrm{U}(1)$-structures on arbitrary $(4n+2)$-manifolds in Section 4.1, we show in Section 4.2 that the 3-form $\gamma \in \Omega^3(Z;\mathbb{C})$ defines such a structure on the twistor space. Then, in Sections 4.3 and 4.4, we study various classes of submanifolds of Z, establishing a series of relationships between those in Z and those in M.

In Section 5.2, we consider totally complex submanifolds of quaternionic-Kähler manifolds Q and relate them to submanifold geometries in M and Z. Finally, in Section 6, we provide several characterizations of complex isotropic cones in C. This paper also includes two appendices: Appendix A.1 collects some results on the linear algebra of calibrations that we use, and Appendix A.2 gives a brief introduction to metric cones and their associated conical differential forms.

Notation and conventions.

- We often use c_θ, s_θ to denote $\cos\theta, \sin\theta$, respectively, for brevity.
- Repeated indices are summed over all of their allowed values unless explicitly stated otherwise. The symbol ε_{pqr} is the permutation symbol on three letters, so it vanishes if any two indices are equal, and it equals $\mathrm{sgn}(\sigma)$ if $p, q, r = \sigma(1), \sigma(2), \sigma(3)$.
- A superscript on a manifold always denotes its *real* dimension.

- For a manifold M, we use $\mathrm{C}(M) = \mathbb{R}^+ \times M$ with metric $dr^2 + r^2 g_M$ to denote the metric cone over M, as discussed in Appendix A.2.
- If L is a submanifold of M, then NL denotes its normal bundle. Submanifolds are assumed to be embedded. (Much of what we discuss works for immersed submanifolds, but not everything. See also Remark 5.15.) Unless stated otherwise, all submanifolds are assumed to be connected and orientable and thus have exactly two orientations.
- We use interchangeably the terms *semi-calibration* and *comass one*. That is, a differential form α is a *calibration* if it is a semi-calibration that satisfies $d\alpha = 0$.
- The twistor space Z^{4n+2} and the quaternionic-Kähler Q^{4n} are in general *orbifolds*. However, we avoid technical complications and work only over the *smooth* parts of Z and Q. That is, all submanifolds are assumed to not pass through any orbifold points of Z or Q.

2 Calibrated geometry in hyperkähler manifolds

Let C^{4n+4} be a hyperkähler manifold with $n \geq 1$. The hyperkähler structure on C consists of the following data:

- a Riemannian metric g_C;
- a triple of integrable almost-complex structures $(I_1, I_2, I_3) = (I, J, K)$ satisfying the quaternionic relations $I_1 I_2 = I_3$, etc., each of which is orthogonal with respect to g_C;
- a triple of closed 2-forms $(\omega_1, \omega_2, \omega_3)$ given by $\omega_p(X, Y) = g_C(I_p X, Y)$.

Note that ω_p is a Kähler form with respect to I_p, so in particular it is of type $(1,1)$ with respect to I_p. This means that $\omega_p(I_p X, I_p Y) = \omega(X, Y)$ and thus $g_C(X, Y) = \omega_p(X, I_p Y)$. We also have

$$\omega_p(I_q X, Y) = g_C(I_p I_q X, Y) = \varepsilon_{pqr} g_C(I_r X, Y) = \varepsilon_{pqr}\omega_r(X, Y). \tag{2.1}$$

In fact, we have an S^2-family of Kähler structures: for any $v = (v_1, v_2, v_3) \in S^2$, we can take $I_v = \sum_{p=1}^3 v_p I_p$ and $\omega_v(X, Y) = g_C(I_v X, Y)$.

One can show that C inherits a triple of complex-symplectic forms $\sigma_1, \sigma_2, \sigma_3 \in \Omega^2(C; \mathbb{C})$ via

$$\sigma_1 := \omega_2 + i\omega_3, \qquad \sigma_2 := \omega_3 + i\omega_1, \qquad \sigma_3 := \omega_1 + i\omega_2.$$

A calculation shows that σ_1 is of I_1-type $(2,0)$, and analogously for σ_2, σ_3. It follows that each σ_p is a *holomorphic* symplectic form with respect to I_p.

Further, C inherits the following triple of $(2n+2)$-forms $\Upsilon_1, \Upsilon_2, \Upsilon_3$:

$$\Upsilon_1 = \frac{1}{(n+1)!}\sigma_1^{n+1}, \qquad \Upsilon_2 = \frac{1}{(n+1)!}\sigma_2^{n+1}, \qquad \Upsilon_3 = \frac{1}{(n+1)!}\sigma_3^{n+1}.$$

Each Υ_p is a *holomorphic volume form* with respect to I_p, so that $(g_C, I_p, \omega_p, \Upsilon_p)$ is a Calabi–Yau structure on C. More generally, fixing I_1 as a reference, by considering the holomorphic volume form $e^{i(n+1)\theta}\Upsilon_1 = \frac{1}{(n+1)!}(e^{i\theta}\sigma_1)^{n+1}$, we obtain an S^1-family of Calabi–Yau structures with respect to I_1. Since $e^{i\theta}\sigma_1 = (c_\theta\omega_2 - s_\theta\omega_3) + i(s_\theta\omega_2 + c_\theta\omega_3)$, this S^1-family corresponds to rotating the orthogonal pair I_2, I_3 by θ in the equator of S^2 determined by the poles $\pm I_1$.

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Finally, C also admits a quaternionic-Kähler structure via the real 4-form

$$\Lambda = \frac{1}{6}\omega_1^2 + \frac{1}{6}\omega_2^2 + \frac{1}{6}\omega_3^2.$$

(See Definition 5.2 for our definition of quaternionic Kähler.)

In this section, we recall various classes of distinguished submanifolds of C. Some of these classes – e.g., the complex, Lagrangian, special Lagrangian, and quaternionic – arise from a Calabi–Yau or quaternionic-Kähler structure. Others arise from a complex-symplectic structure, or are otherwise special to the hyperkähler setting.

2.1 Submanifolds via the Calabi–Yau and QK structures

Recall that every hyperkähler manifold is a Kähler manifold in an S^2-family of ways, and given such a choice, it is a Calabi–Yau manifold in an S^1-family of ways. Due to these structures, we may consider the following classes of submanifolds.

Definition 2.1 A submanifold $N^{2k} \subset C^{4n+4}$ is I_1-*complex* if

$$\frac{1}{k!}\omega_1^k\Big|_N = \mathrm{vol}_N.$$

That is, if it is calibrated with respect to $\frac{1}{k!}\omega_1^k$.

It is I_1-*anti-complex*, or $-I_1$-complex, if it is calibrated with respect to $-\frac{1}{k!}\omega_1^k$. Equivalently, if it is I_1-complex when equipped with the opposite orientation.

A submanifold is $\pm I_1$-complex if and only if its tangent spaces are I_1-invariant:

$$I_1(T_xN) = T_xN, \quad \forall x \in N.$$

The definitions of I_2-complex and I_3-complex are analogous.

Definition 2.2 A submanifold $N \subset C^{4n+4}$ is ω_1-*isotropic* if

$$\omega_1|_N = 0.$$

An ω_1-isotropic submanifold satisfies $\dim(N) \leq 2n+2$. An ω_1-*Lagrangian* submanifold is an ω_1-isotropic submanifold of maximal dimension $2n+2$.

Let $X, Y \in TL$. Since $\omega_1(X,Y) = g(I_1X, Y)$, we see that L is ω_1-isotropic if and only if $I_1(TL) \subseteq NL$. If N has dimension $2n+2$, then $I_1(TL) = NL$ if and only if L is ω_1-Lagrangian. We use these facts repeatedly.

Definition 2.3 Fix $\theta \in [0, 2\pi)$. A $(2n+2)$-dimensional submanifold $N^{2n+2} \subset C^{4n+4}$ is called Υ_1-*special Lagrangian of phase* $e^{i\theta}$ if

$$\mathrm{Re}(e^{-i\theta}\Upsilon_1)\big|_N = \mathrm{vol}_N.$$

Equivalently [20, Corollary 1.11], there exists an orientation on N^{2n+2} making it Υ_1-special Lagrangian of phase $e^{i\theta}$ if and only if

$$\mathrm{Im}(e^{-i\theta}\Upsilon_1)\big|_N = 0, \qquad \omega_1|_N = 0.$$

When the phase is left unspecified, we assume it to be $e^{i\theta} = 1$.

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Remark 2.4 Every hyperkähler manifold is also quaternionic-Kähler, and such manifolds admit a distinguished class of *quaternionic submanifolds*. However, Gray [18] proved that such submanifolds are always totally geodesic. We will not consider quaternionic submanifolds in this paper.

2.2 Submanifolds via the hyperkähler structure

In addition to the submanifolds discussed above, hyperkähler manifolds also admit three more notable classes of submanifolds: the complex isotropic, special isotropic, and generalized Cayley submanifolds. We discuss each of these in turn.

2.2.1 Complex isotropic submanifolds

Definition 2.5 A $2k$-dimensional submanifold $L^{2k} \subset C^{4n+4}$ is called I_1-*complex isotropic* if it is both I_1-complex and σ_1-isotropic. That is, if

$$\frac{1}{k!}\omega_1^k\Big|_L = \mathrm{vol}_L, \qquad \sigma_1|_L = 0.$$

Said another way, L is I_1-complex, ω_2-isotropic, and ω_3-isotropic:

$$\frac{1}{k!}\omega_1^k\Big|_L = \mathrm{vol}_L, \qquad \omega_2|_L = 0, \qquad \omega_3|_L = 0.$$

An I_1-*complex Lagrangian* submanifold $L^{2n+2} \subset C^{4n+4}$ is an I_1-complex isotropic submanifold of maximal dimension $2n+2$. That is, an I_1-complex Lagrangian submanifold is simultaneously I_1-complex, ω_2-Lagrangian, and ω_3-Lagrangian. The definitions of I_2- and I_3-complex isotropic (resp. complex Lagrangian) are analogous.

Complex isotropic submanifolds are interesting from several points of view. For example, in algebraic geometry, one often considers holomorphic symplectic manifolds that are fibered by complex Lagrangians, as in [29]. As another example, Doan and Rezchikov [11] use complex Lagrangians as part of a hyperkähler Floer theory. In the differential geometry literature, complex isotropic submanifolds have been studied by, for example, Bryant and Harvey [9], Hitchin [22], and Grantcharov and Verbitsky [17].

Proposition 2.6 *Let $L^{2k} \subset C^{4n+4}$ be a $2k$-dimensional submanifold. The following are equivalent:*

(1) *L is I_1-complex, ω_2-isotropic, and ω_3-isotropic.*
(2) *L is I_1-complex and ω_2-isotropic.*

Proof One direction is immediate. For the converse, suppose L is I_1-complex and ω_2-isotropic. Let $X \in TL$, so that $I_1X \in TL$, and thus $-I_3X = I_2(I_1X) \in NL$. Hence, $I_3X \in NL$. This shows that L is ω_3-isotropic. ■

In the complex Lagrangian case, we can say more:

Proposition 2.7 *Let $L^{2n+2} \subset C^{4n+4}$ be a $(2n+2)$-dimensional submanifold. The following are equivalent:*

(1) *L is I_1-complex, ω_2-Lagrangian, and ω_3-Lagrangian.*
(2) *L is I_1-complex and ω_2-Lagrangian.*

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(3) *L is ω_2-Lagrangian and ω_3-Lagrangian.*
(4) *L is I_1-complex, Υ_2-special Lagrangian of phase i^{n+1}, and Υ_3-special Lagrangian of phase* 1.

Proof The equivalence (i) $\iff$ (ii) was observed above. It is clear that (i) $\implies$ (iii). For (iii) $\implies$ (i), suppose that L is ω_2- and ω_3-Lagrangian. Let $X \in TL$, so that $I_3X \in NL$, and thus $I_1X = I_2(I_3X) \in TL$. Hence, L is I_1-complex.

It is clear that (iv) $\implies$ (i). For (i) $\implies$ (iv), suppose that L is I_1-complex, ω_2-Lagrangian, and ω_3-Lagrangian. Then L satisfies $\frac{1}{(n+1)!}\omega_1^{n+1}\big|_L = \mathrm{vol}_L$ and $\omega_2|_L = 0$ and $\omega_3|_L = 0$. Recalling that

$$(-i)^{n+1}\Upsilon_2 = \frac{1}{(n+1)!}(\omega_1 - i\omega_3)^{n+1}, \qquad \Upsilon_3 = \frac{1}{(n+1)!}(\omega_1 + i\omega_2)^{n+1},$$

we have

$$\mathrm{Re}((-i)^{n+1}\Upsilon_2)\big|_L = \frac{1}{(n+1)!}\omega_1^{n+1}\Big|_L = \mathrm{vol}_L, \quad \mathrm{Re}(\Upsilon_3)|_L = \frac{1}{(n+1)!}\omega_1^{n+1}\Big|_L = \mathrm{vol}_L.$$

∎

2.2.2 Special isotropic submanifolds

The following definition is due to Bryant and Harvey [9]. We prove that these forms are calibrations in Theorem A.6 in the Appendix.

Definition 2.8 The *special isotropic forms* are the $2k$-forms $\Theta_{I,2k}, \Theta_{J,2k}, \Theta_{K,2k} \in \Omega^{2k}(C)$ defined by

$$\Theta_{I,2k} = \frac{1}{k!}\mathrm{Re}(\sigma_1^k), \qquad \Theta_{J,2k} = \frac{1}{k!}\mathrm{Re}(\sigma_2^k), \qquad \Theta_{K,2k} = \frac{1}{k!}\mathrm{Re}(\sigma_3^k).$$

A $2k$-dimensional submanifold $N^{2k} \subset C^{4n+4}$ is $\Theta_{I,2k}$-*special isotropic* if it is calibrated by $\Theta_{I,k}$:

$$\Theta_{I,2k}|_N = \mathrm{vol}_N.$$

The definitions of $\Theta_{J,2k}$- and $\Theta_{K,2k}$-special isotropic $2k$-manifold are analogous.

Let us highlight the cases $2k = 2, 4, 2n + 2$.

Example 2.1

(1) For $2k = 2$, the special isotropic 2-forms are

$$\Theta_{I,2} = \omega_2, \qquad \Theta_{J,2} = \omega_3, \qquad \Theta_{K,2} = \omega_1.$$

In particular, a $\Theta_{I,2}$-special isotropic 2-fold is the same as an I_2-complex 2-fold.

(2) For $2k = 4$, the special isotropic 4-forms are

$$\Theta_{I,4} = \frac{1}{2}(\omega_2^2 - \omega_3^2), \qquad \Theta_{J,4} = \frac{1}{2}(\omega_3^2 - \omega_1^2), \qquad \Theta_{K,4} = \frac{1}{2}(\omega_1^2 - \omega_2^2).$$

In particular, if L is an I_1-complex isotropic 4-fold, then L is both $-\Theta_{J,4}$-special isotropic and $\Theta_{K,4}$-special isotropic.

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(3) For $2k = 2n+2$, the special isotropic $(2n+2)$-forms are

$$\Theta_{I,2n+2} = \mathrm{Re}(\Upsilon_1), \qquad \Theta_{J,2n+2} = \mathrm{Re}(\Upsilon_2), \qquad \Theta_{K,2n+2} = \mathrm{Re}(\Upsilon_3).$$

In particular, a $\Theta_{I,2n+2}$-special isotropic $(2n+2)$-fold is the same as an Υ_1-special Lagrangian, which explains the name "special isotropic."

At present, it appears that little is known about special isotropic $2k$-folds in hyperkähler $(4n+4)$-manifolds when $2 < 2k < 2n+2$.

2.2.3 Cayley 4-folds

The following definition is due to Bryant and Harvey [9], though our sign conventions are opposite to theirs.

Definition 2.9 The *generalized Cayley 4-forms* are the 4-forms $\Phi_1, \Phi_2, \Phi_3 \in \Omega^4(C)$ defined by

$$\Phi_1 = -\frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 + \frac{1}{2}\omega_3^2, \quad \Phi_2 = \frac{1}{2}\omega_1^2 - \frac{1}{2}\omega_2^2 + \frac{1}{2}\omega_3^2, \quad \Phi_3 = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 - \frac{1}{2}\omega_3^2.$$

Note that

$$\Phi_2 = \frac{1}{2}\omega_1^2 - \Theta_{I,4} = \frac{1}{2}\omega_3^2 + \Theta_{K,4} = -\frac{1}{2}\omega_2^2 + \frac{1}{2}(\omega_1^2 + \omega_3^2), \tag{2.2}$$

and similarly for cyclic permutations. A four-dimensional submanifold $N^4 \subset C^{4n+4}$ is Φ_2-*Cayley* if it is calibrated by Φ_2:

$$\Phi_2|_N = \mathrm{vol}_N.$$

The definitions of Φ_1-Cayley and Φ_3-Cayley are analogous.

Remark 2.10 Bryant and Harvey [9, Lemma 2.14] computed that the $\mathrm{SO}(4n+4)$-stabilizer of the generalized Cayley 4-forms in $\mathbb{R}^{4n+4}$ are

$$\mathrm{Stab}(\Phi_1) = \begin{cases} \mathrm{Spin}(7), & \text{if } n = 1, \\ \mathrm{Sp}(n+1)\mathrm{O}(2), & \text{if } n \geq 2. \end{cases}$$

This above definition was inspired by $\mathrm{Spin}(7)$-geometry, as we now recall. If $(X^8, (g, \omega, I, \Upsilon))$ is a Calabi–Yau 8-manifold, where $\omega \in \Omega^2(X)$ is the Kähler form and $\Upsilon \in \Omega^4(X; \mathbb{C})$ is the holomorphic volume form, then X inherits a torsion-free $\mathrm{Spin}(7)$-structure via the following formula:

$$\Phi = \frac{1}{2}\omega^2 - \mathrm{Re}(\Upsilon). \tag{2.3}$$

The real 4-form $\Phi \in \Omega^4(X)$ is called the *Cayley 4-form*, and a four-dimensional submanifold $N \subset X$ satisfying $\Phi|_N = \mathrm{vol}_N$ is called *Cayley*. The following fact is well known, but we include a proof for completeness.

Proposition 2.11 *Let $(X^8, (g, \omega, I, \Upsilon))$ be a Calabi–Yau 8-manifold, and equip X with its induced $\mathrm{Spin}(7)$-structure. Let $N^4 \subset X$ be a four-dimensional submanifold.*

(1) *If N is complex, then N is Cayley.*
(2) *If N is special Lagrangian of phase $e^{i\pi} = -1$, then N is Cayley.*

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Proof If N is complex, each tangent space T_xN admits a basis of the form $\{e_1, Ie_1, e_2, Ie_2\}$. Then $v_k = e_k - iIe_k$ is of type $(1,0)$ for $k = 1,2$, and $T_xN = e_1 \wedge Ie_1 \wedge e_2 \wedge Ie_2$ is a multiple of $v_1 \wedge \overline{v_1} \wedge v_2 \wedge \overline{v_2}$ and thus of type $(2,2)$. Since $\mathrm{Re}(\Upsilon)$ is type $(4,0)+(0,4)$, it vanishes on T_xN. But $\frac{1}{2}\omega^2$ restricts to the volume form on T_xN, so by (2.3), N is calibrated by Φ.

If N is special Lagrangian with phase -1, it is calibrated by $-\mathrm{Re}(\Upsilon)$. Since it is also Lagrangian, $\frac{1}{2}\omega^2$ vanishes on N, and thus, again by (2.3), N is calibrated by Φ. ∎

When the ambient space is hyperkähler, Bryant and Harvey showed that the above fact can be generalized to higher dimensions in the following sense.

Proposition 2.12 ([9, Theorem 8.20]) *Let C^{4n+4} be a hyperkähler $(4n+4)$-manifold. Let $L^4 \subset C^{4n+4}$ be a four-dimensional submanifold. Then:*

(1) *If N is I_1-complex or I_3-complex, then N is Φ_2-Cayley.*
(2) *If N is $-\Theta_{I,4}$-special isotropic or $\Theta_{K,4}$-special isotropic, then N is Φ_2-Cayley.*
(3) *If N is I_1-complex isotropic, then N is simultaneously I_1-complex, $-\Theta_{J,4}$-special isotropic, and $\Theta_{K,4}$-special isotropic, and hence is Φ_2-Cayley.*

Proof Parts (a) and (b) are contained in [9, Theorem 8.20]. It is easy to see from (2.2) that (a) holds. For example, if N is I_1-complex, then $\frac{1}{2}\omega_1^2$ restricts to the volume form, but $-\Theta_{I,4} = -\mathrm{Re}(\frac{1}{2}\sigma_1^2)$ is of I_1-type $(4,0)+(0,4)$, and thus vanishes on N since the tangent spaces of N are of I_1-type $(2,2)$. Part (b) is less obvious, and uses a normal form for the tangent spaces of N. Details are given in [9, Sections 2 and 3]. Part (c) is immediate from the first two. ∎

Remark 2.13 Note that every calibration $\phi \in \Omega^k(C)$ discussed in this section is stabilized by the Lie group $\mathrm{Sp}(n+1)$, which acts transitively on the unit sphere in $T_xC \simeq \mathbb{R}^{4n+4}$. Consequently, at any point $x \in C$, every unit vector $v \in T_xC$ lies in some ϕ-calibrated k-plane.

2.3 Bookkeeping: summary of forms on C

Starting in the next section, we will assume that the hyperkähler manifold C^{4n+4} is a metric cone, say $C = \mathrm{C}(M)$ for some Riemannian $(4n+3)$-manifold M. Studying the geometry of M and its relationship with C will require the introduction of further tensors and differential forms. So, before continuing, we briefly summarize the tensors and forms already defined on C:

g_C	Riemannian metric
I_1, I_2, I_3	Complex structures
$\omega_1, \omega_2, \omega_3$	Kähler 2-forms
$\Upsilon_1, \Upsilon_2, \Upsilon_3$	Complex volume $(2n+2)$-forms
$\sigma_1, \sigma_2, \sigma_3$	Complex symplectic 2-forms
$\Theta_{I,2k}, \Theta_{J,2k}, \Theta_{K,2k}$	Special isotropic $2k$-forms
Φ_1, Φ_2, Φ_3	Cayley 4-forms
Λ	Quaternionic 4-form

3 Calibrated geometry in 3-Sasakian manifolds

If $(C^{4n+4}, g_C) = (M \times \mathbb{R}^+, dr^2 + r^2 g_M)$ is a hyperkähler cone, then its link M^{4n+3} inherits a 3-*Sasakian structure*, as we recall in Section 3.1. Then, in Sections 3.2 and 3.3, we explain how each of the calibrated geometries of C discussed previously has a semi-calibrated counterpart in the 3-Sasakian link M.

In Section 3.4, we recall that M is the total space of a natural S^1-bundle $p_1 : M \to Z$. The base space, Z^{4n+2}, called a *twistor space*, admits both Kähler–Einstein and nearly Kähler structures. It is interesting to ask exactly how much geometric structure the map $p_1 : M \to Z$ preserves. In this regard, we discover that every 3-Sasakian manifold M admits a natural $\mathbb{C}$-valued 3-form $\Gamma_1 \in \Omega^3(M; \mathbb{C})$ that descends to a 3-form on Z (Proposition 3.21). Later, in Section 4.2, we will prove that the descended 3-form endows Z with a canonical $\mathrm{Sp}(n)\mathrm{U}(1)$-structure.

Finally, in Theorem 3.20, we observe that $\mathrm{Re}(\Gamma_1) \in \Omega^3(M)$ is a semi-calibration, and classify the $\mathrm{Re}(\Gamma_1)$-calibrated 3-folds in terms of more familiar geometries.

3.1 3-Sasakian manifolds as links

Definition 3.1 Let M be an odd-dimensional manifold. An *almost contact metric structure* on M is a triple $(g_M, \alpha, \mathrm{J})$ consisting of a Riemannian metric g_M, a 1-form $\alpha \in \Omega^1(M)$, and an endomorphism $\mathrm{J} \in \Gamma(\mathrm{End}(TM))$ satisfying

$$\alpha(\mathrm{J}X) = 0, \quad \mathrm{J}(A) = 0, \quad \mathrm{J}^2|_{\mathrm{Ker}(\alpha)} = -\mathrm{Id}, \quad g_M(\mathrm{J}X, \mathrm{J}Y) = g_M(X, Y) - \alpha(X)\alpha(Y),$$

where $A := \alpha^\sharp \in \Gamma(TM)$ is the *Reeb vector field*. It follows that $\alpha(A) = 1$.

Thus, if M is equipped with an almost contact metric structure, then each tangent space splits as

$$T_x M = \mathbb{R}A|_x \oplus \mathrm{Ker}(\alpha|_x).$$

Further, restricting to the hyperplane $\mathrm{Ker}(\alpha|_x) \subset T_x M$, the endomorphism $\mathrm{J} : \mathrm{Ker}(\alpha|_x) \to \mathrm{Ker}(\alpha|_x)$ is a g_M-orthogonal complex structure. Thus, the hyperplane field $\mathrm{Ker}(\alpha) \subset TM$ is naturally endowed with the Hermitian structure $(g_M, \mathrm{J}, \Omega)$, where $\Omega := g_M(\mathrm{J}\cdot, \cdot)$ is the corresponding nondegenerate 2-form.

Definition 3.2 Let M be a $(4n+3)$-manifold. An $(\mathrm{Sp}(n) \times 3)$-*structure* (or *almost 3-contact metric structure*) on M consists of data $(g_M, (\alpha_1, \alpha_2, \alpha_3), (\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3))$ such that:

- each triple $(g_M, \alpha_p, \mathrm{J}_p)$ is an almost contact metric structure ($p = 1, 2, 3$); and
- letting $A_p := \alpha_p^\sharp \in \Gamma(TM)$ denote the corresponding Reeb fields, we require

$$\mathrm{J}_p \circ \mathrm{J}_q - \alpha_p \otimes A_q = \varepsilon_{pqr} \mathrm{J}_r - \delta_{pq}\,\mathrm{Id},$$
$$\mathrm{J}_p(A_q) = \varepsilon_{pqr} A_r.$$

Note that there is no sum over r in the above equations. For example, the above equations say $\mathrm{J}_1(A_1) = 0$, $\mathrm{J}_1(A_2) = A_3$, $\mathrm{J}_1(A_3) = -A_2$, that $\mathrm{J}_1^2 = -\mathrm{Id}$ on $\mathrm{Ker}(\alpha_1)$, and that $\mathrm{J}_1\mathrm{J}_2 = \mathrm{J}_3$. Similarly for cyclic permutations of 1, 2, 3.

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Let M^{4n+3} carry an $(\mathrm{Sp}(n) \times 3)$-structure. We make three remarks. First, for each $p = 1, 2, 3$, the tangent bundle splits as

$$TM = \mathbb{R}A_p \oplus \mathrm{Ker}(\alpha_p), \tag{3.1}$$

and the hyperplane field $\mathrm{Ker}(\alpha_p) \subset TM$ carries a Hermitian structure $(g_M, \mathsf{J}_p, \Omega_p)$, where $\Omega_p := g_M(\mathsf{J}_p\cdot, \cdot)$. In fact, each $\mathrm{Ker}(\alpha_p)$ is also endowed with the complex volume form $\Psi_p \in \Lambda^{2n+1,0}(\mathrm{Ker}(\alpha_p))$ given by

$$\begin{aligned} \Psi_1 &= (\alpha_2 + i\alpha_3) \wedge \frac{1}{n!}(\Omega_2 + i\Omega_3)^n, \\ \Psi_2 &= (\alpha_3 + i\alpha_1) \wedge \frac{1}{n!}(\Omega_3 + i\Omega_1)^n, \\ \Psi_3 &= (\alpha_1 + i\alpha_2) \wedge \frac{1}{n!}(\Omega_1 + i\Omega_2)^n. \end{aligned} \tag{3.2}$$

Second, considering (3.1) for $p = 1, 2, 3$ simultaneously, we see that the tangent bundle splits further as

$$TM = \widetilde{\mathsf{H}} \oplus \widetilde{\mathsf{V}}, \tag{3.3}$$

where

$$\widetilde{\mathsf{H}} = \mathrm{Ker}(\alpha_1, \alpha_2, \alpha_3), \qquad \widetilde{\mathsf{V}} = \mathbb{R}A_1 \oplus \mathbb{R}A_2 \oplus \mathbb{R}A_3.$$

Note that the $4n$-plane field $\widetilde{\mathsf{H}} \subset TM$ is preserved by the three endomorphisms $\mathsf{J}_1, \mathsf{J}_2, \mathsf{J}_3$. In fact, the restrictions of $\mathsf{J}_1, \mathsf{J}_2, \mathsf{J}_3$ to $\widetilde{\mathsf{H}}$ are g_M-orthogonal complex structures that satisfy the quaternionic relations $\mathsf{J}_1\mathsf{J}_2 = \mathsf{J}_3$, etc.

Third, we consider the relationship between the structure on a manifold (M^{4n+3}, g_M) and that of its metric cone

$$C^{4n+4} = \mathrm{C}(M) = (\mathbb{R}^+ \times M, g_{\mathrm{C}} = dr^2 + r^2 g_M).$$

In one direction, if (M, g_M) is equipped with a compatible $(\mathrm{Sp}(n) \times 3)$-structure $(g_M, (\alpha_p), (\mathsf{J}_p))$, then the $(4n+4)$-manifold C inherits a Riemannian metric g_{C}, a triple of g_{C}-orthogonal almost-complex structures (I_1, I_2, I_3) satisfying $I_1 I_2 = I_3$, etc., and a triple of nondegenerate 2-forms ω_p defined by

$$g_{\mathrm{C}} = dr^2 + r^2 g_M, \qquad \omega_p(X, Y) = g_{\mathrm{C}}(I_p X, Y),$$
$$I_p(X) = \begin{cases} \mathsf{J}_p X - \alpha_p(X) r\partial_r, & \text{if } X \in TM, \\ A_p, & \text{if } X = r\partial_r, \end{cases}$$

where $X, Y \in TC$. A computation shows that for each $p = 1, 2, 3$,

$$\omega_p = r\, dr \wedge \alpha_p + r^2 \Omega_p. \tag{3.4}$$

Altogether, the data $(g_{\mathrm{C}}, (\omega_1, \omega_2, \omega_3), (I_1, I_2, I_3))$ are an almost hyper-Hermitian structure on C.

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Conversely, if the metric cone $(C^{4n+4}, g_C = dr^2 + r^2 g_M)$ carries an almost hyper-Hermitian structure $(g_C, (\omega_1, \omega_2, \omega_3), (I_1, I_2, I_3))$ that is conical in the sense of Definition A.9, namely that

$$\mathscr{L}_{r\partial_r}(\omega_p) = 2\omega_p, \quad p = 1, 2, 3,$$

then its link (M, g_M) inherits a compatible $(\mathrm{Sp}(n) \times 3)$-structure $(g_M, (\alpha_p), (\mathsf{J}_p))$ via

$$\alpha_p = (r\partial_r \lrcorner\, \omega_p)|_M, \qquad \mathsf{J}_p = \begin{cases} I_p, & \text{on } \mathrm{Ker}(\alpha_p), \\ 0, & \text{on } \mathbb{R}A_p. \end{cases}$$

This relationship leads to the following definition:

Definition 3.3 Let M be a $(4n+3)$-manifold. A 3-*Sasakian structure* on M is an $(\mathrm{Sp}(n) \times 3)$-structure $(g_M, (\alpha_p), (\mathsf{J}_p))$ for which the induced almost hyper-Hermitian structure $(g_C, (\omega_p), (I_p))$ on its metric cone $\mathrm{C}(M) = \mathbb{R}^+ \times M$ hyperkähler.

Note that this is equivalent to requiring that the 2-forms $\omega_1, \omega_2, \omega_3$ are all closed. (See, for example, [21, Section 2].)

3.1.1 Distinguished forms on 3-Sasakian manifolds

For the remainder of this work, M^{4n+3} will denote a 3-Sasakian $(4n+3)$-manifold with 3-Sasakian structure $(g_M, (\alpha_1, \alpha_2, \alpha_3), (\mathsf{J}_1, \mathsf{J}_2, \mathsf{J}_3))$. The induced conical hyperkähler structure on $C^{4n+4} = \mathbb{R}^+ \times M$ will be denoted $(g_C, (\omega_1, \omega_2, \omega_3), (I_1, I_2, I_3))$. In this section, we record some of the distinguished differential forms on M and compute their exterior derivatives.

To begin, we consider the contact 1-forms $\alpha_1, \alpha_2, \alpha_3 \in \Omega^1(M)$ and the transverse Kähler forms $\Omega_1, \Omega_2, \Omega_3 \in \Omega^2(M)$ defined by $\Omega_p(X, Y) = g_M(\mathsf{J}_p X, Y)$. By (3.4), we may compute

$$0 = d\omega_p = d(r\,dr \wedge \alpha_p) + d(r^2 \Omega_p) = r\,dr \wedge (-d\alpha_p + 2\Omega_p) + r^2 d\Omega_p,$$

which implies that

$$d\alpha_p = 2\Omega_p, \qquad d\Omega_p = 0. \tag{3.5}$$

(The first equation in (3.5) shows that each α_p is indeed a contact form. That is, that $\alpha_p \wedge (d\alpha_p)^{2n+1}$ is nowhere zero.)

Next, we decompose the 2-forms $\Omega_1, \Omega_2, \Omega_3$ according to the splitting

$$\Lambda^2(T^*M) = \Lambda^2(\widetilde{\mathsf{V}}^*) \oplus (\widetilde{\mathsf{V}} \otimes \widetilde{\mathsf{H}}) \oplus \Lambda^2(\widetilde{\mathsf{H}}^*).$$

One can show that each Ω_p has no component in $\widetilde{\mathsf{V}}^* \otimes \widetilde{\mathsf{H}}^*$ and that the $\Lambda^2(\widetilde{\mathsf{V}}^*)$-component of Ω_1 is $\alpha_2 \wedge \alpha_3$. Letting $\kappa_1, \kappa_2, \kappa_3$ denote the $\Lambda^2(\widetilde{\mathsf{H}}^*)$-component of Ω_p, we arrive at the formulas

$$\Omega_1 = \alpha_2 \wedge \alpha_3 + \kappa_1, \qquad \Omega_2 = \alpha_3 \wedge \alpha_1 + \kappa_2, \qquad \Omega_3 = \alpha_1 \wedge \alpha_2 + \kappa_3. \tag{3.6}$$

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Taking d of (3.6) and using (3.5) shows that

$$
\begin{aligned}
d\kappa_1 &= 2(\alpha_2 \wedge \Omega_3 - \alpha_3 \wedge \Omega_2) = 2(\alpha_2 \wedge \kappa_3 - \alpha_3 \wedge \kappa_2), \\
d\kappa_2 &= 2(\alpha_3 \wedge \Omega_1 - \alpha_1 \wedge \Omega_3) = 2(\alpha_3 \wedge \kappa_1 - \alpha_1 \wedge \kappa_3), \\
d\kappa_3 &= 2(\alpha_1 \wedge \Omega_2 - \alpha_2 \wedge \Omega_1) = 2(\alpha_1 \wedge \kappa_2 - \alpha_2 \wedge \kappa_1).
\end{aligned} \tag{3.7}
$$

Finally, recalling the transverse complex volume forms $\Psi_1, \Psi_2, \Psi_3 \in \Omega^{2n+1}(M;\mathbb{C})$ of (3.2), we compute

(3.8)

$$
d\Psi_1 = \frac{2}{n!}(\Omega_2 + i\Omega_3)^{n+1}, \quad d\Psi_2 = \frac{2}{n!}(\Omega_3 + i\Omega_1)^{n+1}, \quad d\Psi_3 = \frac{2}{n!}(\Omega_1 + i\Omega_2)^{n+1}.
$$

To conclude this section, we summarize the relationships between various forms on the hyperkähler cone C^{4n+4} and those on its 3-Sasakian link M^{4n+3}.

Proposition 3.4 *We have*

$$
\omega_1 = r\,dr \wedge \alpha_1 + r^2\,\Omega_1, \tag{3.9}
$$

$$
\frac{1}{2}\omega_1^2 = r^3\,dr \wedge (\alpha_1 \wedge \Omega_1) + r^4\,\frac{1}{2}\Omega_1^2, \tag{3.10}
$$

$$
\frac{1}{k!}\omega_1^k = r^{2k-1}\,dr \wedge \frac{1}{(k-1)!}(\alpha_1 \wedge \Omega_1^{k-1}) + r^{2k}\,\frac{1}{k!}\Omega_1^k. \tag{3.11}
$$

Consequently,

$$
\begin{aligned}
\Upsilon_1 &= r^{2n+1}\,dr \wedge \Psi_1 + r^{2n+2}\,\frac{1}{(n+1)!}(\Omega_2 + i\Omega_3)^{n+1}, \\
\Theta_{I,4} &= r^3\,dr \wedge (\alpha_2 \wedge \Omega_2 - \alpha_3 \wedge \Omega_3) + r^4\frac{1}{2}(\Omega_2^2 - \Omega_3^2), \\
\Phi_1 &= r^3\,dr \wedge (-\alpha_1 \wedge \Omega_1 + \alpha_2 \wedge \Omega_2 + \alpha_3 \wedge \Omega_3) + r^4\,\frac{1}{2}(-\Omega_1^2 + \Omega_2^2 + \Omega_3^2), \\
\Lambda &= r^3\,dr \wedge \frac{1}{3}(\alpha_1 \wedge \Omega_1 + \alpha_2 \wedge \Omega_2 + \alpha_3 \wedge \Omega_3) + r^4\,\frac{1}{6}(\Omega_1^2 + \Omega_2^2 + \Omega_3^2).
\end{aligned} \tag{3.12}
$$

Proof Each of these follows from a straightforward calculation. ∎

3.2 Submanifolds via the Sasaki–Einstein structure

By analogy with our discussion in Sections 2.1 and 2.2, we now consider the various classes of submanifolds of M. We begin with those defined in terms of a Sasaki–Einstein structure.

By Remark 2.13, we can apply Proposition A.1 to (3.11) with k replaced by $k+1$. We deduce that for $p = 1,2,3$, the $(2k+1)$-forms

$$
\frac{1}{k!}(\alpha_p \wedge \Omega_p^k) \in \Omega^{2k+1}(M)
$$

are semi-calibrations. Their calibrated submanifolds are called I_p*-CR submanifolds*. To be precise:

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Proposition 3.5 *Let $L^{2k+1} \subset M^{4n+3}$ be a $(2k+1)$-dimensional submanifold. We say L is I_1-CR if any of the following equivalent conditions holds:*

(1) $\mathrm{C}(L) \subset C$ *is I_1-complex. That is, each tangent space of* $\mathrm{C}(L)$ *is I_1-invariant.*
(2) $\mathrm{C}(L)$ *is (up to a change of orientation)* $\frac{1}{(k+1)!}\omega_1^{k+1}$*-calibrated:*

$$\frac{1}{(k+1)!}\,\omega_1^{k+1}\Big|_{\mathrm{C}(L)} = \mathrm{vol}_{\mathrm{C}(L)}.$$

(3) *Each tangent space T_xL is* J_1*-invariant and contains the Reeb vector A_1.*
(4) *L satisfies (up to a change of orientation) that*

$$\frac{1}{k!}(\alpha_1 \wedge \Omega_1^k)\Big|_L = \mathrm{vol}_L.$$

Proof The equivalences (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) are well known. The equivalence (ii) $\Longleftrightarrow$ (iv) follows from Proposition A.1. ∎

Proposition 3.6 *Let $L^k \subset M^{4n+3}$ be a submanifold. We say L is α_1-isotropic (resp. α_1-Legendrian if $k = 2n+1$) if any of the following equivalent conditions holds:*

(1) $\mathrm{C}(L)$ *is ω_1-isotropic:* $\omega_1|_{\mathrm{C}(L)} = 0$.
(2) $\alpha_1|_L = 0$.
(3) $\alpha_1|_L = 0$ *and* $\Omega_1|_L = 0$.

In particular, an α_1-isotropic submanifold $L \subset M$ satisfies $\dim(L) \leq 2n+1$.

Proof The first equation in (3.5) shows the equivalence (ii) $\Longleftrightarrow$ (iii). The equivalence (i) $\Longleftrightarrow$ (iii) follows from (3.9). ∎

Next, from formula (3.12) together with Proposition A.1 and Remark 2.13, we observe that for $p = 1,2,3$ and a constant $e^{i\theta} \in S^1$, the $(2n+1)$-forms

$$\mathrm{Re}(e^{-i\theta}\Psi_p) \in \Omega^{2n+1}(M)$$

are semi-calibrations. Their calibrated submanifolds are called *Ψ_p-special Legendrian submanifolds of phase $e^{i\theta}$*. We observe:

Proposition 3.7 *Let $L^{2n+1} \subset M^{4n+3}$ be a $(2n+1)$-dimensional submanifold. We say L is Ψ_1-special Legendrian if any of the following equivalent conditions holds:*

(1) $\mathrm{C}(L)$ *is (up to a change of orientation) Υ_1-special Lagrangian:* $\mathrm{Re}(\Upsilon_1)|_{\mathrm{C}(L)} = \mathrm{vol}_{\mathrm{C}(L)}$.
(2) $\mathrm{C}(L)$ *satisfies* $\omega_1|_{\mathrm{C}(L)} = 0$ *and* $\mathrm{Im}(\Upsilon_1)|_{\mathrm{C}(L)} = 0$.
(3) *L satisfies (up to a change of orientation) that* $\mathrm{Re}(\Psi_1)|_L = \mathrm{vol}_L$.
(4) *L satisfies* $\alpha_1|_L = 0$ *and* $\mathrm{Im}(\Psi_1)|_L = 0$.

Proof The equivalence (i) $\Longleftrightarrow$ (ii) is well known. The equivalence (ii) $\Longleftrightarrow$ (iv) follows from equation (3.12) and Proposition 3.6. The equivalence (i) $\Longleftrightarrow$ (iii) follows from (3.12), Remark 2.13, and Proposition A.1. ∎

3.3 Submanifolds via the 3-Sasakian structure

We now turn to those submanifolds of M whose definition requires more than the Sasaki–Einstein structure. Here, we will discuss the *CR isotropic*, *special isotropic*, and *associative* submanifolds.

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3.3.1 CR isotropic submanifolds

Proposition 3.8 *Let $L^{2k+1} \subset M^{4n+3}$ be a $(2k+1)$-dimensional submanifold, $1 \le k \le n$. We say L is I_1-CR isotropic (resp. I_1-CR* Legendrian *if $k = n$) if any of the following equivalent conditions holds:*

(1) $\mathrm{C}(L) \subset C$ *is I_1-complex, ω_2-isotropic, and ω_3-isotropic.*
(2) $\mathrm{C}(L) \subset C$ *is I_1-complex and ω_2-isotropic.*
(3) *L is I_1-CR, α_2-isotropic, and α_3-isotropic.*
(4) *L is I_1-CR and α_2-isotropic.*

Proof The equivalence (i) $\Longleftrightarrow$ (ii) was shown in Proposition 2.6. The equivalences (i) $\Longleftrightarrow$ (iii) and (ii) $\Longleftrightarrow$ (iv) both follow directly from Propositions 3.5 and 3.6. ■

In the CR Legendrian case, we can say more:

Corollary 3.9 *Let $L^{2n+1} \subset M^{4n+3}$ be a $(2n+1)$-dimensional submanifold. The following are equivalent:*

(1) $\mathrm{C}(L)$ *is I_1-complex, ω_2-Lagrangian, and ω_3-Lagrangian (i.e.,* $\mathrm{C}(L)$ *is I_1-complex Lagrangian).*
(2) $\mathrm{C}(L)$ *is ω_2-Lagrangian and ω_3-Lagrangian.*
(3) $\mathrm{C}(L)$ *is I_1-complex, Υ_2-special Lagrangian of phase i^{n+1}, and Υ_3-special Lagrangian of phase* 1.
(4) *L is I_1-CR, α_2-Legendrian, and α_3-Legendrian (i.e., L is I_1-CR Legendrian).*
(5) *L is α_2-Legendrian and α_3-Legendrian.*
(6) *L is I_1-CR, Ψ_2-special Legendrian of phase i^{n+1}, and Ψ_3-special Lagrangian of phase* 1.

Proof The equivalence (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) was shown in Proposition 2.7. The equivalence (i) $\Longleftrightarrow$ (iv) was shown in Proposition 3.8. Finally, (ii) $\Longleftrightarrow$ (v) follows from Proposition 3.6, and (iii) $\Longleftrightarrow$ (vi) follows from Proposition 3.7. ■

Examples of CR isotropic submanifolds can be constructed via Example 5.2 together with Corollary 5.12.

3.3.2 Special isotropic submanifolds

Definition 3.10 The *special isotropic forms* on M are the real $(2k-1)$-forms $\theta_{I,2k-1}, \theta_{J,2k-1}, \theta_{K,2k-1} \in \Omega^{2k-1}(M)$ defined by

$$\theta_{I,2k-1} := (r\partial_r \lrcorner \Theta_{I,2k})|_M, \quad \theta_{J,2p-1} := (r\partial_r \lrcorner \Theta_{J,2k})|_M, \quad \theta_{K,2k-1} := (r\partial_r \lrcorner \Theta_{K,2k})|_M.$$

In particular, for $2k-1 = 1, 3, 2n+1$, these are

$$\begin{aligned} \theta_{I,1} &= \alpha_2, \\ \theta_{I,3} &= \alpha_2 \wedge \Omega_2 - \alpha_3 \wedge \Omega_3 = \alpha_2 \wedge \kappa_2 - \alpha_3 \wedge \kappa_3, \\ \theta_{I,2n+1} &= \mathrm{Re}(\Psi_1). \end{aligned}$$

By Remark 2.13, Proposition A.1, and Theorem A.6, the special isotropic forms $\theta_{I,2k-1}, \theta_{J,2k-1}, \theta_{K,2k-1}$ are semi-calibrations.

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Proposition 3.11 *Let $L^{2k-1} \subset M^{4n+3}$ be a $(2k-1)$-dimensional submanifold, $1 \leq k \leq n+1$. We say L is $\theta_{I,2k-1}$-special isotropic if either of the following equivalent conditions holds:*

(1) $\mathrm{C}(L) \subset C$ *is* $\Theta_{I,2k}$*-special isotropic.*
(2) L *is* $\theta_{I,2k-1}$*-special isotropic.*

Proof This follows from Remark 2.13 and Proposition A.1. ∎

3.3.3 Associative 3-folds

The following definition is due to Bryant and Harvey [9].

Definition 3.12 The *generalized associative* 3*-forms* are the real 3-forms $\phi_1, \phi_2, \phi_3 \in \Omega^3(M)$ defined by

$$\phi_1 = -\alpha_1 \wedge \Omega_1 + \alpha_2 \wedge \Omega_2 + \alpha_3 \wedge \Omega_3,$$
$$\phi_2 = \alpha_1 \wedge \Omega_1 - \alpha_2 \wedge \Omega_2 + \alpha_3 \wedge \Omega_3,$$
$$\phi_3 = \alpha_1 \wedge \Omega_1 + \alpha_2 \wedge \Omega_2 - \alpha_3 \wedge \Omega_3.$$

Equivalently,

$$\phi_1 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \kappa_1 + \alpha_2 \wedge \kappa_2 + \alpha_3 \wedge \kappa_3,$$
$$\phi_2 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \kappa_1 - \alpha_2 \wedge \kappa_2 + \alpha_3 \wedge \kappa_3,$$
$$\phi_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \kappa_1 + \alpha_2 \wedge \kappa_2 - \alpha_3 \wedge \kappa_3,$$

where the κ_j were defined in (3.6). A three-dimensional submanifold $L^3 \subset M^{4n+3}$ is ϕ_1*-associative* if it is calibrated by ϕ_1:

$$\phi_1|_L = \mathrm{vol}_L.$$

The definitions of ϕ_2-associative and ϕ_3-associative are analogous.

Observing that

$$\Phi_1 = r^3\, dr \wedge \phi_1 + r^4\, \frac{1}{2}(-\Omega_1^2 + \Omega_2^2 + \Omega_3^2),$$

we obtain:

Proposition 3.13 *Let $L^3 \subset M^{4n+3}$ be a three-dimensional submanifold. The following are equivalent:*

(1) $\mathrm{C}(L) \subset C$ *is* Φ_1*-Cayley.*
(2) $L \subset M$ *is* ϕ_1*-associative.*

Proof This follows from Remark 2.13 and Proposition A.1. ∎

Finally, we remark on the relationships between the above submanifolds. Let us recall that a manifold is called Sasaki–Einstein if its cone is Calabi–Yau and that a 7-manifold is called nearly parallel G_2 if its cone is a $\mathrm{Spin}(7)$-manifold. Suppose now that $(Y^7, (g, \alpha, \mathrm{J}, \Psi))$ is a Sasaki–Einstein 7-manifold. It is well known that Y inherits a nearly parallel G_2-structure by the following formula:

$$\phi = \alpha \wedge \Omega - \mathrm{Re}(\Psi).$$

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The real 3-form $\phi \in \Omega^3(M)$ is called the *associative* 3-*form*, and a three-dimensional submanifold $\Sigma^3 \subset M$ satisfying $\phi|_\Sigma = \mathrm{vol}_\Sigma$ is called *associative*. The following fact is well known, although we prove a more general result in Proposition 3.15.

Proposition 3.14 *Let $(Y^7, (g, \alpha, \mathrm{J}, \Psi))$ be a Sasaki–Einstein 7-manifold, and equip Y with its induced nearly parallel G_2-structure ϕ. Let $L^3 \subset Y$ be a three-dimensional submanifold. Then:*

(1) *If L is CR, then L is associative.*
(2) *If L is special Legendrian of phase $e^{i\pi} = -1$, then L is associative.*

When the ambient space is 3-Sasakian, the above fact generalizes to higher dimensions in the following way.

Proposition 3.15 *Let M^{4n+3} be a 3-Sasakian $(4n+3)$-manifold. Let $L^3 \subset M$ be a three-dimensional submanifold. Then:*

(1) *If L is I_1-CR or I_3-CR, then L is ϕ_2-associative.*
(2) *If L is $-\theta_{I,3}$-special isotropic or $\theta_{K,3}$-special isotropic, then L is ϕ_2-associative.*
(3) *If L is I_1-CR isotropic, then L is simultaneously I_1-CR, $-\theta_{J,3}$-special isotropic, and $\theta_{K,3}$-special isotropic, and hence is ϕ_2-associative.*

Proof (a) If $L \subset M$ is I_1-CR (resp. I_3-CR), then Proposition 3.5 implies that its cone $\mathrm{C}(L) \subset C$ is I_1-complex (resp. I_3-complex). By Proposition 2.12(a), $\mathrm{C}(L)$ is Φ_2-Cayley, so by Proposition 3.13, L is ϕ_2-associative.

(b) If $L \subset M$ is $-\theta_{I,3}$-special isotropic (resp. $\theta_{K,3}$-special isotropic), then Proposition 3.11 implies that its cone $\mathrm{C}(L) \subset C$ is $-\Theta_{I,4}$-special isotropic (resp. $\Theta_{K,4}$-special isotropic). By Proposition 2.12(b), $\mathrm{C}(L)$ is Φ_2-Cayley, so by Proposition 3.13, L is ϕ_2-associative.

(c) If L is I_1-CR isotropic, then Proposition 3.8 implies that $\mathrm{C}(L) \subset C$ is I_1-complex isotropic, and the result follows from an argument analogous to those used in parts (a) and (b). Alternatively, if L is I_1-CR isotropic, then by definition, L is I_1-CR, α_2-isotropic, and α_3-isotropic. Recalling that

$$\theta_{J,3} = \alpha_3 \wedge \Omega_3 - \alpha_1 \wedge \Omega_1 \qquad\qquad \theta_{K,3} = \alpha_1 \wedge \Omega_1 - \alpha_2 \wedge \Omega_2,$$

we observe that L is $-\theta_{J,3}$- and $\theta_{K,3}$-special isotropic. ∎

Remark 3.16 Where associative 3-folds in 3-Sasakian manifolds M^{4n+3} are concerned, the case $n = 1$ has received the most attention in light of the connection to G_2-geometry. Recently, several studies have considered the two one-parameter families of *squashed* associative 3-forms on M^7 given by

$$-\phi^-_{1,t} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + t^2(-\alpha_1 \wedge \kappa_1 + \alpha_2 \wedge \kappa_2 + \alpha_3 \wedge \kappa_3),$$
$$\phi^+_{1,t} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - t^2(\alpha_1 \wedge \kappa_1 + \alpha_2 \wedge \kappa_2 + \alpha_3 \wedge \kappa_3).$$

See, for example, [6], [23], or [24].

3.3.4 Summary

The following table summarizes the relationships discussed above.

$\dim(\mathrm{C}(L))$	Cone $\mathrm{C}(L) \subset C$	Link $L \subset M$	$\dim(L)$
$2k$	I_1-complex	I_1-CR	$2k-1$
$2n+2$	ω_1-Lagrangian	α_1-Legendrian	$2n+1$
$\leq 2n+2$	ω_1-isotropic	α_1-isotropic	$\leq 2n+1$
$2n+2$	Υ_1-special Lagrangian	Ψ_1-special Legendrian	$2n+1$
$2n+2$	I_1-complex Lagrangian	I_1-CR Legendrian	$2n+1$
$2k$	I_1-complex isotropic	I_1-CR isotropic	$2k-1$
$2k$	$\Theta_{I,2k}$-special isotropic	$\theta_{I,2k-1}$-special isotropic	$2k-1$
4	Φ_1-Cayley	ϕ_1-associative	3

With the exception of α_1-Legendrian and α_1-isotropic submanifolds, all of the "link" submanifolds $L \subset M^{4n+3}$ that appear in the table are minimal (i.e., have zero mean curvature), because a calibrated cone is minimal, and the link of a minimal cone is minimal.

3.4 3-Sasakian manifolds as circle bundles

From now on, 3-Sasakian $(4n+3)$-manifolds M are assumed to be compact. Above, we viewed M as the link of a hyperkähler cone C. In this section, we adopt a different perspective, viewing M as the total space of a circle bundle. The starting point is the following result.

Theorem 3.17 (Boyer-Galicki [8], Theorems 7.5.1, 13.2.5, 13.3.1) *Let M be a compact 3-Sasakian $(4n+3)$-manifold. For $v = (v_1, v_2, v_3) \in S^2$, let $A_v = v_1A_1 + v_2A_2 + v_3A_3$ denote the corresponding Reeb field. Then:*

(1) *Each A_v defines a locally free S^1-action on M and quasi-regular foliation $\mathcal{F}_v \subset M$. Let $Z_v := M/\mathcal{F}_v$ denote the corresponding leaf space, and let $p_v \colon M \to Z_v$ denote the projection.*

(2) *The projection $p_v \colon M \to Z_v$ is a principal S^1-orbibundle with connection 1-form $\alpha_v = \sum v^i \alpha_i$, and it is an orbifold Riemannian submersion.*

(3) *For $v, v' \in S^2$, there is a diffeomorphism $Z_v \approx Z_{v'}$. In fact, each Z_v may be identified with the (orbifold) twistor space Z of the quaternionic-Kähler $4n$-orbifold $Q = M/\mathcal{F}_A$, where $\mathcal{F}_A$ is the three-dimensional foliation determined by the vector fields A_1, A_2, A_3.*

Thus, every compact 3-Sasakian $(4n+3)$-manifold M has a natural S^2-family of projections $p_v \colon M^{4n+3} \to Z^{4n+2}$. For definiteness, we choose to work with $p_1 := p_{(1,0,0)} \colon M \to Z$, with respect to which $\alpha_1 \in \Omega^1(M)$ is a connection 1-form. On M, the

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choice of p_1 preferences the splitting $TM = \mathbb{R}A_1 \oplus \operatorname{Ker}(\alpha_1)$. On the hyperkähler cone $C^{4n+4} = \mathrm{C}(M)$, our choice distinguishes the Kähler structure $(g_{\mathrm{C}}, I_1, \omega_1)$.

3.4.1 The 3-forms $\Gamma_1, \Gamma_2, \Gamma_3$ and 4-forms Ξ_1, Ξ_2, Ξ_3

We now introduce $\mathbb{C}$-valued 3-forms $\Gamma_1, \Gamma_2, \Gamma_3 \in \Omega^3(M;\mathbb{C})$ and $\mathbb{R}$-valued 4-forms $\Xi_1, \Xi_2, \Xi_3 \in \Omega^4(M)$ that will play a key role in understanding the structure on the twistor space Z. These forms do not appear to have been studied before. Recalling the 2-forms κ_j defined in (3.6), we define

$$\begin{aligned} \Gamma_1 &= (\alpha_2 - i\alpha_3) \wedge (\kappa_2 + i\kappa_3), \\ \Gamma_2 &= (\alpha_3 - i\alpha_1) \wedge (\kappa_3 + i\kappa_1), \\ \Gamma_3 &= (\alpha_1 - i\alpha_2) \wedge (\kappa_1 + i\kappa_2), \end{aligned} \tag{3.13}$$

and

$$\Xi_1 = \kappa_2^2 + \kappa_3^2, \qquad \Xi_2 = \kappa_3^2 + \kappa_1^2, \qquad \Xi_3 = \kappa_1^2 + \kappa_2^2. \tag{3.14}$$

Note that the real and imaginary parts of Γ_1 are given by

$$\begin{aligned} \operatorname{Re}(\Gamma_1) &= \alpha_2 \wedge \kappa_2 + \alpha_3 \wedge \kappa_3, \\ \operatorname{Im}(\Gamma_1) &= \alpha_2 \wedge \kappa_3 - \alpha_3 \wedge \kappa_2. \end{aligned} \tag{3.15}$$

Their exterior derivatives are given by:

Proposition 3.18 *We have*

$$\begin{aligned} d\operatorname{Re}(\Gamma_1) &= 2\Xi_1 - 4\alpha_2 \wedge \alpha_3 \wedge \kappa_1, \\ d\operatorname{Im}(\Gamma_1) &= 0, \\ d\Xi_1 &= -4\kappa_1 \wedge \operatorname{Im}(\Gamma_1). \end{aligned}$$

Proof This is a straightforward computation using the definitions (3.15) and (3.14) and the exterior derivative formulas (3.5) and (3.7). ∎

Remark 3.19 We remark in passing that one can compute

$$\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = \tfrac{1}{2} d(\alpha_{123} + \alpha_1 \wedge \kappa_1 + \alpha_2 \wedge \kappa_2 + \alpha_3 \wedge \kappa_3),$$

showing that the natural 4-form $\kappa_1^2 + \Xi_1 = \kappa_1^2 + \kappa_2^2 + \kappa_3^2$ is exact.

To clarify the geometric meaning of $\Gamma_1 \in \Omega^3(M;\mathbb{C})$, we consider the 2-form

$$\widetilde{\Omega}_1 := 2\kappa_1 - \alpha_2 \wedge \alpha_3.$$

Using equations (3.5)–(3.7), (3.15), and Proposition 3.18, we derive the identities

$$\begin{aligned} d\widetilde{\Omega}_1 &= 3\operatorname{Im}(2\Gamma_1), \\ d\operatorname{Re}(2\Gamma_1) &= 2(2\Xi_1 - 4\alpha_2 \wedge \alpha_3 \wedge \kappa_1). \end{aligned}$$

When $n = 1$, in which case $\dim(Z) = 6$ and $\dim(M) = 7$, there is a coincidence $\kappa_1^2 = \kappa_2^2 = \kappa_3^2$, which implies $\Xi_1 = 2\kappa_1^2$, and therefore

$$\begin{aligned} d\widetilde{\Omega}_1 &= 3\operatorname{Im}(2\Gamma_1), \\ d\operatorname{Re}(2\Gamma_1) &= 2\widetilde{\Omega}_1^2, \end{aligned}$$

which is familiar from the geometry of nearly Kähler 6-manifolds [26]. So, when $n = 1$, the forms $\widetilde{\Omega}_1 \in \Omega^2(M)$ and $2\Gamma_1 \in \Omega^3(M;\mathbb{C})$ are the pullbacks via $p_1 \colon M^7 \to Z^6$ of the nearly Kähler 2-form and complex volume form on Z, respectively.

In Sections 4.1 and 4.2, we will see that aspects of this picture persist in higher dimensions. That is, for any $n \geq 1$, the 2-form $\widetilde{\Omega}_1$ is the pullback of the nearly Kähler 2-form, while $2\Gamma_1$ is the pullback of a natural 3-form that (together with other geometric data) defines an $\mathrm{Sp}(n)\mathrm{U}(1)$-structure on Z. When $n = 1$, the $\mathrm{Sp}(1)\mathrm{U}(1) \cong \mathrm{U}(2)$-structure on Z induces the familiar $\mathrm{SU}(3)$-structure, but when $n > 1$ the group $\mathrm{Sp}(n)\mathrm{U}(1)$ is not contained in $\mathrm{SU}(2n+1)$.

3.4.2 $\mathrm{Re}(\Gamma_1)$-calibrated 3-folds

The real parts of the 3-forms $\Gamma_1, \Gamma_2, \Gamma_3 \in \Omega^3(M;\mathbb{C})$ turn out to be semi-calibrations (Corollary 4.11), and thus give rise to a distinguished class of 3-folds of M. The following theorem characterizes these submanifolds; we defer the proof to Section 4.4, where the result is restated as Theorem 4.31.

Theorem 3.20 *Let $L^3 \subset M^{4n+3}$ be a three-dimensional submanifold. The following are equivalent:*

(1) $\mathrm{C}(L)$ *is a $(c_\theta I_2 + s_\theta I_3)$-complex isotropic 4-fold for some constant $e^{i\theta} \in S^1$.*
(2) *L is a $(c_\theta I_2 + s_\theta I_3)$-CR isotropic 3-fold for some constant $e^{i\theta} \in S^1$.*
(3) *L is $\mathrm{Re}(\Gamma_1)$-calibrated.*

Examples of $\mathrm{Re}(\Gamma_1)$-calibrated submanifolds can be constructed via Example 5.2 together with Theorem 6.3.

3.4.3 Descent to Z

To conclude this section, we observe that certain differential forms defined on M descend to the twistor space Z via the map $p_1 \colon M \to Z$. For this, we recall that a k-form $\phi \in \Omega^k(M)$ is called *p_1-semibasic* if $\iota_X \phi = 0$ for all $X \in \mathrm{Ker}((p_1)_*)$. Since the fibers of $p_1 \colon M \to Z$ are connected, it is a standard fact that a k-form $\phi \in \Omega^k(M)$ descends to Z if and only if both ϕ and $d\phi$ are p_1-semibasic.

Proposition 3.21 *Consider the projection $p := p_1 \colon M \to Z$.*

(1) *There exist $\mathbb{R}$-valued differential 2-forms $\omega_{\mathrm{V}}, \omega_{\mathrm{H}}, \omega_{\mathrm{KE}}, \omega_{\mathrm{NK}} \in \Omega^2(Z)$ satisfying*

$$\alpha_2 \wedge \alpha_3 = p^*(\omega_{\mathrm{V}}), \qquad \kappa_1 + \alpha_2 \wedge \alpha_3 = \Omega_1 = p^*(\omega_{\mathrm{KE}}),$$
$$\kappa_1 = p^*(\omega_{\mathrm{H}}), \qquad 2\kappa_1 - \alpha_2 \wedge \alpha_3 = \widetilde{\Omega}_1 = p^*(\omega_{\mathrm{NK}}).$$

(2) *There exist a $\mathbb{C}$-valued differential 3-form $\gamma \in \Omega^3(Z;\mathbb{C})$ and an $\mathbb{R}$-valued differential 4-form $\xi \in \Omega^4(Z)$ satisfying*

$$\Gamma_1 = p^*(\gamma),$$
$$\Xi_1 = p^*(\xi).$$

Proof (a) By equations (3.5) and (3.7), we have

$$d(\alpha_2 \wedge \alpha_3) = -2(\alpha_2 \wedge \kappa_3 - \alpha_3 \wedge \kappa_2), \qquad d\kappa_1 = 2(\alpha_2 \wedge \kappa_3 - \alpha_3 \wedge \kappa_2).$$

Therefore, both $\alpha_2 \wedge \alpha_3$ and $d(\alpha_2 \wedge \alpha_3)$ are p_1-semibasic, and similarly for κ_1 and $d\kappa_1$.
(b) By Proposition 3.18, we have

$$d\Gamma_1 = 2(\kappa_2^2 + \kappa_3^2) - 4\alpha_2 \wedge \alpha_3 \wedge \kappa_1, \qquad d\Xi_1 = -4\kappa_1 \wedge (\alpha_2 \wedge \kappa_3 - \alpha_3 \wedge \kappa_2).$$

Therefore, both Γ_1 and $d\Gamma_1$ are p_1-semibasic, and similarly for Ξ_1 and $d\Xi_1$. ∎

Remark 3.22 By contrast, one can check that the following forms on M do *not* descend via $p_1 \colon M \to Z$ to forms on Z:

$$\begin{array}{llll} \kappa_2, \kappa_3, & \Gamma_2, \Gamma_3, & \alpha_1, \alpha_2, \alpha_3, & \phi_1, \phi_2, \phi_3, \\ \Omega_2, \Omega_3, & \Xi_2, \Xi_3, & \Psi_1, \Psi_2, \Psi_3. & \end{array}$$

Remark 3.23 One must be careful to distinguish the 3-form $\Gamma_1 = (\alpha_2 - i\alpha_3) \wedge (\kappa_2 + i\kappa_3)$ from the special isotropic 3-form

$$(r\partial_r \lrcorner \tfrac{1}{2}\sigma_1^2)|_M = (\alpha_2 + i\alpha_3) \wedge (\kappa_2 + i\kappa_3).$$

While Γ_1 descends to Z, the special isotropic 3-form $(r\partial_r \lrcorner \frac{1}{2}\sigma_1^2)|_M$ does not, because its exterior derivative has α_1 terms. Note that for $n = 1$, the object $(r\partial_r \lrcorner \frac{1}{2}\sigma_1^2)|_M = \Psi_1$ is a 3-form on M^7 whose real part calibrates special Legendrian 3-folds.

4 Calibrated geometry in twistor spaces

We now turn to the submanifold theory of twistor spaces Z, organizing our discussion as follows. In Section 4.1, we briefly discuss $\mathrm{Sp}(n)\mathrm{U}(1)$-geometry on arbitrary $(4n+2)$-manifolds Y^{4n+2}. Then, in Section 4.2 (Theorem 4.7), we prove that every twistor space Z^{4n+2} admits a canonical $\mathrm{Sp}(n)\mathrm{U}(1)$-structure, which (among other data) entails a distinguished 3-form $\gamma \in \Omega^3(Z; \mathbb{C})$. In Proposition 4.10, we prove that $\mathrm{Re}(\gamma) \in \Omega^3(Z)$ is a semi-calibration, and devote Section 4.3 to the study of $\mathrm{Re}(\gamma)$-calibrated 3-folds. In a certain sense (Proposition 4.10(b)), these are higher-codimension generalizations of special Lagrangian 3-folds in six-dimensional nearly Kähler twistor spaces.

Finally, in Section 4.4, we study the relationships between submanifolds of M^{4n+3} and those in Z^{4n+2}. More specifically, distinguishing the map $p_1 \colon M \to Z$, we consider how various submanifolds $\Sigma^k \subset Z$ behave under the operations of p_1-circle bundle lift $p_1^{-1}(\Sigma)^{k+1} \subset M$ and p_1-horizontal lift $\widehat{\Sigma}^k \subset M$.

We remind the reader that as mentioned in the introduction, we only consider submanifolds of Z that do not meet any orbifold points.

4.1 $\mathrm{Sp}(n)\mathrm{U}(1)$-structures

Let Y^{4n+2} be a smooth $(4n+2)$-manifold with $n \geq 1$.

Definition 4.1 A $(\mathrm{U}(2n) \times \mathrm{U}(1))$-*structure* on Y^{4n+2} is an almost-Hermitian structure (g, J_+, ω_+) together with a distribution of J_+-invariant $4n$-planes $\mathsf{H} \subset$

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TY. Equivalently, it is an almost-Hermitian structure (g, J_+, ω_+) together with an orthogonal splitting

$$TY = \mathsf{H} \oplus \mathsf{V},$$

where $\mathsf{H} \subset TY$ and $\mathsf{V} \subset TY$ are J_+-invariant subbundles with $\mathrm{rank}(\mathsf{H}) = 4n$ and $\mathrm{rank}(\mathsf{V}) = 2$.

Given a $(\mathrm{U}(2n) \times \mathrm{U}(1))$-structure $(g, J_+, \omega_+, \mathsf{H})$, we split (g, J_+, ω_+) into horizontal and vertical parts as follows:

$$g = g_{\mathsf{H}} + g_{\mathsf{V}}, \qquad J_+ = J_+|_{\mathsf{H}} + J_+|_{\mathsf{V}}, \qquad \omega_+ = \omega_{\mathsf{H}} + \omega_{\mathsf{V}}.$$

Further, we can extend it to a one-parameter family $(g(t), J_+, \omega_+(t), \mathsf{H})$ by defining

$$g(t) = t^2 g_{\mathsf{H}} + g_{\mathsf{V}}, \qquad \omega_+(t) = t^2 \omega_{\mathsf{H}} + \omega_{\mathsf{V}}.$$

Moreover, by reversing the orientation of the vertical subbundle $\mathsf{V} \subset TY$, we obtain a *second* one-parameter family $(g(t), J_-, \omega_-(t), \mathsf{H})$ by defining

$$J_- = J_+|_{\mathsf{H}} - J_+|_{\mathsf{V}}, \qquad \omega_-(t) = t^2 \omega_{\mathsf{H}} - \omega_{\mathsf{V}}.$$

For calculations on Y, we will need local frames adapted to the geometry of the $(\mathrm{U}(2n) \times \mathrm{U}(1))$-structure. To be precise:

Definition 4.2 A $(\mathrm{U}(2n) \times \mathrm{U}(1))$-*coframe at* $y \in Y$ is a g-orthonormal coframe

$$(\rho, \mu) := (\rho_{10}, \rho_{11}, \rho_{12}, \rho_{13}, \ldots, \rho_{n0}, \rho_{n1}, \rho_{n2}, \rho_{n3}, \mu_2, \mu_3) : T_y Y \to \mathbb{R}^{4n} \times \mathbb{R}^2$$

for which

$$\omega_{\mathsf{V}}|_y = \mu_2 \wedge \mu_3, \qquad \omega_{\mathsf{H}}|_y = \sum_{j=1}^{n} (\rho_{j0} \wedge \rho_{j1} + \rho_{j2} \wedge \rho_{j3}).$$

For example, we will soon recall (Theorem 4.6) that every twistor space Z^{4n+2} admits a natural $(\mathrm{U}(2n) \times \mathrm{U}(1))$-structure. In fact, we will show (Theorem 4.7) that twistor spaces admit an additional piece of data:

Definition 4.3 Let Y^{4n+2} be a $(4n+2)$-manifold with a $(\mathrm{U}(2n) \times \mathrm{U}(1))$-structure $(g, J_+, \omega_+, \mathsf{H})$. A *compatible* $\mathrm{Sp}(n)\mathrm{U}(1)$-*structure* is a complex 3-form $\gamma \in \Omega^3(Y; \mathbb{C})$ with the following property: At each $y \in Y$, there exists a $(\mathrm{U}(2n) \times \mathrm{U}(1))$-coframe (ρ, μ) such that

$$\gamma|_y = (\mu_2 - i\mu_3) \wedge \sum_{j=1}^{n} (\rho_{j0} + i\rho_{j1}) \wedge (\rho_{j2} + i\rho_{j3}).$$

Note that if γ is a compatible $\mathrm{Sp}(n)\mathrm{U}(1)$-structure, then γ has J_+-type $(2,1)$ and J_--type $(3,0)$.

To justify this terminology, we make a digression into linear algebra. Consider the following $\mathrm{Sp}(n)\mathrm{U}(1)$-representation on $\mathbb{R}^{4n+2}$. For $(A, \lambda) \in \mathrm{Sp}(n) \times \mathrm{U}(1)$ and $(h, z) \in \mathbb{H}^n \oplus \mathbb{C}$, define

$$(A, \lambda) \cdot (h, z) := (Ah\lambda^{-1}, \lambda^{-2} z). \tag{4.1}$$

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Identify $\mathbb{H}^n \simeq \mathbb{C}^{2n}$ by writing $h = h_1 + jh_2$ with $h_1, h_2 \in \mathbb{C}^n$. This identification endows $\mathbb{H}^n$ with the complex structure given by right multiplication by i, which in turn yields an embedding $\iota\colon \mathrm{Sp}(n) \to \mathrm{U}(2n)$. In this way, the representation (4.1) induces an embedding

$$\begin{aligned}\mathrm{Sp}(n)\mathrm{U}(1) &\to \mathrm{U}(2n) \times \mathrm{U}(1)\\ (A, \lambda) &\mapsto (\iota(A)\lambda^{-1}, \lambda^{-2}).\end{aligned}$$

The image of this map is

$$\{(B, \nu) \in \mathrm{U}(2n) \times \mathrm{U}(1) \colon \nu^{-1/2} B \in \mathrm{Sp}(n)\}. \tag{4.2}$$

Since $\mathrm{Sp}(n)$ contains the element $-\mathrm{Id}$, the condition $\nu^{-1/2}B \in \mathrm{Sp}(n)$ does not depend on the choice of square root.

Let $(e_{10}, e_{11}, e_{12}, e_{13}, \ldots, e_{n0}, e_{n1}, e_{n2}, e_{n3}, f_2, f_3)$ denote the standard basis of $\mathbb{R}^{4n+2}$, and let $(e^{10}, e^{11}, e^{12}, e^{13}, \ldots, e^{n0}, e^{n1}, e^{n2}, e^{n3}, f^2, f^3)$ denote its dual basis. We identify $\mathbb{R}^{4n+2} \simeq \mathbb{C}^{2n} \oplus \mathbb{C}$ via the complex structure J_0 whose Kähler form is

$$\omega_0 = f^2 \wedge f^3 + \sum (e^{j0} \wedge e^{j1} + e^{j2} \wedge e^{j3}).$$

Identifying $\mathbb{C}^{2n} \simeq \mathbb{H}^n$, the standard hyperkähler triple on $\mathbb{H}^n$ is

$$\begin{aligned}\beta_1 &= \sum (e^{j0} \wedge e^{j1} + e^{j2} \wedge e^{j3}), \quad \beta_2 = \sum (e^{j0} \wedge e^{j2} - e^{j1} \wedge e^{j3}),\\ \beta_3 &= \sum (e^{j0} \wedge e^{j3} + e^{j1} \wedge e^{j2}).\end{aligned} \tag{4.3}$$

We consider the 3-form $\gamma_0 \in \Lambda^3((\mathbb{R}^{4n+2})^*)$ given by

$$\gamma_0 = (f^2 - i f^3) \wedge (\beta_2 + i\beta_3).$$

Then:

Proposition 4.4 *With respect to the standard* $(\mathrm{U}(2n) \times \mathrm{U}(1))$*-action on* $\mathbb{R}^{4n+2}$*, the stabilizer of* $\gamma_0 \in \Lambda^3((\mathbb{R}^{4n+2})^*)$ *is the subgroup* $\mathrm{Sp}(n)\mathrm{U}(1) \leq \mathrm{U}(2n) \times \mathrm{U}(1)$ *given by (4.2).*

Proof Let $(B, \nu) \in \mathrm{U}(2n) \times \mathrm{U}(1)$, and set $\tau = f^2 - i f^3$ and $\beta = \beta_2 + i\beta_3$. Since τ has J_0-type $(0, 1)$ and β has J_0-type $(2, 0)$, we have

$$\nu^* \tau = \nu^{-1} \tau, \qquad \nu^* \beta = \nu^2 \beta,$$

and hence

$$(B, \nu)^* \gamma_0 = (B, \nu)^* (\tau \wedge \beta) = \nu^* \tau \wedge B^* \beta = \tau \wedge (\nu^{-1/2} B)^* \beta.$$

If $(B, \nu) \in \mathrm{Sp}(n)\mathrm{U}(1)$, then $\nu^{-1/2} B \in \mathrm{Sp}(n)$ by (4.2). Thus, since $\mathrm{Sp}(n)$ stabilizes $\beta_1, \beta_2, \beta_3$, we get

$$(B, \nu)^* \gamma_0 = \tau \wedge \beta = \gamma_0.$$

Conversely, if $(B, \nu) \in \mathrm{U}(2n) \times \mathrm{U}(1)$ stabilizes γ_0, then

$$\tau \wedge \beta = \tau \wedge (\nu^{-1/2} B)^* \beta.$$

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Contracting both sides with the vector $f_2 + i f_3$ implies that $\beta = (\nu^{-1/2} B)^* \beta$, so that $\nu^{-1/2} B \in \mathrm{U}(2n)$ stabilizes β. Since the $\mathrm{U}(2n)$-stabilizer of β is $\mathrm{Sp}(n)$, we deduce that $\nu^{-1/2} B \in \mathrm{Sp}(n)$, and hence $(B, \nu) \in \mathrm{Sp}(n)\mathrm{U}(1)$. ∎

Example 4.1 The case $n = 1$ is particularly special. Let Y^6 be a 6-manifold with a $(\mathrm{U}(2) \times \mathrm{U}(1))$-structure $(g, J_+, \omega_+, \mathsf{H})$. By definition, a compatible $\mathrm{Sp}(1)\mathrm{U}(1)$-structure is a complex 3-form $\gamma \in \Omega^3(Y; \mathbb{C})$ such that at each $y \in Y$, there exists a $(\mathrm{U}(2) \times \mathrm{U}(1))$-coframe $(\rho_0, \rho_1, \rho_2, \rho_3, \mu_2, \mu_3) \colon T_y Y \to \mathbb{R}^4 \times \mathbb{R}^2$ for which

$$\gamma|_y = (\mu_2 - i\mu_3) \wedge (\rho_0 + i\rho_1) \wedge (\rho_2 + i\rho_3).$$

So, γ is a nonvanishing 3-form of J_--type $(3,0)$ satisfying

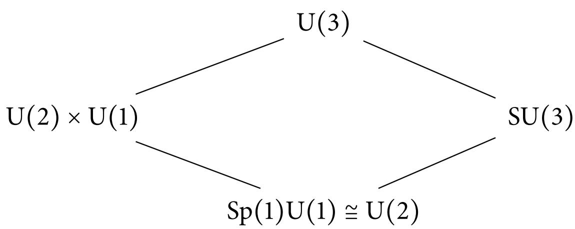
$$-\frac{i}{8} \gamma \wedge \overline{\gamma} = \mu_2 \wedge \mu_3 \wedge \rho_0 \wedge \rho_1 \wedge \rho_2 \wedge \rho_3.$$

As such, $\gamma \in \Omega^3(Y; \mathbb{C})$ defines an $\mathrm{SU}(3)$-structure on Y.

Alternatively, the presence of a compatible $\mathrm{SU}(3)$-structure on Y^6 follows abstractly from the following group isomorphism of $\mathrm{Sp}(1)\mathrm{U}(1) \cong \mathrm{U}(2)$ onto a subgroup of $\mathrm{SU}(3)$. Using $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$, we have

$$\mathrm{Sp}(1)\mathrm{U}(1) = \left\{ \begin{pmatrix} B & 0 \\ 0 & \nu \end{pmatrix} : \nu^{-1/2} B \in \mathrm{SU}(2) \right\} \cong \left\{ \begin{pmatrix} T & 0 \\ 0 & (\det T)^{-1} \end{pmatrix} : T \in \mathrm{U}(2) \right\} \leq \mathrm{SU}(3),$$
$$\begin{pmatrix} B & 0 \\ 0 & \det B \end{pmatrix} \mapsto \begin{pmatrix} B & 0 \\ 0 & (\det B)^{-1} \end{pmatrix}.$$

(This is a group homomorphism because $\mathrm{U}(1)$ is abelian.) The situation is described by the following diagram:



Remark 4.5 The notation in this remark is that made standard in the monograph of Salamon [28]. Let $T = \mathsf{H} \oplus \mathsf{V} \simeq \mathbb{R}^{4n+2}$ denote the (real) $\mathrm{Sp}(n)\mathrm{U}(1)$-representation of (4.1). Let $E \simeq \mathbb{C}^{2n}$ denote the standard complex $\mathrm{Sp}(n)$-representation, and let $L \simeq \mathbb{C}$ denote the standard complex $\mathrm{U}(1)$-representation. Then, by refining the splitting $\Lambda^2(T^*) = \Lambda^2(\mathsf{H}^*) \otimes (\mathsf{H}^* \otimes \mathsf{V}^*) \otimes \Lambda^2(\mathsf{V}^*)$, one can decompose the space of real 2-forms into $\mathrm{Sp}(n)\mathrm{U}(1)$-irreducible representations as follows:

$$\begin{aligned}
\Lambda^2(\mathsf{H}^*) &\cong \mathbb{R}\omega_{\mathsf{H}} \oplus [\mathrm{Sym}^2(E)] \oplus [\Lambda^2_0(E)] \oplus [\![L^2]\!] \oplus [\![\Lambda^2_0(E) \otimes L^2]\!] \\
\mathsf{H}^* \otimes \mathsf{V}^* &\cong [\![E \otimes L^3]\!] \oplus [\![E \otimes L]\!] \\
\Lambda^2(\mathsf{V}^*) &\cong \mathbb{R}\omega_{\mathsf{V}}.
\end{aligned}$$

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Alternatively, by refining the J_+-type splitting $\Lambda^2(T^*) = [\![\Lambda^{2,0}]\!] \oplus [\Lambda^{1,1}]$, one obtains

$$[\![\Lambda^{2,0}]\!] \cong [\![\Lambda^2_0(E) \otimes L^2]\!] \oplus [\![L^2]\!] \oplus [\![E \otimes L^3]\!]$$
$$[\Lambda^{1,1}] \cong \mathbb{R}\omega_{\mathsf{V}} \oplus \mathbb{R}\omega_{\mathsf{H}} \oplus [\mathrm{Sym}^2_0(E)] \oplus [\Lambda^2_0(E)] \oplus [\![E \otimes L]\!].$$

4.2 The geometry of twistor spaces

We now return to the study of twistor spaces Z. The following fact is well known:

Theorem 4.6 *Let M^{4n+3} be a 3-Sasakian manifold, and fix a projection $p = p_1 \colon M \to Z$. The quotient Z admits a $(\mathrm{U}(2n) \times \mathrm{U}(1))$-structure $(g, J_+, \omega_+, \mathsf{H})$ for which:*

- *$(g(1),\, J_+,\, \omega_+(1))$ is Kähler–Einstein with positive scalar curvature.*
- *$\left(g(\sqrt{2}),\, J_-,\, \omega_-(\sqrt{2})\right)$ is nearly Kähler.*
- *$p_*(\widetilde{\mathsf{H}}) = \mathsf{H}$ and $p_*(\mathrm{span}(A_2, A_3)) = \mathsf{V}$.*

Proof The Kähler–Einstein structure is very well known and has been extensively studied. The statement about the nearly Kähler structure is [8, Theorem 14.3.9]. Details can be found in [4] or [25]. ∎

From now on, the twistor space Z will carry the $(\mathrm{U}(2n) \times \mathrm{U}(1))$-structure $(g, J_+, \omega_+, \mathsf{H})$ described in the previous proposition. We will write

$$(g_{\mathrm{KE}}, J_{\mathrm{KE}}, \omega_{\mathrm{KE}}) := (g(1),\, J_+,\, \omega_+(1)),$$
$$(g_{\mathrm{NK}}, J_{\mathrm{NK}}, \omega_{\mathrm{NK}}) := (g(\sqrt{2}),\, J_-,\, \omega_-(\sqrt{2})).$$

In particular,

$$\omega_{\mathrm{KE}} = \omega_{\mathsf{H}} + \omega_{\mathsf{V}}, \qquad \omega_{\mathrm{NK}} = 2\omega_{\mathsf{H}} - \omega_{\mathsf{V}}. \tag{4.4}$$

We now recover the important observation of Alexandrov [3] that Z naturally admits *even more* structure:

Theorem 4.7 *Let Z be a twistor space with its $(\mathrm{U}(2n) \times \mathrm{U}(1))$-structure $(g_{\mathrm{KE}}, J_{\mathrm{KE}}, \omega_{\mathrm{KE}}, \mathsf{H})$. Then Z naturally admits a compatible* $\mathrm{Sp}(n)\mathrm{U}(1)$*-structure $\gamma \in \Omega^3(Z;\mathbb{C})$.*

Proof By Proposition 3.21(b), there exists a unique 3-form $\gamma \in \Omega^3(Z;\mathbb{C})$ satisfying

$$p^*(\gamma) = \Gamma_1 = (\alpha_2 - i\alpha_3) \wedge (\kappa_2 + i\kappa_3).$$

This 3-form is an $\mathrm{Sp}(n)\mathrm{U}(1)$-structure. ∎

In Section 5.1, we will give a second proof of Theorem 4.7 from the perspective of quaternionic-Kähler geometry. For now, using Proposition 3.21, we can compute the following exterior derivatives:

$$\begin{aligned}
d\omega_{\mathsf{V}} &= -\mathrm{Im}(2\gamma), & d\omega_{\mathrm{KE}} &= 0, & d\,\mathrm{Re}(\gamma) &= 2\xi - 4\omega_{\mathsf{H}} \wedge \omega_{\mathsf{V}}, \\
d\omega_{\mathsf{H}} &= \mathrm{Im}(2\gamma), & d\omega_{\mathrm{NK}} &= 3\,\mathrm{Im}(2\gamma), & d\,\mathrm{Im}(\gamma) &= 0, \\
&&&& d\xi &= -4\omega_{\mathsf{H}} \wedge \mathrm{Im}(\gamma).
\end{aligned}$$

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Example 4.2 When $n = 1$, there is a coincidence $\xi = 2\omega_{\mathsf{H}}^2$. Therefore, in this case, using that $\omega_{\mathrm{NK}}^2 = (2\omega_{\mathsf{H}} - \omega_{\mathsf{V}})^2 = 4\omega_{\mathsf{H}}^2 - 4\omega_{\mathsf{H}} \wedge \omega_{\mathsf{V}} = 2\xi - 4\omega_{\mathsf{H}} \wedge \omega_{\mathsf{V}}$, we recover the equations

$$d\omega_{\mathrm{NK}} = 3\,\mathrm{Im}(2\gamma),$$
$$d\,\mathrm{Re}(2\gamma) = 2\,\omega_{\mathrm{NK}} \wedge \omega_{\mathrm{NK}},$$

familiar from the theory of nearly Kähler 6-manifolds.

4.3 $\mathrm{Re}(\gamma)$-calibrated 3-folds

Let Z^{4n+2} be a twistor space equipped with the $(\mathrm{U}(2n) \times \mathrm{U}(1))$-structure $(g_{\mathrm{KE}}, J_{\mathrm{KE}}, \omega_{\mathrm{KE}}, \mathsf{H})$. With respect to this structure, one can consider several classes of submanifolds of Z, such as:

- J_{KE}-complex (resp. J_{NK}-complex) submanifolds.
- Horizontal submanifolds (i.e. those tangent to H).
- ω_{KE}-isotropic (resp. ω_{NK}-isotropic) submanifolds.

These submanifolds have been the subject of numerous studies, particularly when $\dim(Z) = 6$. However, since we have now shown that Z admits a compatible $\mathrm{Sp}(n)\mathrm{U}(1)$-structure $\gamma \in \Omega^3(Z;\mathbb{C})$, twistor spaces also admit a distinguished class of 3-folds. In this section, we explore these.

We begin by showing that $\mathrm{Re}(\gamma) \in \Omega^3(Z)$ is a semi-calibration, for which we need a preliminary lemma.

Lemma 4.8 *For any horizontal unit vector $v \in \mathsf{H}$, the 2-form $\iota_v(\mathrm{Re}(\gamma)) \in \Omega^2(Z)$ is a semi-calibration. Moreover, its calibrated 2-planes lie in the 6-plane $L \oplus \mathsf{V}$, where L is the quaternionic line spanned by v.*

Proof It suffices to work at a fixed point $z \in Z$. Let (ρ, μ) be an $\mathrm{Sp}(n)\mathrm{U}(1)$-coframe at z as in Definition 4.3. We may then write $\gamma|_z = \tau \wedge (\beta_2 + i\beta_3)$, where

$$\tau = \mu_2 - i\mu_3, \quad \beta_2 = \sum_{i=1}^{n}(\rho_{j0} \wedge \rho_{j2} - \rho_{j1} \wedge \rho_{j3}), \quad \beta_3 = \sum_{i=1}^{n}(\rho_{j0} \wedge \rho_{j3} + \rho_{j1} \wedge \rho_{j2}).$$

Define complex structures J_2 and J_3 on $\mathsf{H}|_z$ by declaring

$$J_2(\rho_{j0}) = \rho_{j2}, \qquad J_3(\rho_{j0}) = \rho_{j3},$$
$$J_2(\rho_{j1}) = -\rho_{j3}, \qquad J_3(\rho_{j1}) = \rho_{j2},$$

which implies $J_+J_2 = J_3$ and $g(J_2\cdot, \cdot) = \beta_2$ and $g(J_3\cdot, \cdot) = \beta_3$. Note that τ, β_2, β_3, as well as J_2, J_3, depend on the choice of $\mathrm{Sp}(n)\mathrm{U}(1)$-frame.

Now, let $v \in \mathsf{H}$ be a horizontal unit vector. Let $w = J_2 v$, so that

$$\begin{aligned}\iota_v\gamma = \iota_v(\tau \wedge (\beta_2 + i\beta_3)) = \tau \wedge \iota_v(\beta_2 + i\beta_3) &= \tau \wedge (g(J_2 v, \cdot) + i g(J_+J_2 v, \cdot)) \\ &= \tau \wedge (w^\flat - iJ_+ w^\flat) \\ &= (\mu_2 - i\mu_3) \wedge (w^\flat - iJ_- w^\flat)\end{aligned}$$

since $J_+ = J_-$ on horizontal vectors. This 2-form is decomposable and has J_--type $(2,0)$. Moreover, $\{\mu_2, \mu_3, w^\flat, J_- w^\flat\}$ is an orthonormal set. Thus, this 2-form is a standard complex volume form, and hence its real part is a semi-calibration. ■

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Remark 4.9 The above proof shows slightly more, namely that the $\iota_v(\mathrm{Re}(\gamma))$-calibrated 2-planes lie in the 4-plane $\mathrm{span}(w, J_+ w) \oplus \mathsf{V} = \mathrm{span}(J_2 v, J_3 v) \oplus \mathsf{V}$.

Proposition 4.10 *The 3-form $\mathrm{Re}(\gamma) \in \Omega^3(Z)$ is a semi-calibration. Moreover, let $E \in \mathrm{Gr}_3^+(TZ)$ be an oriented 3-plane.*

(1) *E is $\mathrm{Re}(\gamma)$-calibrated if and only if $E = \mathbb{R}v \oplus E'$ for some $v \in E \cap \mathsf{H}$ and some 2-plane E' that is $\iota_v(\mathrm{Re}(\gamma))$-calibrated.*
(2) *If E is a $\mathrm{Re}(\gamma)$-calibrated 3-plane, there is a quaternionic line $L \subset \mathsf{H}$ such that E is contained in $L \oplus \mathsf{V}$.*
(3) *If E is $\mathrm{Re}(\gamma)$-calibrated, then E is ω_{NK}-isotropic.*

Proof If $E \in \mathrm{Gr}_3^+(T_z Z)$ is an oriented 3-plane at $z \in Z$, then $\dim(E \cap \mathsf{H}) \geq 1$, so there exists a unit vector $v \in E \cap \mathsf{H}$, and we may orthogonally split $E = \mathbb{R}v \oplus E'$. Then

$$(\mathrm{Re}(\gamma))(E) = (\iota_v \mathrm{Re}(\gamma))(E') \leq 1$$

by Lemma 4.8, so the comass of $\mathrm{Re}(\gamma)$ is at most 1. Now, let v be a horizontal unit vector and let E' be an $\iota_v(\mathrm{Re}(\gamma))$-calibrated 2-plane, which exists by Lemma 4.8. Then $E = \mathbb{R}v \oplus E'$ is $\mathrm{Re}(\gamma)$-calibrated, which shows that $\mathrm{Re}(\gamma)$ has comass equal to one. Further, we have seen that an oriented 3-plane E is $\mathrm{Re}(\gamma)$-calibrated if and only if E' is $\iota_v(\mathrm{Re}(\gamma))$-calibrated, which proves (a).

Part (b) follows from Remark 4.9. Finally, since γ is of J_--type $(3,0)$, part (c) follows from Proposition A.5. ∎

Returning to the 3-Sasakian manifold M^{4n+3}, we can now establish the following:

Corollary 4.11 *The 3-form $\mathrm{Re}(\Gamma_1) \in \Omega^3(M)$ is a semi-calibration.*

Proof Recall that $p_1 \colon M \to Z$ is a Riemannian submersion, that $\mathrm{Re}(\Gamma_1) = p_1^*(\mathrm{Re}(\gamma))$, and that $\mathrm{Re}(\gamma) \in \Omega^3(Z)$ has comass one. The result now follows from Proposition A.4. ∎

Remark 4.12 We pause to make two remarks. First, Proposition 4.10 shows that $\mathrm{Re}(\gamma)$-calibrated 3-folds $L^3 \subset Z^{4n+2}$ are ω_{NK}-isotropic. However, we emphasize that such 3-folds need not be ω_{KE}-isotropic in general. Later (Theorem 5.16), we will characterize the $\mathrm{Re}(\gamma)$-calibrated 3-folds $L \subset Z$ satisfying $\omega_{\mathrm{KE}}|_L = 0$.

Second, we clarify that Proposition 4.10 asserts $\mathrm{Re}(\gamma)$ is a semi-calibration with respect to the metric g_{KE}. Therefore, by Proposition A.3, the 3-form $\mathrm{Re}(t^2\gamma)$ is a semi-calibration with respect to the metric $g(t) = t^2 g_{\mathsf{H}} + g_{\mathsf{V}}$. In particular, $\mathrm{Re}(2\gamma)$ is a semi-calibration with respect to $g_{\mathrm{NK}} = 2g_{\mathsf{H}} + g_{\mathsf{V}}$.

4.3.1 A normal form for $\mathrm{Re}(\gamma)$-calibrated 3-planes

We now aim to establish a normal form for $\mathrm{Re}(\gamma)$-calibrated 3-planes in Z. Since the subsequent discussion is a matter of linear algebra, we work in $\mathbb{R}^{4n+2} \simeq \mathbb{H}^n \oplus \mathbb{C}$. As we have done previously, we let

$$(e_{10}, e_{11}, e_{12}, e_{13}, \ldots, e_{n0}, e_{n1}, e_{n2}, e_{n3}, f_2, f_3)$$

denote the standard basis of $\mathbb{R}^{4n+2}$, let $(e^{10}, e^{11}, \ldots, f^2, f^3)$ denote its dual basis, let $\beta_1, \beta_2, \beta_3$ be the standard hyperkähler triple on $\mathbb{H}^n$ as in (4.3), and consider the 3-form $\gamma_0 \in \Lambda^3((\mathbb{R}^{4n+2})^*)$ given by

$$\gamma_0 = (f^2 - i f^3) \wedge (\beta_2 + i\beta_3).$$

Now, for $e^{i\theta} \in S^1$, define the 2-plane

$$V_\theta = \mathrm{span}\left(c_\theta(-f_2 - e_{13}) + s_\theta(-f_3 - e_{12}), s_\theta(-f_2 + e_{13}) + c_\theta(-f_3 + e_{12})\right). \tag{4.5}$$

In particular, we highlight

$$V_{\frac{\pi}{4}} = \mathrm{span}\left(f_2 + f_3 + e_{12} + e_{13}, f_2 + f_3 - e_{12} - e_{13}\right). \tag{4.6}$$

Proposition 4.13 *Consider the* $\mathrm{Sp}(n)\mathrm{U}(1)$*-action on* $\mathbb{H}^n \oplus \mathbb{C}$ *given in* (4.1)*. Let* $E \subset \mathbb{H}^n \oplus \mathbb{C}$ *be a* $\mathrm{Re}(\gamma_0)$*-calibrated 3-plane. Then there exist* $(A, \lambda) \in \mathrm{Sp}(n)\mathrm{U}(1)$ *and a unique* $\theta \in [0, \frac{\pi}{4}]$ *such that* $(A, \lambda) \cdot E = \mathbb{R}e_{10} \oplus V_\theta$*. Moreover, the following are equivalent:*

(1) $\dim(E \cap \mathbb{H}^n) = 2$.
(2) $E = (E \cap \mathbb{H}^n) \oplus (E \cap \mathbb{C})$.
(3) E *is* ω_{KE}*-isotropic.*
(4) $\theta = \frac{\pi}{4}$.

Proof Let $E \subset \mathbb{H}^n \oplus \mathbb{C}$ be a $\mathrm{Re}(\gamma_0)$-calibrated 3-plane. By Proposition 4.10, there exists a quaternionic line $L \subset \mathbb{H}^n$ for which $E \subset L \oplus \mathbb{C}$. Since the subgroup $\mathrm{Sp}(n) \leq \mathrm{Sp}(n)\mathrm{U}(1)$ acts transitively on the quaternionic lines of $\mathbb{H}^n$, there exists $A_0 \in \mathrm{Sp}(n)$ such that $A_0 \cdot L = L_0$, where L_0 is the standard quaternionic line

$$L_0 = \mathrm{span}(e_{10}, e_{11}, e_{12}, e_{13}).$$

Thus, $(A_0, 1) \cdot E \subset L_0 \oplus \mathbb{C}$, so we can without loss of generality suppose that $E \subset L_0 \oplus \mathbb{C}$.

Now, $L_0 \oplus \mathbb{C}$ is a complex 3-plane, and the restriction of γ_0 to $L_0 \oplus \mathbb{C}$ is a complex volume form. Thus, the problem reduces to finding a normal form for special Lagrangian 3-planes in a complex 3-space with respect to the action of $\mathrm{Sp}(1)\mathrm{U}(1) \cong \mathrm{U}(2)$. Such a normal form was established in [5, Proposition 3.2]. (Translating between notations, the $b_1, ib_1, b_2, ib_2, b_3, ib_3$ of [5] corresponds to our $e_{10}, e_{11}, e_{12}, e_{13}, f_2, -f_3$.)

For $\theta \in [0, \frac{\pi}{4}]$, write $W_\theta = \mathbb{R}e_{10} \oplus V_\theta$. We observe that the conditions (a), (b), and (c) above are invariant under the action of $\mathrm{Sp}(n)\mathrm{U}(1)$, so it is enough to verify that for W_θ they are equivalent to $\theta = \frac{\pi}{4}$. If $\theta = \frac{\pi}{4}$, we have

$$W_{\frac{\pi}{4}} = \mathrm{span}(e_{10},\ e_{12} + e_{13},\ f_2 + f_3) = (W_{\frac{\pi}{4}} \cap \mathbb{H}^n) \oplus (W_{\frac{\pi}{4}} \cap \mathbb{C}),$$

so both (a) and (b) hold. If $\theta \neq \frac{\pi}{4}$, then one can compute from (4.5) that $\dim(W_\theta \cap \mathbb{H}^n) = 1$. Since a $\mathrm{Re}(\gamma_0)$-calibrated 3-plane cannot contain any complex lines, we have $\dim(W_\theta \cap \mathbb{C}) < 2$, and hence

$$\dim((W_\theta \cap \mathbb{H}^n) \oplus (W_\theta \cap \mathbb{C})) = \dim(W_\theta \cap \mathbb{H}^n) + \dim(W_\theta \cap \mathbb{C}) < 3 = \dim(W_\theta),$$

so both (a) and (b) do not hold.

With respect to the above basis, we have $\omega_{\mathrm{KE}} = \beta_1 + f^2 \wedge f^3$. Letting

$$v_2 = c_\theta(-f_2 - e_{13}) + s_\theta(-f_3 - e_{12}), \quad v_3 = s_\theta(-f_2 + e_{13}) + c_\theta(-f_3 + e_{12}),$$

so $V_\theta = \mathrm{span}(v_2, v_3)$ and $W_\theta = \mathbb{R}e_{10} \oplus V_\theta$, a computation shows that

$$\omega_{\mathrm{KE}}(e_{10}, v_2) = \omega_{\mathrm{KE}}(e_{10}, v_3) = 0, \qquad \omega_{\mathrm{KE}}(v_2, v_3) = 2(c_\theta^2 - s_\theta^2),$$

so $\omega_{\mathrm{KE}}|_{W_\theta} = 0$ if and only if $\theta = \frac{\pi}{4}$. ■

4.3.2 HV compatibility

Definition 4.14 A submanifold $\Sigma^k \subset Z^{4n+2}$ is called *HV-compatible* if at each $x \in \Sigma$, we have

$$T_x\Sigma = (T_x\Sigma \cap \mathsf{H}) \oplus (T_x\Sigma \cap \mathsf{V}).$$

HV compatibility is a rather stringent condition. Nevertheless, we now observe that certain natural classes of submanifolds of Z automatically satisfy it.

Proposition 4.15 *Let $\Sigma^k \subset Z^{4n+2}$ be a submanifold, $1 \leq k \leq 2n+1$.*

(1) *If Σ is HV-compatible, then Σ is ω_{KE}-isotropic if and only if Σ is ω_{NK}-isotropic.*
(2) *Suppose* $\dim(\Sigma) = 2n+1$. *If Σ is ω_{KE}-Lagrangian and ω_{NK}-Lagrangian, then Σ is HV-compatible. Moreover,* $\dim(T_z\Sigma \cap \mathsf{H}) = 2n$ *and* $\dim(T_z\Sigma \cap \mathsf{V}) = 1$ *at each $z \in \Sigma$.*
(3) *Suppose* $\dim(\Sigma) = 3$. *If Σ is* $\mathrm{Re}(\gamma)$*-calibrated, then Σ is HV-compatible if and only if Σ is ω_{KE}-isotropic. In this case,* $\dim(T_z\Sigma \cap \mathsf{H}) = 2$ *and* $\dim(T_z\Sigma \cap \mathsf{V}) = 1$ *at each $z \in \Sigma$.*

Proof (a) Suppose Σ is HV-compatible. If Σ is ω_{KE}-isotropic, then (4.4) says that

$$\omega_{\mathsf{V}}|_\Sigma = -\,\omega_{\mathsf{H}}|_\Sigma\,. \tag{4.7}$$

We claim that $\omega_{\mathsf{H}}|_\Sigma = \omega_{\mathsf{V}}|_\Sigma = 0$, which would imply again by (4.4) that Σ is also ω_{NK}-isotropic. Let $u_1, u_2 \in T_x\Sigma$, and decompose them orthogonally as $u_j = u_j^{\mathsf{H}} + u_j^{\mathsf{V}}$, where $u_j^{\mathsf{H}} \in \mathsf{H}$ and $u_j^{\mathsf{V}} \in \mathsf{V}$. Since Σ is HV-compatible, both u_j^{H} and u_j^{V} are in $T_x\Sigma$ for $j = 1, 2$. Using (4.7) and the facts that $\omega_{\mathsf{H}} \in \Lambda^2(\mathsf{H}^*)$ and $\omega_{\mathsf{V}} \in \Lambda^2(\mathsf{V}^*)$, we have

$$\begin{aligned}\omega_{\mathsf{V}}(u_1, u_2) &= \omega_{\mathsf{V}}(u_1^{\mathsf{H}} + u_1^{\mathsf{V}}, u_2^{\mathsf{H}} + u_2^{\mathsf{V}}) = \omega_{\mathsf{V}}(u_1^{\mathsf{V}}, u_2^{\mathsf{V}}) \\ &= -\omega_{\mathsf{H}}(u_1^{\mathsf{V}}, u_2^{\mathsf{V}}) = 0.\end{aligned}$$

The argument in the other direction is essentially the same, with (4.7) replaced by $\omega_{\mathsf{V}}|_\Sigma = 2\omega_{\mathsf{H}}|_\Sigma$.

(b) Let $\Sigma^{2n+1} \subset Z$ be ω_{KE}-Lagrangian and ω_{NK}-Lagrangian, so that $\omega_{\mathsf{V}}|_\Sigma = 0$ and $\omega_{\mathsf{H}}|_\Sigma = 0$. Fix $z \in \Sigma$, let $\pi_{\mathsf{H}}\colon T_zZ \to \mathsf{H}$ and $\pi_{\mathsf{V}}\colon T_zZ \to \mathsf{V}$ denote the projection maps, so that

$$T_z\Sigma \subset \pi_{\mathsf{H}}(T_z\Sigma) \oplus \pi_{\mathsf{V}}(T_z\Sigma).$$

Let $(\rho, \mu)\colon T_zZ \to \mathbb{R}^{4n+2}$ be an $\mathrm{Sp}(n)\mathrm{U}(1)$-coframe at z. Since $\mu^2 \wedge \mu^3|_\Sigma = \omega_{\mathsf{V}}|_\Sigma = 0$, we have $\mu^2 \wedge \mu^3|_{\pi_{\mathsf{V}}(T_z\Sigma)} = 0$, so that $\dim(\pi_{\mathsf{V}}(T_z\Sigma)) \leq 1$. Moreover, since ω_{H} is a

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nondegenerate 2-form on the $4n$-plane H, the condition $\omega_{\mathsf{H}}|_{\pi_{\mathsf{H}}(T_z\Sigma)} = 0$ implies that $\dim(\pi_{\mathsf{H}}(T_z\Sigma)) \leq 2n$. Therefore, since

$$\dim(\pi_{\mathsf{H}}(T_z\Sigma) \oplus \pi_{\mathsf{V}}(T_z\Sigma)) = \dim(\pi_{\mathsf{H}}(T_z\Sigma)) + \dim(\pi_{\mathsf{V}}(T_z\Sigma)) \leq 2n + 1 = \dim(T_z\Sigma),$$

we deduce that $T_z\Sigma = \pi_{\mathsf{H}}(T_z\Sigma) \oplus \pi_{\mathsf{V}}(T_z\Sigma)$, which implies the result.

(c) This is immediate from Proposition 4.13. ∎

4.3.3 Other phases

Thus far, we have studied the real 3-form $\mathrm{Re}(\gamma) \in \Omega^3(Z)$. More generally, one can consider the S^1-family of real 3-forms $\mathrm{Re}(e^{-i\theta}\gamma)$ for constant $e^{i\theta} \in S^1$. We now explore the corresponding submanifold theory, beginning with a familiar situation:

Example 4.3 Suppose that $n = 1$, so that the twistor space Z is six-dimensional, and $\gamma \in \Omega^3(Z;\mathbb{C})$ is an $\mathrm{Sp}(1)\mathrm{U}(1) = \mathrm{U}(2)$-structure. By the discussion in Examples 4.1 and 4.2, the 3-form γ induces an $\mathrm{SU}(3)$-structure on Z^6 and satisfies

$$d\omega_{\mathrm{NK}} = 3\,\mathrm{Im}(2\gamma),$$
$$d\,\mathrm{Re}(2\gamma) = 2\,\omega_{\mathrm{NK}} \wedge \omega_{\mathrm{NK}}.$$

Now, let $L^3 \subset Z^6$ be an oriented three-dimensional submanifold. It is well known that L is ω_{NK}-Lagrangian if and only if L is γ-special Lagrangian of phase 1. That is,

$$\mathrm{Re}(2\gamma)|_L = \mathrm{vol}_L \iff \mathrm{Im}(2\gamma)|_L = 0 \text{ and } \omega_{\mathrm{NK}}|_L = 0 \iff L \text{ is } \omega_{\mathrm{NK}}\text{-Lagrangian}.$$

More generally, one might wish to consider γ-special Lagrangian 3-folds of other phases $e^{i\theta} \in S^1$. However, it is well known that if $L^3 \subset Z^6$ satisfies $\mathrm{Re}(e^{-i\theta}\gamma)|_L = \mathrm{vol}_L$, then $e^{-i\theta} = \pm 1$.

Example 4.3 is the special case $n = 1$ of the following more general statement, which is new:

Proposition 4.16 *Let $L^3 \subset Z^{4n+2}$ be a three-dimensional submanifold.*

(1) *If L is $\mathrm{Re}(e^{-i\theta}\gamma)$-calibrated, then $e^{i\theta} = \pm 1$.*
(2) *If L is $\mathrm{Re}(\gamma)$-calibrated, then $\omega_{\mathrm{NK}}|_L = 0$ and $\mathrm{Im}(\gamma)|_L = 0$. If $n = 1$, then the converse also holds.*

Proof Suppose that $L \subset Z^{4n+2}$ is $\mathrm{Re}(e^{-i\theta}\gamma)$-calibrated. By the same argument as in Proposition 4.10, we have $\omega_{\mathrm{NK}}|_L = 0$. Since $d\omega_{\mathrm{NK}} = 6\,\mathrm{Im}(\gamma)$, it follows that $\mathrm{Im}(\gamma)|_L = 0$. Therefore,

$$\pm\mathrm{vol}_L = \mathrm{Re}(e^{-i\theta}\gamma)\big|_L = \cos(\theta)\,\mathrm{Re}(\gamma)|_L\,.$$

Since $\mathrm{Re}(\gamma)$ has comass one, it follows that $\cos(\theta) = \pm 1$. (The converse of (b) when $n = 1$ is the well-known result discussed in Example 4.3.) ∎

4.4 Relations between submanifolds in M and Z

We now systematically discuss the relationships between the various classes of submanifolds in Z^{4n+2} and those in M^{4n+3}. Broadly speaking, given a submanifold $\Sigma \subset Z$, there are two natural ways to construct a corresponding submanifold of M. The first is to consider the circle bundle $p_1^{-1}(\Sigma) \subset M$, and the second is to consider its p_1-horizontal lift $\widehat{\Sigma} \subset M$ (provided it exists). We will examine both constructions.

4.4.1 Circle bundle constructions

We begin by considering submanifolds of the form $p_1^{-1}(\Sigma) \subset M$ for some submanifold $\Sigma \subset Z$. First, we consider those that are I_1-CR. In general, Proposition 3.15(a) shows that every I_1-CR 3-fold of M is ϕ_2-associative. For circle bundles, the converse also holds:

Proposition 4.17 *Let $\Sigma^{2k} \subset Z^{4n+2}$ be a submanifold, $2 \leq 2k \leq 4n$. Then Σ is J_+-complex if and only if $p_1^{-1}(\Sigma)$ is I_1-CR. Moreover, in the case of $2k = 2$, these conditions are also equivalent to: $p_1^{-1}(\Sigma)$ is ϕ_2-associative.*

Proof Let $\Sigma \subset Z$ be a submanifold, and set $L = p_1^{-1}(\Sigma) \subset M$. Fix $x \in L$ and let $z = p_1(x) \in \Sigma$. Note that

$$\Sigma \text{ is } J_+\text{-complex} \iff (\omega_{\mathrm{KE}})^k\big|_\Sigma = k!\,\mathrm{vol}_\Sigma,$$
$$L \text{ is } I_1\text{-CR} \iff (\alpha_1 \wedge \Omega_1^k)\big|_L = k!\,\mathrm{vol}_L.$$

Since $A_1 \in T_xL$, we can write $T_xL = \mathbb{R}A_1 \oplus \widetilde{U}$ for some subspace $\widetilde{U} \subset \mathrm{Ker}(\alpha_1)$. Let $\{\widetilde{u}_1, \ldots, \widetilde{u}_{2k-1}\}$ be an orthonormal basis of $\widetilde{U}$ such that $\{A_1, \widetilde{u}_1, \ldots, \widetilde{u}_{2k-1}\}$ is an oriented orthonormal basis of T_xL. Setting $u_j = (p_1)_*(\widetilde{u}_j)$, and noting that

$$p_1|_{\mathrm{Ker}(\alpha_1)} \colon \mathrm{Ker}(\alpha_1)|_x \to T_zZ$$

is an isometry, we see that $\{u_1, \ldots, u_{2k-1}\}$ is an orthonormal basis of $T_z\Sigma$. Therefore, recalling that $\Omega_1 = p_1^*(\omega_{\mathrm{KE}})$, we have

$$\begin{aligned} L \text{ is } I_1\text{-CR} \iff (\alpha_1 \wedge \Omega_1^k)(A_1, \widetilde{u}_1, \ldots, \widetilde{u}_{2k-1}) = k! &\iff \Omega_1^k(\widetilde{u}_1, \ldots, \widetilde{u}_{2k-1}) = k! \\ &\iff \omega_{\mathrm{KE}}^k(u_1, \ldots, u_{2k-1}) = k! \\ &\iff \Sigma \text{ is } J_+\text{-complex.} \end{aligned}$$

Now suppose $k = 1$. Observe that

$$\begin{aligned} \phi_2 &= \alpha_1 \wedge \Omega_1 - \alpha_2 \wedge \Omega_2 + \alpha_3 \wedge \Omega_3 \\ &= \alpha_1 \wedge \Omega_1 - \alpha_2 \wedge \kappa_2 + \alpha_3 \wedge \kappa_3. \end{aligned}$$

Since $\iota_{A_1}(-\alpha_2 \wedge \kappa_2 + \alpha_3 \wedge \kappa_3) = 0$, we have $(-\alpha_2 \wedge \kappa_2 + \alpha_3 \wedge \kappa_3)|_L = 0$. Therefore, we see that $\phi_2|_L = (\alpha_1 \wedge \Omega_1)|_L$, which gives the result. ∎

The previous proposition shows that a circle bundle $p_1^{-1}(\Sigma)$ is I_1-CR if and only if Σ is J_+-complex. In fact, any I_1-CR submanifold is *locally* a circle bundle:

Proposition 4.18 *Let $L^{2k+1} \subset M^{4n+3}$ be a submanifold, $2 \leq 2k \leq 4n$. Then L is I_1-CR if and only if L is locally of the form $p_1^{-1}(\Sigma)$ for some J_+-complex submanifold $\Sigma^{2k} \subset Z^{4n+2}$.*

Proof ($\Longleftarrow$) This follows from Proposition 4.17.

($\Longrightarrow$) Let $L \subset M$ be I_1-CR, and abbreviate $p := p_1$. At each $x \in L$, we have $A_1|_x \in T_xL$, so (short-time) integral curves of A_1 lie in L. That is, at each $x \in L$, there exists an open set $I_x \subset p^{-1}(p(x))$ such that $x \in I_x$ and $I_x \subset L$.

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We claim that $p(L) \subset Z$ is an embedded $2k$-dimensional submanifold of Z. To see this, fix $z \in p(L)$, and let $x \in L$ have $p(x) = z$. Letting ℓ satisfy $(2k+1) + \ell = 4n+3$, we choose a neighborhood $W \subset M$ of x and a chart $\widetilde{\phi}\colon W \to \mathbb{R}^{4n+3} = \mathbb{R}^{2k} \times \mathbb{R} \times \mathbb{R}^{\ell}$ with coordinate functions denoted $\widetilde{\phi} = (t^1, \ldots, t^{2k}, u, v^1, \ldots, v^{\ell})$ such that

$$\widetilde{\phi}(L \cap W) \subset \mathbb{R}^{2k} \times \mathbb{R} \times 0,$$
$$\widetilde{\phi}_*(A_1) = \frac{\partial}{\partial u} \quad \text{on } L \cap W.$$

Since $p\colon M \to Z$ is a submersion, it is an open map, and therefore $p(W) \subset M$ is an open set. Letting $\pi\colon \mathbb{R}^{2k} \times \mathbb{R} \times \mathbb{R}^{\ell} \to \mathbb{R}^{2k} \times \mathbb{R}^{\ell}$ denote the natural projection map, we observe that $\pi \circ \widetilde{\phi}\colon W \to \mathbb{R}^{4n+2} = \mathbb{R}^{2k} \times \mathbb{R}^{\ell}$ descends to a chart $\phi\colon p(W) \to \mathbb{R}^{4n+2} = \mathbb{R}^{2k} \times \mathbb{R}^{\ell}$ such that

$$\phi(p(L) \cap p(W)) \subset \mathbb{R}^{2k} \times 0.$$

This provides slice coordinates at $z \in p(L)$, showing that $p(L) \subset Z$ is an embedded $2k$-fold.

It follows that $p^{-1}(p(L)) \subset M$ is an embedded $(2k+1)$-dimensional submanifold of M, so that $L \subset p^{-1}(p(L))$ is an open set for dimension reasons. That $\Sigma := p(L)$ is J_+-complex follows from Proposition 4.17. ∎

Next, for any submanifold $\Sigma \subset Z$, we note that its circle bundle $p_1^{-1}(\Sigma) \subset M$ is never α_1-isotropic. However, in special situations, it can be α_2-isotropic. In this direction, we first observe:

Lemma 4.19 *Let $\Sigma^k \subset Z^{4n+2}$ be a submanifold with $1 \leq k \leq 2n$. The following are equivalent:*

(1) *$p_1^{-1}(\Sigma)$ is α_2-isotropic.*
(2) *$p_1^{-1}(\Sigma)$ is α_3-isotropic.*
(3) *Σ is horizontal.*

Proof Let $\Sigma \subset Z^{4n+2}$ be a submanifold with $\dim(\Sigma) \leq 2n$, and set $L = p_1^{-1}(\Sigma) \subset M$. Fix $x \in L$ and let $z = p_1(x) \in \Sigma$.

(i) $\Longleftrightarrow$ (ii). Suppose that L is α_2-isotropic at x. By Proposition 3.6, we have both $T_xL \subset \mathrm{Ker}(\alpha_2)$ and $\Omega_2|_{T_xL} = 0$. That is, the subspace $T_xL \subset \mathrm{Ker}(\alpha_2)$ is Ω_2-isotropic. Therefore, since $A_1 \in T_xL$, it follows that $A_3 = -\mathsf{J}_2(A_1)$ is orthogonal to T_xL, and hence $T_xL \subset \mathrm{Ker}(\alpha_3)$, showing that L is α_3-isotropic at x.

(ii) $\Longleftrightarrow$ (iii). Since $A_1 \in T_xL$, we can write $T_xL = \mathbb{R}A_1 \oplus \widetilde{U}$ for some subspace $\widetilde{U} \subset \mathrm{Ker}(\alpha_1)$. Since $p_1|_{\mathrm{Ker}(\alpha_1)}\colon \mathrm{Ker}(\alpha_1)|_x \to T_zZ$ is an isometry, it follows that $(p_1)_*(\widetilde{U}) = T_z\Sigma$. Now, observe that

$$\begin{aligned} T_z\Sigma \subset \mathsf{H} &\iff (p_1)_*(\widetilde{U}) \subset (p_1)_*(\widetilde{\mathsf{H}}) \iff \widetilde{U} \subset \mathrm{Ker}(\alpha_1, \alpha_2, \alpha_3) \\ &\iff T_xL \subset \mathrm{Ker}(\alpha_2, \alpha_3). \end{aligned}$$

Thus, if Σ is horizontal at z, then $T_z\Sigma \subset \mathsf{H}$, so that $T_xL \subset \mathrm{Ker}(\alpha_2, \alpha_3)$, and hence L is both α_2- and α_3-isotropic at x. Conversely, if L is α_2-isotropic at x, then by the previous paragraph, L is also α_3-isotropic at x, so $T_xL \subset \mathrm{Ker}(\alpha_2, \alpha_3)$, and hence Σ is horizontal at z. ∎

Corollary 4.20 *Let $\Sigma^{2k} \subset Z^{4n+2}$ be a submanifold, $2 \le 2k \le 2n$. Then Σ is J_+-complex and horizontal if and only if $p_1^{-1}(\Sigma)$ is I_1-CR isotropic (i.e., I_1-CR, α_2-isotropic, and α_3-isotropic).*

Proof This follows immediately from Proposition 4.17 and Lemma 4.19. ∎

Corollary 4.21 *Let $L^{2k+1} \subset M^{4n+3}$ be a submanifold, $3 \le 2k+1 \le 2n+1$. Then L is I_1-CR isotropic if and only if L is locally of the form $p_1^{-1}(\Sigma)$ for some horizontal J_+-complex submanifold $\Sigma^{2k} \subset Z^{4n+2}$.*

Proof This follows from Proposition 4.18 and Corollary 4.20. ∎

When Σ is $2n$-dimensional, the situation is particularly special:

Corollary 4.22 *Let $\Sigma^{2n} \subset Z^{4n+2}$ be $2n$-dimensional. The following are equivalent:*

(1) *Σ is J_+-complex and horizontal.*
(2) *Σ is horizontal.*
(3) *$p_1^{-1}(\Sigma)$ is α_2-Legendrian.*
(4) *$p_1^{-1}(\Sigma)$ is α_3-Legendrian.*
(5) *$p_1^{-1}(\Sigma)$ is I_1-CR Legendrian (i.e., I_1-CR, α_2-Legendrian, and α_3-Legendrian).*
(6) *$p_1^{-1}(\Sigma)$ is Ψ_2-special Legendrian of phase i^{n+1} and Ψ_3-special Legendrian of phase* 1.

Proof The equivalence (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) is Lemma 4.19. The equivalence (i) $\Longleftrightarrow$ (v) is Corollary 4.20.

It is clear that (v) $\Longrightarrow$ (iv). Conversely, if (iv) holds, then $L := p_1^{-1}(\Sigma)$ is both α_3-Legendrian and α_2-Legendrian, so that $\mathrm{C}(L) \subset C$ is both ω_2-Lagrangian and ω_3-Lagrangian, and therefore $\mathrm{C}(L)$ is I_1-complex Lagrangian. This proves (v).

It remains only to involve condition (vi). For this, note that (v) $\Longrightarrow$ (vi) follows from Corollary 3.9, and (vi) $\Longrightarrow$ (iii) follows from Proposition 3.7. ∎

The results of this subsection can be summarized in the following table.

$\dim(p_1^{-1}(\Sigma))$	S^1-bundle $p_1^{-1}(\Sigma) \subset M$	Base $\Sigma \subset Z$	$\dim(\Sigma)$	Ref.
$2k+1$	I_1-CR	J_+-complex	$2k$	4.18
3	ϕ_2-associative	J_+-complex	2	4.17
$\le 2n+1$	α_2-isotropic	Horizontal	$\le 2n$	4.19
$2n+1$	α_2-Legendrian	(J_+-complex and) horiz.	$2n$	4.22
$2n+1$	Ψ_2-special Legendrian of phase i^{n+1}	(J_+-complex and) horiz.	$2n$	4.22
$2n+1$	I_1-CR Legendrian	(J_+-complex and) horiz.	$2n$	4.22
$2k+1 \le 2n+1$	I_1-CR isotropic	J_+-complex and horiz.	$2k \le 2n$	4.21

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4.4.2 p_1-horizontal lifts

Let $L \subset M^{4n+3}$ be a submanifold, and recall that L is p_1-horizontal if and only if it is α_1-isotropic. In this case, $\dim(L) \leq 2n+1$, and its projection $p_1(L) \subset Z$ is ω_{KE}-isotropic. Conversely:

Proposition 4.23 *Let $\Sigma \subset Z^{4n+2}$ be a submanifold. Then Σ locally lifts to a p_1-horizontal submanifold of M if and only if Σ is ω_{KE}-isotropic. In this case,* $\dim(\Sigma) \leq 2n+1$.

Proof Suppose first that Σ locally lifts to a p_1-horizontal submanifold $\widehat{\Sigma} \subset M$. Since $\widehat{\Sigma}$ is p_1-horizontal, we have that $\alpha_1|_{\widehat{\Sigma}} = 0$. Therefore, Proposition 3.6 implies that $(p_1^*\omega_{\mathrm{KE}})|_{\widehat{\Sigma}} = \Omega_1|_{\widehat{\Sigma}} = 0$, and hence $\omega_{\mathrm{KE}}|_{\Sigma} = 0$.

Conversely, suppose that Σ is ω_{KE}-isotropic. Since $p_1 \colon M \to Z$ is a Riemannian submersion, the restriction of the derivative $(p_1)_* \colon TM \to TZ$ to the p_1-horizontal subbundle $\mathrm{Ker}(\alpha_1) \subset TM$ is an isometric isomorphism. Consider the distribution on M defined by $D := (p_1)_*|^{-1}_{\mathrm{Ker}(\alpha_1)}(T\Sigma) \subset TM$. Since Σ is ω_{KE}-isotropic, we have $\omega_{\mathrm{KE}}|_{T\Sigma} = 0$, and therefore $2\Omega_1|_D = 2(p_1^*\omega_{\mathrm{KE}})|_D = 0$. Since, by (3.5), $2\Omega_1$ is the curvature 2-form of the connection α_1 on the bundle $p_1 \colon M \to Z$, an application of the Frobenius theorem implies that D is locally integrable. By construction, the integral submanifolds of D are (local) p_1-horizontal lifts of Σ. ∎

4.4.3 p_1-horizontality and CR isotropic submanifolds

Note that if $L \subset M$ is p_1-horizontal, then L cannot be I_1-CR. Nevertheless, it is possible for L to be I_2-CR or I_3-CR. Moreover, it is also possible for L to be both p_1- and p_2-horizontal simultaneously. The following proposition elaborates on this.

Proposition 4.24 *Let $L^{2k+1} \subset M$ be a $(2k+1)$-dimensional submanifold,* $3 \leq 2k+1 \leq 2n+1$. *Then:*

(1) *L is I_2-CR and p_1-horizontal if and only if L is I_2-CR isotropic.*
(2) *Suppose* $\dim(L) = 2n+1$. *Then L is I_2-CR and p_1-horizontal $\iff$ L is I_2-CR Legendrian $\iff$ L is p_3-horizontal and p_1-horizontal.*

Proof (a) This follows from Proposition 3.8 (iii) $\iff$ (iv) with indices 1,2,3 replaced by $2, 1, -3$.

(b) This follows from Corollary 3.9 (iv) $\iff$ (v), again with 1,2,3 replaced by $2, 1, -3$. ∎

Now, given a CR isotropic submanifold $L \subset M$, we consider the geometric properties of its projection $p_1(L) \subset Z$. To state the result, we introduce the following notation. For a vertical unit vector $V \in \mathsf{V}_z \subset T_zZ$, we let $\beta_V := \iota_V(\mathrm{Re}\,\gamma)$ denote the induced nondegenerate 2-form on H_z, and let $J_V \in \mathrm{End}(\mathsf{H}_z)$ denote the corresponding complex structure on H_z.

Proposition 4.25 *Let $L^{2k+1} \subset M$ be a $(2k+1)$-dimensional submanifold,* $3 \leq 2k+1 \leq 2n+1$.

(1) *If L is α_1-isotropic and $(-s_\theta\alpha_2 + c_\theta\alpha_3)$-isotropic for some $e^{i\theta} \in S^1$, then $p_1(L) \subset Z$ is ω_{KE}-isotropic and ω_{NK}-isotropic.*

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(2) *If L is $(c_\theta I_2 + s_\theta I_3)$-CR isotropic for some $e^{i\theta} \in S^1$, then $\Sigma := p_1(L) \subset Z$ is ω_{KE}-isotropic, ω_{NK}-isotropic, and HV-compatible. Moreover,* $\dim(T_z\Sigma \cap \mathsf{V}) = 1$ *for all $z \in \Sigma$, and the $2k$-plane $T_z\Sigma \cap \mathsf{H}$ is J_V-invariant for any vertical unit vector $V \in T_z\Sigma \cap \mathsf{V}$.*

Proof (a) Suppose $L \subset M$ is α_1-isotropic and $(-s_\theta\alpha_2 + c_\theta\alpha_3)$-isotropic for some constant $e^{i\theta} \in S^1$. On L, we have $\alpha_1 = 0$ and $-s_\theta\alpha_2 + c_\theta\alpha_3 = 0$. This second equation implies

$$c_\theta\, \alpha_2 \wedge \alpha_3 = 0 \qquad\qquad s_\theta\, \alpha_2 \wedge \alpha_3 = 0,$$

and hence $\alpha_2 \wedge \alpha_3 = 0$ on L. Therefore, $\alpha_1 = 0$ implies $0 = d\alpha_1 = 2\Omega_1 = 2(\alpha_2 \wedge \alpha_3 + \kappa_1) = 2\kappa_1$, so that $\kappa_1 = 0$ on L. We deduce that $\Omega_1|_L = 0$ and $\widetilde{\Omega}_1|_L = 0$. Therefore, on the projection $p_1(L) \subset Z$, we have both $\omega_{\mathrm{KE}}|_{p_1(L)} = 0$ and $\omega_{\mathrm{NK}}|_{p_1(L)} = 0$.

(b) Suppose $L \subset M$ is $(c_\theta I_2 + s_\theta I_3)$-CR isotropic, so that L is α_1-isotropic and $(-s_\theta\alpha_2 + c_\theta\alpha_3)$-isotropic, and $(c_\theta I_2 + s_\theta I_3)$-CR. By part (a), the projection $\Sigma := p_1(L)$ is both ω_{KE}-isotropic and ω_{NK}-isotropic.

Fix $x \in L$, let $z = p(x) \in \Sigma$, set $\widetilde{V} = c_\theta A_2 + s_\theta A_3 \in T_xM$, and let $\mathsf{J}_V = c_\theta \mathsf{J}_2 + s_\theta \mathsf{J}_3$. By assumption, we can write $T_xL = H_L \oplus \mathbb{R}\widetilde{V}$ for some J_V-invariant $2k$-plane $H_L \subset \widetilde{\mathsf{H}}$. It follows that $T_z\Sigma = H_\Sigma \oplus \mathbb{R}V$, where $H_\Sigma := p_*(H_L) \subset \mathsf{H}$ is a horizontal $2k$-plane, and $V = p_*(\widetilde{V}) \in \mathsf{V}$ is a vertical unit vector. In particular, this shows that Σ is HV compatible, and that $\dim(T_z\Sigma \cap \mathsf{V}) = 1$.

Now, since $\mathrm{Re}(\Gamma_1) = p^*(\mathrm{Re}(\gamma))$, we have that $\iota_{\widetilde{V}}(\mathrm{Re}\,\Gamma_1) = p^*(\iota_V(\mathrm{Re}\,\gamma)) = p^*(\beta_V)$ on L. In particular, if $Y \in H_L$ is a horizontal vector tangent to L, then

$$g_{\mathrm{KE}}(p_*\mathsf{J}_V Y, p_*\cdot) = g_M(\mathsf{J}_V Y, \cdot) = \mathrm{Re}(\Gamma_1)(\widetilde{V}, Y, \cdot) = \beta_V(p_*Y, p_*\cdot) = g_{\mathrm{KE}}(J_V p_* Y, p_*\cdot),$$

which shows that

$$p_*\mathsf{J}_Y = J_V p_* \text{ on } H_L. \tag{4.8}$$

Finally, if $X \in T_z\Sigma \cap \mathsf{H} = H_\Sigma$, then $X = p_*(\widetilde{X})$ for some $\widetilde{X} \in H_L$. Since H_L is J_V-invariant, it follows that $\mathsf{J}_V\widetilde{X} \in H_L$. Therefore, $J_V X = J_V p_*(\widetilde{X}) = p_*(\mathsf{J}_V\widetilde{X}) \in p_*(H_L) = H_\Sigma$, which shows that H_Σ is J_V-invariant. ∎

Conversely, we now ask which submanifolds $\Sigma \subset Z$ admit local p_1-horizontal lifts to CR isotropic submanifolds of M. As we now show, the necessary conditions given in Proposition 4.25(b) are in fact sufficient:

Proposition 4.26 *Let $\Sigma^k \subset Z^{4n+2}$ be a submanifold, $3 \le k \le 2n+1$, that is, ω_{KE}-isotropic, ω_{NK}-isotropic, and HV-compatible.*

(1) *If Σ is nowhere tangent to H, then every local p_1-horizontal lift of Σ is α_1-isotropic and $(-s_\theta\alpha_2 + c_\theta\alpha_3)$-isotropic for some constant $e^{i\theta} \in S^1$.*

(2) *If* $\dim(T_z\Sigma \cap \mathsf{V}) = 1$ *for all $z \in \Sigma$, and if $T_z\Sigma \cap \mathsf{H}$ is J_V-invariant for any vertical unit vector $V \in T_z\Sigma \cap \mathsf{V}$, then every local p_1-horizontal lift of Σ is $(c_\theta I_2 + s_\theta I_3)$-CR isotropic for some constant $e^{i\theta} \in S^1$.*

Proof (a) Let $\Sigma \subset Z$ be as in the statement. Since Σ is ω_{KE}-isotropic, Proposition 4.23 implies that Σ locally admits a p_1-horizontal lift to a k-dimensional submanifold $L \subset M$, which is automatically α_1-isotropic. Moreover, since Σ is

HV-compatible, and since $(p_1)_*|_{\mathrm{Ker}(\alpha_1)}: \mathrm{Ker}(\alpha_1) \to TZ$ is an isomorphism that respects the horizontal-vertical splitting, it follows that TL splits as

$$TL = (TL \cap \widetilde{\mathsf{H}}) \oplus (TL \cap \widetilde{\mathsf{V}}). \tag{4.9}$$

Now, note that the system $\omega_{\mathrm{KE}}|_\Sigma = \omega_{\mathrm{NK}}|_\Sigma = 0$ is equivalent to $\omega_{\mathsf{V}}|_\Sigma = \omega_{\mathsf{H}}|_\Sigma = 0$. Since $p_1^*(\omega_{\mathsf{V}}) = \alpha_2 \wedge \alpha_3$, it follows that $\{\alpha_2|_L, \alpha_3|_L\}$ is a linearly dependent set of 1-forms on L. Moreover, since Σ is nowhere tangent to H, it follows that L is nowhere tangent to $\widetilde{\mathsf{H}} = \mathrm{Ker}(\alpha_1, \alpha_2, \alpha_3)$, and thus there is no point of L at which $\alpha_2|_L, \alpha_3|_L$ simultaneously vanish. Therefore, there is an S^1-valued function $e^{i\theta}: L \to S^1$ such that the 1-form

$$\tau_\theta := -s_\theta \alpha_2 + c_\theta \alpha_3$$

vanishes on L. It remains to show that $e^{i\theta}$ is constant on L. For this, we compute on L that

$$0 = d\tau_\theta = d\theta \wedge (-s_\theta \alpha_2 + c_\theta \alpha_3) + 2(c_\theta \kappa_2 + s_\theta \kappa_3),$$

where we have used that $\alpha_1|_L = 0$ to compute $d\alpha_2 = 2\kappa_2$ and $d\alpha_3 = 2\kappa_3$. Now, the first term is in $(T^*L \otimes \widetilde{\mathsf{V}}^*)|_L$, while the second is in $\Lambda^2(\widetilde{\mathsf{H}}^*)|_L$, so by equation (4.9), they vanish independently. In particular, $d\theta \wedge (-s_\theta \alpha_2 + c_\theta \alpha_3) = 0$. Together with the equation $c_\theta \alpha_2 + s_\theta \alpha_3 = 0$ on L, this implies that $d\theta \wedge \alpha_2 = 0$ and $d\theta \wedge \alpha_3 = 0$, which yields $d\theta = 0$, so (since L is assumed connected) θ is constant.

(b) Let $\Sigma \subset Z$ be as in the statement. By part (a), every local p_1-horizontal lift $L \subset M$ of the submanifold $\Sigma \subset Z$ is α_1-isotropic and $(-s_\theta \alpha_2 + c_\theta \alpha_2)$-isotropic for some $e^{i\theta} \in S^1$. Thus, it remains only to show that L is $(c_\theta I_2 + s_\theta I_3)$-CR.

Fix $x \in L$, and let $z = p_1(x) \in \Sigma$. By assumption, we may split $T_z\Sigma = H_\Sigma \oplus \mathbb{R}V$, where $V \in \mathsf{V}$ is a unit vector, and $H_\Sigma \subset \mathsf{H}$ is J_V-invariant. Therefore, since $(p_1)_*$ yields an isomorphism $\mathrm{Ker}(\alpha_1)|_x \to T_zZ$ that respects the horizontal–vertical splittings, we may decompose $TL = H_L \oplus \mathbb{R}\widetilde{V}$, where $H_L \subset \widetilde{\mathsf{H}}$ satisfies $p_*(H_L) = H_\Sigma$ and $\widetilde{V} \in \widetilde{\mathsf{V}}$ satisfies $p_*(\widetilde{V}) = V$.

Now, since L is both α_1-isotropic and $(-s_\theta \alpha_2 + c_\theta \alpha_2)$-isotropic, it follows that $\widetilde{V} = c_\theta A_2 + s_\theta A_3$. Let $\mathsf{J}_V = c_\theta \mathsf{J}_2 + s_\theta \mathsf{J}_3$. If $X \in H_L$, then $p_*X \in H_\Sigma$, so by (4.8) we have $p_*(\mathsf{J}_V X) = J_V(p_*X) \in H_\Sigma = p_*(H_L)$, and therefore $\mathsf{J}_V X \in H_L$. Thus, H_L is J_V-invariant, and so L is $(c_\theta I_2 + s_\theta I_3)$-CR. ∎

In the highest and lowest dimensions, the relationship between CR isotropic submanifolds of M and their projections in Z becomes simpler. Indeed, in the top dimension:

Corollary 4.27

(1) *If $L^{2n+1} \subset M^{4n+3}$ is $(c_\theta I_2 + s_\theta I_3)$-CR Legendrian for some $e^{i\theta} \in S^1$, then $p_1(L) \subset Z$ is ω_{KE}-Lagrangian and ω_{NK}-Lagrangian.*

(2) *Conversely, if $\Sigma^{2n+1} \subset Z^{4n+2}$ is ω_{KE}-Lagrangian and ω_{NK}-Lagrangian, then every local p_1-horizontal lift of Σ is $(c_\theta I_2 + s_\theta I_3)$-CR Legendrian for some $e^{i\theta} \in S^1$.*

Proof (a) This follows from Proposition 4.25.

(b) Suppose $\Sigma \subset Z$ is ω_{KE}-Lagrangian and ω_{NK}-Lagrangian. By Proposition 4.15(b), it follows that Σ is HV compatible, and that $\dim(T_z\Sigma \cap \mathsf{V}) = 1$ at each $z \in \Sigma$. Therefore, by Proposition 4.26(a), every local p_1-horizontal lift $L \subset M$ is α_1-Legendrian and

$(-s_\theta\alpha_2 + c_\theta\alpha_3)$-Legendrian for some constant $e^{i\theta} \in S^1$. By Corollary 3.9(v) $\Longrightarrow$ (iv), it follows that L is $(c_\theta I_2 + s_\theta I_3)$-CR Legendrian. ∎

Corollary 4.28

(1) *If $L^3 \subset M^{4n+3}$ is $(c_\theta I_2 + s_\theta I_3)$-CR isotropic for some $e^{i\theta} \in S^1$, then $p_1(L) \subset Z$ is (up to a change of orientation) $\mathrm{Re}(\gamma)$-calibrated and ω_{KE}-isotropic.*
(2) *Conversely, if $\Sigma^3 \subset Z^{4n+2}$ is $\mathrm{Re}(\gamma)$-calibrated and ω_{KE}-isotropic, then every local p_1-horizontal lift of Σ is $(c_\theta I_2 + s_\theta I_3)$-CR isotropic for some $e^{i\theta} \in S^1$.*

Proof (a) Let $L^3 \subset M^{4n+3}$ be a $(c_\theta I_2 + s_\theta I_3)$-CR isotropic 3-fold. By Proposition 4.25(b), $\Sigma := p_1(L) \subset Z$ is ω_{KE}-isotropic, so it remains only to show that Σ is $\mathrm{Re}(\gamma)$-calibrated.

Fix $z \in \Sigma$. Again, by Proposition 4.25(b), we may decompose $T_z\Sigma = H_\Sigma \oplus \mathbb{R}V$ for some 2-plane $H_\Sigma \subset \mathsf{H}$ and vertical unit vector $T \in \mathsf{V}_z$. Let $N \in \mathsf{V}_z$ be the vertical unit vector such that $\{T, N\}$ is an oriented orthonormal basis of V_z, and let $\beta_T, \beta_N \in \Lambda^2(\mathsf{H}_z^*)$ be the induced nondegenerate 2-forms from γ. Since H_Σ is J_V-invariant, it follows that $\beta_V|_{H_\Sigma} = \pm\mathrm{vol}_{H_\Sigma}$. Therefore,

$$\mathrm{Re}(\gamma)|_{T_z\Sigma} = (T^\flat \wedge \beta_T + N^\flat \wedge \beta_N)\big|_{T_z\Sigma} = \pm\mathrm{vol}_{V_\Sigma} \wedge \mathrm{vol}_{H_\Sigma} + 0 = \pm\mathrm{vol}_\Sigma.$$

(b) Suppose $\Sigma^3 \subset Z^{4n+2}$ is $\mathrm{Re}(\gamma)$-calibrated and ω_{KE}-isotropic. By Proposition 4.15(c), it follows that Σ is HV compatible, so we may write $T_z\Sigma = H_\Sigma \oplus V_\Sigma$, where $H_\Sigma \subset \mathsf{H}$ and $V_\Sigma \subset \mathsf{V}$. The same proposition shows that $\dim(V_\Sigma) = 1$. Now, let $V \in V_\Sigma$ be a unit vector, let $\beta_V = \iota_V(\mathrm{Re}(\gamma))$ denote the induced nondegenerate 2-form on H_z, and let J_V be the corresponding complex structure on H_z. Since $\mathrm{Re}(\gamma)|_\Sigma = \mathrm{vol}_\Sigma = \mathrm{vol}_{V_\Sigma} \wedge \mathrm{vol}_{H_\Sigma}$, it follows that $\beta_V|_{H_\Sigma} = \pm\mathrm{vol}_{H_\Sigma}$, which proves that H_Σ is J_V-invariant. Therefore, Proposition 4.26(b) gives the result. ∎

4.4.4 p_1-horizontality of special isotropic submanifolds

By Proposition 3.15(b), every $-\theta_{I,3}$-special isotropic 3-fold is ϕ_2-associative. Moreover, since $\iota_{A_1}(-\theta_{I,3}) = 0$ by Definition 3.10, Proposition A.2 implies that every $-\theta_{I,3}$-special isotropic 3-fold is p_1-horizontal. We now observe that these necessary conditions are sufficient:

Proposition 4.29 *Let $L^{2k+1} \subset M^{4n+3}$ be a $(2k+1)$-dimensional submanifold, $3 \le 2k + 1 \le 2n + 1$.*

(1) *If L is $\theta_{I,2k+1}$-special isotropic, then L is p_1-horizontal.*
(2) *If L is Ψ_1-special Legendrian, then L is p_1-horizontal.*
(3) *Suppose $\dim(L) = 3$. Then L is $-\theta_{I,3}$-special isotropic if and only if L is ϕ_2-associative and p_1-horizontal.*

Proof (a) Since $\iota_{A_1}(\theta_{I,2k+1}) = 0$, Proposition A.2 gives the result.

(b) This is simply part (a) in the case of $\dim(L) = 2n + 1$.

(c) Suppose $\dim(L) = 3$. Then

$$\begin{aligned} &L \text{ is } \phi_2\text{-associative and } p_1\text{-horizontal} \\ &\iff (\alpha_1 \wedge \Omega_1 - \alpha_2 \wedge \Omega_2 + \alpha_3 \wedge \Omega_3)|_L = \mathrm{vol}_L \ \text{ and } \ \alpha_1|_L = 0 \end{aligned}$$

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and

$$L \text{ is } -\theta_{I,3}\text{-special isotropic} \iff (-\alpha_2 \wedge \Omega_2 + \alpha_3 \wedge \Omega_3)|_L = \mathrm{vol}_L$$

The result is now immediate. ∎

Example 4.4 For $n = 1$, Proposition 4.29(c) is the well-known fact that a 3-fold $L^3 \subset M^7$ is ϕ_2-associative and p_1-horizontal if and only if it is Ψ_1-special Legendrian of phase -1.

4.4.5 $\mathrm{Re}(\Gamma_1)$-calibrated 3-folds of M

We now observe that $\mathrm{Re}(\Gamma_1)$-calibrated 3-folds $L^3 \subset M^{4n+3}$ are always p_1-horizontal, and describe their projections $p_1(L) \subset Z$. Namely:

Proposition 4.30 *If $L^3 \subset M^{4n+3}$ is $\mathrm{Re}(\Gamma_1)$-calibrated, then L is p_1-horizontal (equivalently, α_1-isotropic). Moreover:*

(1) *If $L^3 \subset M^{4n+3}$ is $\mathrm{Re}(\Gamma_1)$-calibrated, then L is locally a p_1-horizontal lift of a 3-fold in Z that is both $\mathrm{Re}(\gamma)$-calibrated and ω_{KE}-isotropic.*
(2) *Conversely, if $\Sigma^3 \subset Z^{4n+2}$ is both $\mathrm{Re}(\gamma)$-calibrated and ω_{KE}-isotropic, then Σ locally lifts to a $\mathrm{Re}(\Gamma_1)$-calibrated 3-fold in M.*

Proof Let $L \subset M$ be a $\mathrm{Re}(\Gamma_1)$-calibrated 3-fold. Since $\mathrm{Re}(\Gamma_1) = \alpha_2 \wedge \kappa_2 + \alpha_3 \wedge \kappa_3$, we have $\iota_{A_1}(\mathrm{Re}(\Gamma_1)) = 0$. In view of the splitting $TM = \mathbb{R}A_1 \oplus \mathrm{Ker}(\alpha_1)$, Proposition A.2 implies that $TL \subset \mathrm{Ker}(\alpha_1)$, so that L is p_1-horizontal (equivalently, $\alpha_1|_L = 0$).

Parts (a) and (b) now follow from Proposition 4.23 and the fact that $\Gamma_1 = p_1^*(\gamma)$. ∎

We are now in a position to prove Theorem 3.20, which classifies the $\mathrm{Re}(\Gamma_1)$-calibrated 3-folds in terms of more familiar geometries.

Theorem 4.31 *Let $L^3 \subset M^{4n+3}$ be a three-dimensional submanifold. The following are equivalent:*

(1) *$C(L)$ is a $(c_\theta I_2 + s_\theta I_3)$-complex isotropic 4-fold for some constant $e^{i\theta} \in S^1$.*
(2) *L is a $(c_\theta I_2 + s_\theta I_3)$-CR isotropic 3-fold for some constant $e^{i\theta} \in S^1$.*
(3) *L is locally of the form $p_v^{-1}(S)$ for some horizontal J_+-complex curve $S \subset Z$ and some $v = (0, c_\theta, s_\theta)$.*
(4) *L is locally a p_1-horizontal lift of a 3-fold $\Sigma^3 \subset Z$ that is $\mathrm{Re}(\gamma)$-calibrated and ω_{KE}-isotropic.*
(5) *L is $\mathrm{Re}(\Gamma_1)$-calibrated.*

Proof (i) $\iff$ (ii). This follows from Proposition 3.8.

(ii) $\iff$ (iii). This is Corollary 4.21.

(ii) $\iff$ (iv). This is Corollary 4.28.

(iv) $\iff$ (v). This is Proposition 4.30. ∎

5 Submanifolds of quaternionic Kähler manifolds

Thus far, we have studied twistor spaces Z as S^1-quotients of 3-Sasakian manifolds M. In Section 5.1, we adopt a different perspective, viewing Z as the total space of a canonical S^2-bundle $\tau\colon Z \to Q$ over a quaternionic-Kähler manifold Q^{4n}. This leads to an alternative construction of the $\mathrm{Sp}(n)\mathrm{U}(1)$-structure on Z, including the 3-form $\gamma \in \Omega^3(Z;\mathbb{C})$.

In Section 5.2, we turn our attention to *totally complex* submanifolds of Q^{4n}, a class that is intimately related to the (semi-)calibrated geometries of previous sections. To explain these relations, we will recall that a totally complex submanifold $U^{2k} \subset Q^{4n}$ admits two distinct lifts to Z, namely its *τ-horizontal lift* $\widetilde{U}^{2k} \subset Z$, and its *geodesic circle bundle lift* $\mathcal{L}(U)^{2k+1} \subset Z$.

Given such a circle bundle lift $\mathcal{L}(U) \subset Z$, we will prove (Corollary 5.12) that its local p_1-horizontal lifts to M are $(c_\theta I_2 + s_\theta I_3)$-CR isotropic. The main result of this section (Theorem 5.14) is that the converse also holds: If $L \subset M$ is a compact $(c_\theta I_2 + s_\theta I_3)$-CR isotropic submanifold, then L is a p_1-horizontal lift of some circle bundle $\mathcal{L}(U)$. As an application, we prove (Theorem 5.17) that every compact $(2n+1)$-fold $\Sigma \subset Z$ that is Lagrangian with respect both ω_{KE} and ω_{NK} is of the form $\mathcal{L}(U)$, thereby generalizing a result of Storm [30] to higher dimensions.

We remind the reader that as mentioned in the introduction, we only consider submanifolds of Q that do not meet any orbifold points.

5.1 Quaternionic Kähler manifolds

Let Q^{4n} be a smooth $4n$-manifold, $n \geq 1$.

Definition 5.1 An *almost quaternionic-Hermitian structure* (or $\mathrm{Sp}(n)\mathrm{Sp}(1)$-*structure*) on Q is a pair (g_Q, E) consisting of an orientation and a Riemannian metric g_Q, and a rank 3 subbundle $E \subset \mathrm{End}(TQ)$ such that:

(1) At each $q \in Q$, there exists a local frame (j_1, j_2, j_3) of E, called an *admissible frame*, satisfying the quaternionic relations $j_1 j_2 = j_3$ and $j_1^2 = j_2^2 = j_3^2 = -\mathrm{Id}$.
(2) Every $j \in E$ acts by isometries: $g_Q(jX, jY) = g_Q(X, Y)$, for all $X, Y \in TQ$.

Equivalently, an almost quaternionic-Hermitian structure may be defined as 4-form $\Pi \in \Omega^4(Q)$ such that at each $q \in Q$, there exists a coframe $L\colon T_qQ \to \mathbb{R}^{4n}$ for which $\Pi|_q = \frac{1}{6}L^*(\beta_1^2 + \beta_2^2 + \beta_3^2)$, where $\{\beta_1, \beta_2, \beta_3\}$ is the standard hyperkähler triple on $\mathbb{R}^{4n} = \mathbb{H}^n$. (See [27] or [8] for details.)

Definition 5.2 Let $n \geq 2$. An almost quaternionic-Hermitian structure (g_Q, E) is *quaternionic-Kähler (QK)* if $E \subset \mathrm{End}(TQ)$ is a parallel subbundle (with respect to the connection ∇ induced by g_Q). That is, if σ is a local section of E, then $\nabla\sigma$ is also a local section of E. An equivalent condition is that the 4-form $\Pi \in \Omega^4(Q)$ is g_Q-parallel.

For $n = 1$, we say (Q^4, g_Q) is *quaternionic-Kähler* if the metric g_Q is Einstein and anti-self-dual.

Remark 5.3 It is well known that if (g_Q, E) is a QK structure, then $\mathrm{Hol}(g_Q) \leq \mathrm{Sp}(n)\mathrm{Sp}(1)$. Conversely, for $n \geq 2$, if g is a Riemannian metric on Q with $\mathrm{Hol}(g) \leq \mathrm{Sp}(n)\mathrm{Sp}(1)$, then there exists a g-parallel rank 3 subbundle $E \subset \mathrm{End}(TQ)$ such that (g, E) is a QK structure.

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5.1.1 The twistor space

From now on, (Q^{4n}, g_Q, E) denotes a quaternionic-Kähler $4n$-manifold with positive scalar curvature. The *twistor space of* Q is the $(4n+2)$-manifold

$$Z := \{j \in E \colon j^2 = -\mathrm{Id}\}.$$

The obvious projection map $\tau \colon Z \to Q$ is then an S^2-bundle, and we let $\mathsf{V} \subset TZ$ denote the (rank 2) vertical bundle. The Levi–Civita connection of g_Q induces a connection on the vector bundle $E \subset \mathrm{End}(TZ)$, and hence a connection on the S^2-subbundle $Z \subset E$, thereby yielding a $4n$-plane field $\mathsf{H} \subset TZ$ such that

$$TZ = \mathsf{H} \oplus \mathsf{V}.$$

We now recall the Kähler–Einstein structure $(g_{\mathrm{KE}}, \omega_{\mathrm{KE}}, J_{\mathrm{KE}})$ on Z. First, define a Riemannian metric g_{KE} by requiring that $g_{\mathrm{KE}}(\mathsf{H}, \mathsf{V}) = 0$ and

(1) For $X, Y \in \mathsf{H}$, we have $g_{\mathrm{KE}}(X, Y) = g_Q(\tau_* X, \tau_* Y)$.
(2) On V, the metric g_{KE} is induced by the fiber metric $\langle \cdot, \cdot \rangle$ on $E \subset \mathrm{End}(TZ)$ under the identifications $\mathsf{V}_z \simeq T_z(Z_{\tau(z)}) \subset T_z(E_{\tau(z)}) \simeq E_{\tau(z)}$.

Next, define an almost-complex structure J_{KE} on Z by requiring that both H and V are J_{KE}-invariant, and

(1) On H_z, we set $J_{\mathrm{KE}} = (\tau_*|_{\mathsf{H}_z})^{-1} \circ z \circ \tau_*$.
(2) On V_z, identifying vertical vectors $X \in \mathsf{V}_z \simeq T_z(Z_{\tau(z)})$ with endomorphisms $j_X \in z^\perp = \{j \in E_{\tau(z)} \colon \langle j, z \rangle = 0\}$, we set $J_{\mathrm{KE}} X = z \circ j_X$.

We let $\omega_{\mathrm{KE}}(X, Y) = g_{\mathrm{KE}}(J_{\mathrm{KE}} X, Y)$. It is well known [27] that the triple $(g_{\mathrm{KE}}, \omega_{\mathrm{KE}}, J_{\mathrm{KE}})$ is a Kähler–Einstein structure.

Remark 5.4 The $(\mathrm{U}(2n) \times \mathrm{U}(1))$-structure $(g_{\mathrm{KE}}, \omega_{\mathrm{KE}}, J_{\mathrm{KE}}, \mathsf{H})$ just defined on Z coincides with the one described in Section 4.2. In brief, if Q^{4n} is a quaternionic-Kähler manifold of positive scalar curvature, then its *Konishi bundle* $M^{4n+3} = F_{\mathrm{SO}(3)}(E)$, which is the $\mathrm{SO}(3)$-frame bundle of the rank 3 vector bundle $E \to Q$, admits a 3-Sasakian structure, from which one can recover the $(\mathrm{U}(2n) \times \mathrm{U}(1))$-structure on Z. For details, see [8, Sections12.2 and 13.3.2].

Recall from Theorem 4.7 that there exists a canonical $\mathrm{Sp}(n)\mathrm{U}(1)$-structure $\gamma \in \Omega^3(Z; \mathbb{C})$ on the twistor space $(Z, g_{\mathrm{KE}}, J_{\mathrm{KE}}, \mathsf{H})$. We end this section by giving a different proof of the existence of this $\mathrm{Sp}(n)\mathrm{U}(1)$-structure, working directly from the projection $\tau \colon Z \to Q$, without reference to M. At a point $z \in Z$, choose an admissible frame (z, j_2, j_3) at $\tau(z) \in Q$. Via the isomorphism

$$\mathsf{V}_z \simeq z^\perp = \{j \in E_{\tau(z)} \colon \langle j, z \rangle = 0\},$$

the points $j_2, j_3 \in E_{\tau(z)}$ define vertical vectors at z, and hence (via the metric) 1-forms $\mu_2, \mu_3 \in \Lambda^1(\mathsf{V}^*|_z)$ at z. On the other hand,

$$J_2 := (\tau_*|_{\mathsf{H}_z})^{-1} \circ j_3 \circ \tau_* \qquad J_3 := -(\tau_*|_{\mathsf{H}_z})^{-1} \circ j_2 \circ \tau_* \tag{5.1}$$

are g_{KE}-orthogonal complex structures on H_z, and hence yield 2-forms $\beta_2 := g_{\mathrm{KE}}(J_2\cdot,\cdot)$ and $\beta_3 := g_{\mathrm{KE}}(J_3\cdot,\cdot)$ on H_z. We can now define a $\mathbb{C}$-valued 3-form γ at $z \in Z$ by

$$\gamma := (\mu_2 - i\mu_3) \wedge (\beta_2 + i\beta_3). \tag{5.2}$$

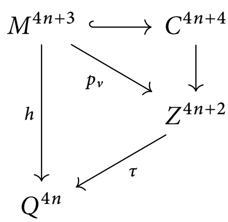
This 3-form is independent of the choice (j_2, j_3). That is, one can check that if $(z, \widetilde{j}_2, \widetilde{j}_3) = (z, c_\theta j_2 + s_\theta j_3, -s_\theta j_2 + c_\theta j_3)$ is another admissible frame at $\tau(z)$, then the corresponding 1-forms $\widetilde{\mu}_2, \widetilde{\mu}_3$ on V_z and 2-forms $\widetilde{\beta}_2, \widetilde{\beta}_3$ on H_z satisfy

$$(\widetilde{\mu}_2 - i\widetilde{\mu}_3) \wedge (\widetilde{\beta}_2 + i\widetilde{\beta}_3) = (\mu_2 - i\mu_3) \wedge (\beta_2 + i\beta_3).$$

Remark 5.5 In fact, there is a natural one-parameter family of 3-forms on Z given by $e^{i\theta}\gamma \in \Omega^3(Z;\mathbb{C})$ for constants $e^{i\theta} \in S^1$. In particular, the 3-form defined by (5.2) agrees with that of Section 4.2 (viz., Theorem 4.7) up to a constant $\lambda \in S^1$. The 90° rotation in formula (5.1) relating (J_2, J_3) to (j_2, j_3) was chosen to arrange for $\lambda = 1$. (This follows from Theorem 6.3 and Proposition 4.16.)

5.1.2 The diamond diagram

Altogether, the various spaces we have considered can be summarized by the *diamond diagram*:



Example 5.1

- The *flat model* is $(C, M, Z, Q) = (\mathbb{H}^{n+1}, \mathbb{S}^{4n+3}, \mathbb{CP}^{2n+1}, \mathbb{HP}^n)$, in which each $p_v\colon \mathbb{S}^{4n+3} \to \mathbb{CP}^{2n+1}$ for $v \in S^2$ is a complex Hopf fibration, $h\colon \mathbb{S}^{4n+3} \to \mathbb{HP}^n$ is the quaternionic Hopf fibration, and $\tau\colon \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ is the classical twistor fibration.
- Perhaps the second simplest family of examples is

$$(M, Z, Q) = \big(\mathbb{S}(T^*\mathbb{CP}^{n+1}),\ \mathbb{P}(T^*\mathbb{CP}^{n+1}),\ \mathrm{Gr}_2(\mathbb{C}^{n+2})\big),$$

  where $\mathbb{P}(T^*\mathbb{CP}^{n+1})$ and $\mathbb{S}(T^*\mathbb{CP}^{n+1})$ refer to the projectivized cotangent bundle and unit sphere subbundle of the cotangent bundle of $\mathbb{CP}^{n+1}$, respectively [33]. In the case of $n = 1$, these spaces are $(M^7, Z^6, Q^4) = (N_{1,1}, \frac{\mathrm{SU}(3)}{T^2}, \mathbb{CP}^2)$, where $N_{1,1} = \frac{\mathrm{SU}(3)}{\mathrm{U}(1)}$ is an exceptional Aloff–Wallach space.
- An exceptional example is $(M^{11}, Z^{10}, Q^8) = \Big(\frac{\mathrm{G}_2}{\mathrm{Sp}(1)_+}, \frac{\mathrm{G}_2}{\mathrm{U}(2)_+}, \frac{\mathrm{G}_2}{\mathrm{SO}(4)}\Big)$. Here, M^{11} and Z^{10} should not be confused with $\frac{\mathrm{G}_2}{\mathrm{Sp}(1)_-} \cong V_2(\mathbb{R}^7)$ and $\frac{\mathrm{G}_2}{\mathrm{U}(2)_-} \cong \mathrm{Gr}_2(\mathbb{R}^7)$. See [8, Example 13.6.8].

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5.2 Totally complex submanifolds

We now turn to the various submanifolds of a quaternionic-Kähler manifold (Q^{4n}, g_Q, E), continuing to assume that g_Q has positive scalar curvature.

Definition 5.6 A submanifold $U^{2k} \subset Q^{4n}$ is *almost-complex* if there exists a section $i \in \Gamma(Z|_U)$ such that $i(T_u U) = T_u U$ for all $u \in U$.

We will be particularly interested in the following subclass of almost-complex submanifolds.

Definition 5.7 A submanifold $U^{2k} \subset Q^{4n}$, for $1 \le k \le 2n$, is called *totally complex* if there exists a section $i \in \Gamma(Z|_U)$ such that at each $u \in U$:

(1) $i(T_u U) = T_u U$.
(2) For all $j \in Z_u$ with $\langle j, i \rangle = 0$, we have $j(T_u U) \subset (T_u U)^{\perp}$.

A totally complex submanifold $U \subset Q^{4n}$ is called *maximal* if $\dim(U) = 2n$.

Totally complex submanifolds were introduced by Funabashi [16], who proved that they are minimal (zero-mean curvature) provided $n \ge 2$.

Example 5.2

- In $Q = \mathbb{HP}^n$, the maximal totally complex submanifolds with parallel second fundamental form were classified by Tsukada [32]. The list consists of the two infinite families
$$\mathbb{CP}^n \to \mathbb{HP}^n \qquad \mathbb{CP}^1 \times \frac{\mathrm{SO}(n+1)}{\mathrm{SO}(2) \times \mathrm{SO}(n-1)} \to \mathbb{HP}^n \quad (n \ge 2)$$
and four sporadic exceptions (in $\mathbb{HP}^6, \mathbb{HP}^9, \mathbb{HP}^{15}$, and $\mathbb{HP}^{27}$). Bedulli, Gori, and Podestà [7] proved that a maximal totally complex submanifold of $\mathbb{HP}^n$ is homogeneous if and only if it appears on Tsukada's list.
- If $Q = \mathrm{Gr}_2(\mathbb{C}^{n+2})$, the maximal totally complex submanifolds that are homogeneous have been recently classified by Tsukada [33].
- If Q is a quaternionic symmetric space, the maximal totally complex submanifolds that are totally geodesic have been classified by Takeuchi [31].

Remark 5.8 Totally complex submanifolds are also studied by Alekseevsky and Marchiafava [1, 2]. In particular, they prove the following results for almost-complex submanifolds $U^{2k} \subset Q^{4k}$:

- If $k \ge 2$ (so that $n \ge 2$), then
$$\nabla_X i = 0, \ \forall X \in TU \iff U \text{ is totally-complex} \iff \left(U, g_Q|_U, i|_U\right) \text{ is Kähler.}$$
For this reason, totally complex submanifolds U of real dimension ≥ 4 are sometimes called "Kähler submanifolds" in the literature.
- If $k = 1$ and $n \ge 2$, then the equivalence
$$\nabla_X i = 0, \ \forall X \in TU \iff U \text{ is totally complex}$$
continues to hold. By contrast, the condition that $\left(U, g_Q|_U, i|_U\right)$ be Kähler is automatic.

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- If $k = 1$ and $n = 1$, then every oriented surface $U^2 \subset Q^4$ is totally complex, and $\left(U, g_Q|_U, i|_U\right)$ is Kähler. By contrast, $\nabla_X i = 0$ for all $X \in TU$ is equivalent to U being *superminimal* (or *infinitesimally holomorphic*), a condition on the second fundamental form (see, e.g., [10, 14, 15]).

5.2.1 The horizontal lift

Given a totally complex submanifold $U^{2k} \subset Q^{4n}$, there are two natural ways to lift U to a submanifold of the twistor space Z. The first of these is the *horizontal lift* $\widetilde{U} \subset Z$, defined as the union of

$$\widetilde{U}_p := \left\{z \in Z_p : z(T_pU) = T_pU\right\}$$

for $p \in U$. The following results were proved in [31, Theorem 4.1], and later generalized in [2, Theorem 4.2 and Proposition 4.7].

Lemma 5.9 [2] *Let $U \subset Q$ be a submanifold, let $i \in \Gamma(Z|_U)$ be a section over U, and let $N = i(U) \subset Z$ be its image. Then $N \subset Z$ is J_{KE}-complex and horizontal if and only if (U, i) is almost-complex and $\nabla_V i = 0$ for all $V \in TU$.*

Proof ($\Longrightarrow$) Suppose N is J_{KE}-complex and horizontal. Fix $u \in U$, and let $z = i(u) \in N$. Let $X \in T_uU$, and write $X = \tau_*(\widetilde{X})$ for some $\widetilde{X} \in T_zN$. Since $T_zN \subset T_zZ$ is complex, we have $J_{\mathrm{KE}}\widetilde{X} \in T_zN$. Since $\widetilde{X}$ is horizontal, we may calculate $i(u)(X) = z(\tau_*\widetilde{X}) = \tau_*(J_{\mathrm{KE}}\widetilde{X}) \in \tau_*(T_zN) = T_uU$. This shows that (U, i) is almost-complex. Moreover, since $N = i(U)$ is horizontal, it follows that $\nabla_V i = 0$ for all $V \in TU$.

($\Longleftarrow$) Suppose (U, i) is almost-complex and $\nabla_V i = 0$ for all $V \in TU$. Since i is a parallel section, its image N is horizontal. Now, fix $z \in N$, write $z = i(u)$ for $u \in U$, and let $Y \in T_zN$. Since (U, i) is almost-complex, we have $i(u)(\tau_* Y) \in T_uU$. Therefore, since Y is horizontal, we have $\tau_*(J_{\mathrm{KE}} Y) = i(u)(\tau_* Y) \in T_uU = \tau_*(T_zN)$. Since $\tau_* : \mathsf{H}_z \to T_uQ$ is an isomorphism, it follows that $J_{\mathrm{KE}} Y \in T_zN$, which proves that N is J_{KE}-complex. ∎

Theorem 5.10 [2, 31] *Let $\Sigma^{2k} \subset Z^{4n+2}$ be a submanifold, where $1 \leq k \leq n$. Then Σ is J_{KE}-complex and horizontal if and only if Σ is locally of the form $\widetilde{U}$ for some totally complex $U^{2k} \subset Q^{4n}$ (resp. a superminimal surface $U^2 \subset Q^4$ if $n = 1$).*

Proof ($\Longleftarrow$) Suppose that Σ is locally of the form $\widetilde{U}$ for some totally complex $U \subset Q$ (resp. superminimal surface if $n = 1$). By definition, U is almost-complex, so there exists a section $i \in \Gamma(Z|_U)$ such that $i(TU) = TU$, and hence $\widetilde{U} = i(U) \cup -i(U)$. Moreover, by Remark 5.8, we have $\nabla_V i = 0$ for all $V \in TU$. Therefore, by Lemma 5.9, the submanifolds $i(U)$ and $-i(U)$ are J_{KE}-complex and horizontal, and hence Σ is, too.

($\Longrightarrow$) Suppose that Σ is J_{KE}-complex and horizontal. Since Σ is horizontal, the Implicit Function Theorem implies that Σ is locally of the form $i(U)$ for some horizontal section $i \in \Gamma(Z|_U)$ over some submanifold $U \subset Q$. By Lemma 5.9, (U, i) is almost-complex and $\nabla_V i = 0$. Thus, by Remark 5.8, U is totally complex (and, in addition, superminimal if $n = 1$). ∎

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5.2.2 The circle bundle lift

Let $U^{2k} \subset Q^{4n}$ be totally complex. The second natural lift of U is the *circle bundle lift* $\mathcal{L}(U) \subset Z$, defined as the union of

$$\mathcal{L}(U)|_p := \{j \in Z_p : j(T_pU) \subset (T_pU)^\perp\}$$

for $p \in U$. Each fiber $\mathcal{L}(U)|_p$ is a great circle in the 2-sphere Z_p.

The circle bundle lift was introduced by Ejiri and Tsukada [12], who proved that if $U^{2k} \subset Q^{4n}$ is totally complex and $k \geq 2$, then $\mathcal{L}(U) \subset Z$ is a minimal submanifold that is both ω_{KE}-isotropic and HV-compatible. In particular, if $\dim(U) = 2n \geq 4$, then $\mathcal{L}(U) \subset Z$ is a minimal ω_{KE}-Lagrangian. In the case of $k = n = 1$, circle bundle lifts of superminimal surfaces $U^2 \subset Q^4$ were studied by Storm [30].

We now explore these submanifolds further. Recall that if $V \in \mathsf{V}_z$ is a vertical unit vector, we let $\beta_V := \iota_V(\text{Re}(\gamma)) \in \Lambda^2(\mathsf{H}_z^*)$ denote the induced nondegenerate 2-form on H_z, and let J_V be the corresponding complex structure on H_z.

Theorem 5.11 *Let $U^{2k} \subset Q^{4n}$ be a submanifold with $1 \leq k \leq n$. If U is totally complex and $n \geq 2$, or if U is superminimal and $n = 1$, then $\mathcal{L} := \mathcal{L}(U)$ satisfies the following:*

(1) *$\mathcal{L} \subset Z$ is ω_{KE}-isotropic, ω_{NK}-isotropic, HV-compatible, and satisfies $\dim(T_z\mathcal{L} \cap \mathsf{V}) = 1$ at every $z \in \mathcal{L}$.*
(2) *For any unit vector $V \in T_z\mathcal{L} \cap \mathsf{V}$, the $2k$-plane $T_z\mathcal{L} \cap \mathsf{H}$ is J_V-invariant.*

Proof Suppose U is totally complex if $n \geq 2$, or superminimal if $n = 1$, and set $\mathcal{L} := \mathcal{L}(U)$. In either case, there exists a section $i \in \Gamma(Z|_U)$ such that $i(TU) = TU$ and $\nabla_X i = 0$ for all $X \in TU$.

(a) Following [12, Proof of Lemma 2.1], we orthogonally decompose $E|_U = \mathbb{R}i \oplus E'$. If $\sigma \in \Gamma(E')$ is a local section, then $\langle \sigma, i \rangle = 0$, so that $\langle \nabla_X \sigma, i \rangle = 0$, and thus $\nabla_X \sigma \in \Gamma(E')$. Thus, $E' \subset E|_U$ is a parallel subbundle. Since $\mathcal{L} \subset E'$ is the unit sphere subbundle, it follows that $\mathcal{L} \subset E'$ is a parallel fiber subbundle. This implies that $T\mathcal{L} = H_\Sigma \oplus V_\Sigma$ for subbundles $H_\Sigma \subset \mathsf{H}$ and $V_\Sigma \subset \mathsf{V}$, meaning that $\mathcal{L}$ is HV compatible.

We now show that $\mathcal{L}$ is ω_{KE}-isotropic and ω_{NK}-isotropic. Fix $z \in \mathcal{L}$, and recall that

$$\omega_{\text{KE}} = \omega_{\mathsf{H}} + \omega_{\mathsf{V}}, \qquad \omega_{\text{NK}} = 2\omega_{\mathsf{H}} - \omega_{\mathsf{V}}.$$

Since $\mathcal{L}$ is HV compatible and $\dim(T_z\mathcal{L} \cap \mathsf{V}) = 1$, it follows that $\omega_{\mathsf{V}}|_{\mathcal{L}} = 0$. Moreover, if $X, Y \in T_z\mathcal{L} \cap \mathsf{H}$, then

$$\omega_{\mathsf{H}}(X, Y) = g_{\text{KE}}(J_{\text{KE}}X, Y) = g_Q(\tau_* J_{\text{KE}}X, \tau_* Y) = g_Q(z(\tau_* X), \tau_* Y) = 0,$$

where in the last step we used that $z(T_{\tau(z)}U) \subset (T_{\tau(z)}U)^\perp$. This shows that $\omega_{\mathsf{H}}|_{\mathcal{L}} = 0$, and therefore $\omega_{\text{KE}}|_{\mathcal{L}} = 0$ and $\omega_{\text{NK}}|_{\mathcal{L}} = 0$.

(b) Fix $z \in \mathcal{L}$, let $u = \tau(z)$, and let $V \in T_z\mathcal{L} \cap \mathsf{V}$ be a vertical unit vector. Let $j \in \mathcal{L}|_u \cap z^\perp$ denote the point on the great circle $\mathcal{L}|_u$ that corresponds to V under the natural isomorphism $\mathsf{V}_z \simeq z^\perp$. Set $i = z \circ j$, so that (z, j, i) is an admissible frame of E_u. By (5.1), we have

$$J_V = (\tau_*|_{\mathsf{H}_z})^{-1} \circ i \circ \tau_*.$$

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Since U is totally complex, the $2k$-plane $T_uU \subset T_uQ$ is i-invariant. Therefore, if $X \in T_z\mathcal{L} \cap \mathsf{H}$, then $i(\tau_*X) \in T_uU$, so that $J_VX = (\tau_*|_{\mathsf{H}_z})^{-1}(i(\tau_*X)) \in T_z\mathcal{L} \cap \mathsf{H}$, proving that $T_z\mathcal{L} \cap \mathsf{H}$ is J_V-invariant. ∎

5.2.3 Circle bundle lifts and CR isotropic submanifolds

We now prove that circle bundle lifts $\mathcal{L}(U) \subset Z$ are intimately related to CR isotropic submanifolds $L \subset M$. Indeed, the geometric properties of $\mathcal{L}(U)$ established in Theorem 5.11 are precisely those needed for its p_1-horizontal lift to be CR isotropic. That is:

Corollary 5.12 *Let $U^{2k} \subset Q^{4n}$ be a submanifold with $1 \le k \le n$. If U is totally complex and $n \ge 2$, or if U is superminimal and $n = 1$, then $\mathcal{L}(U) \subset Z$ admits local p_1-horizontal lifts to M, and every such lift is $(c_\theta I_2 + s_\theta I_3)$-CR isotropic for some $e^{i\theta} \in S^1$.*

Proof This follows from Theorem 5.11 and Proposition 4.26(b). ∎

We now aim to establish a converse in the case where L is compact. For this, we need a technical lemma.

Lemma 5.13 *Let $\Sigma^k \subset Z^{4n+2}$ be a compact submanifold. If Σ is ω_{KE}-isotropic, HV-compatible, and $\dim(T_z\Sigma \cap \mathsf{V}) = 1$ for all $z \in \Sigma$, then $U := \tau(\Sigma) \subset Q^{4n}$ is a $(k-1)$-dimensional submanifold, and $\tau|_\Sigma : \Sigma \to U$ is an S^1-bundle whose fibers are geodesics in Z with respect to the Kähler–Einstein metric.*

Proof Since Σ is HV-compatible and $\dim(T_z\Sigma \cap \mathsf{V}) = 1$ for all $z \in \Sigma$, it follows that $\dim(T_z\Sigma \cap \mathsf{H}) = k - 1$. Therefore, the map $\tau|_\Sigma : \Sigma \to Q$ has constant rank $k-1$. By the Constant Rank Theorem, each fiber $\tau|_\Sigma^{-1}(\tau(z)) \subset \Sigma$ is an embedded 1-manifold, and therefore (since Σ is compact) is an at most countable union of disjoint circles.

We claim that each S^1-fiber is a geodesic. For this, note that since Σ is ω_{KE}-isotropic, it admits local p_1-horizontal lifts to M. Let $L \subset M$ be such a lift. Since Σ is HV-compatible and $p_1 : M \to Z$ respects the horizontal–vertical splitting, we may write $TL = H_L \oplus \mathbb{R}\widetilde{V}$, where $H_L \subset \widetilde{\mathsf{H}}$ and $\widetilde{V} \in \widetilde{\mathsf{V}}$. Moreover, Proposition 4.26(a) implies that L is α_1-isotropic and $(-s_\theta\alpha_2 + c_\theta\alpha_3)$-isotropic for some constant $e^{i\theta} \in S^1$. Therefore, $\widetilde{V} = c_\theta A_2 + s_\theta A_3$ is a Reeb vector field, so its integral curves are geodesics in M. Consequently, the integral curves of $(p_1)_*(\widetilde{V}) \in \mathsf{V} \subset TZ$ are geodesics in Z (and hence geodesics in L), and these are precisely the S^1-fibers $\tau|_\Sigma^{-1}(\tau(z)) \subset \Sigma$.

Consequently, since Σ is compact, each S^1-fiber $\tau|_\Sigma^{-1}(\tau(z)) \subset \Sigma$ is an at most countable union of disjoint *great* circles in the twistor 2-sphere. Since any two great circles in a round 2-sphere intersect, it follows that each S^1-fiber consists of a single great circle.

It remains to show that $U := \tau(\Sigma)$ is a $(k-1)$-dimensional submanifold of Q. For this, note that since Σ is a union of great circles, each of which is the p_1-image of a Reeb circle in M, it admits a free S^1-action. (The action is free because we are working on the regular part of M.) Therefore, the quotient Σ/S^1 admits the structure of smooth $(k-1)$-manifold, and the projection $\pi: \Sigma \to \Sigma/S^1$ is a smooth quotient map.

Now, let $\widehat{\tau}: \Sigma \to U$ denote the map $\tau|_\Sigma$ with restricted codomain, equip $U \subset Q$ with the subspace topology, and let $\iota: U \hookrightarrow Q$ be the inclusion map. If $V \subset U$ is open, then $V = U \cap W$ for some open set $W \subset Q$, and hence $\widehat{\tau}^{-1}(V) = \Sigma \cap \tau^{-1}(W)$ is open

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subset of Σ, which proves that $\widehat{\tau}$ is continuous. Since $\widehat{\tau}$ is a continuous surjection from a compact domain, it follows that $\widehat{\tau}$ is a quotient map. Since π and $\widehat{\tau}$ are quotient maps that are constant on each other's fibers, there exists a unique homeomorphism $F\colon \Sigma/S^1 \to U$ such that $\widehat{\tau} = F \circ \pi$. Choosing a smooth local section $\sigma\colon Y \to \Sigma$ of π, where $Y \subset \Sigma/S^1$ is an open set, we observe that $\tau|_\Sigma \circ \sigma\colon Y \to Q$ is a smooth map of rank $k-1$, which implies that $\iota \circ F\colon \Sigma/S^1 \to Q$ is also a smooth map of rank $k-1$, and therefore a smooth embedding whose image is U. ■

The converse to Corollary 5.12 is now given by the following.

Theorem 5.14 *Let $L^{2k+1} \subset M^{4n+3}$ be a compact submanifold, $1 \le k \le n$. If L is $(c_\theta I_2 + s_\theta I_3)$-CR isotropic for some $e^{i\theta} \in S^1$, and if $p_1(L) \subset Z$ is embedded, then $p_1(L) = \mathcal{L}(U)$ for some totally complex submanifold $U^{2k} \subset Q^{4n}$ (resp. a superminimal surface $U^2 \subset Q^4$ if $n = 1$).*

Proof Suppose that $L \subset M$ is a compact $(c_\theta I_2 + s_\theta I_3)$-CR isotropic $(2k+1)$-fold for some constant $e^{i\theta} \in S^1$ and that $\Sigma := p_1(L) \subset Z$ is embedded. By Proposition 4.25(b), Σ is ω_{KE}-isotropic, ω_{NK}-isotropic, HV-compatible, and $\dim(T_z\Sigma \cap \mathsf{V}) = 1$ for all $z \in \Sigma$. Therefore, Lemma 5.13 implies that $U := \tau(\Sigma) \subset Q$ is a $2k$-dimensional submanifold, and $\tau|_\Sigma : \Sigma \to U$ is an S^1-bundle with geodesic fibers.

Fix $z \in \Sigma$ and let $u = \tau(z)$. Since Σ is HV compatible and $\dim(T_z\Sigma \cap \mathsf{V}) = 1$, we can orthogonally split

$$T_z\Sigma = H_\Sigma \oplus \mathbb{R}V,$$

where $V \in \mathsf{V}_z$ is a vertical unit vector, and $H_\Sigma \subset \mathsf{H}_z$ is $2k$-dimensional. On H_z, let $\beta_V := \iota_V(\mathrm{Re}\,\gamma)$ denote the induced nondegenerate 2-form, and let J_V denote the corresponding complex structure. By Proposition 4.25(b), the $2k$-plane $H_\Sigma \subset \mathsf{H}_z$ is J_V-invariant.

Now, the S^1-fiber $\tau|_\Sigma^{-1}(u) \subset \Sigma$ is a great circle through z in the 2-sphere $Z_u = \tau^{-1}(u)$. Let $j \in \tau|_\Sigma^{-1}(u) \cap z^\perp$ be the point on this circle that corresponds to V under the natural isomorphism $\mathsf{V}_z \simeq z^\perp$. Setting $i = z \circ j$, we see that (z, j, i) is an admissible frame of E_u, which is the fiber over u of the bundle E from Definition 5.1. See Figure 1. We also have $\tau|_\Sigma^{-1}(u) = \{k \in Z_u \colon \langle k, i\rangle = 0\}$, and

$$J_V = (\tau_*|_{\mathsf{H}_z})^{-1} \circ i \circ \tau_*.$$

In particular, the J_V-invariance of the $2k$-plane $H_\Sigma \subset \mathsf{H}_z$ implies that $T_uU \subset T_uQ$ is i-invariant.

Now, let $X_1, X_2 \in T_uU$, and let $\widetilde{X}_j = (\tau_*|_{H_\Sigma})^{-1}(X_j) \in H_\Sigma$. Since Σ is ω_{KE}-isotropic, and $\omega_{\mathrm{KE}} = f^2 \wedge f^3 + \beta_1$, it follows that the $2k$-plane H_Σ is β_1-isotropic. Therefore,

$$g_Q(zX_1, X_2) = g_Q(\tau_*(J_1\widetilde{X}_1), \tau_*\widetilde{X}_2) = g_{\mathrm{KE}}(J_1\widetilde{X}_1, \widetilde{X}_2) = \beta_1(\widetilde{X}_1, \widetilde{X}_2) = 0,$$

which shows that $z(T_uU) \subset (T_uU)^\perp$. Finally, if $X \in T_uU$, then $iX \in T_uU$, so $jX = -z(iX) \in (T_uU)^\perp$, demonstrating that $j(T_uU) \subset (T_uU)^\perp$. This proves that U is totally complex and that

$$\tau|_\Sigma^{-1}(u) = \{k \in Z_u \colon \langle k, i\rangle = 0\} = \{k \in Z_u \colon k(T_uU) \subset (T_uU)^\perp\} = \mathcal{L}(U)|_u.$$

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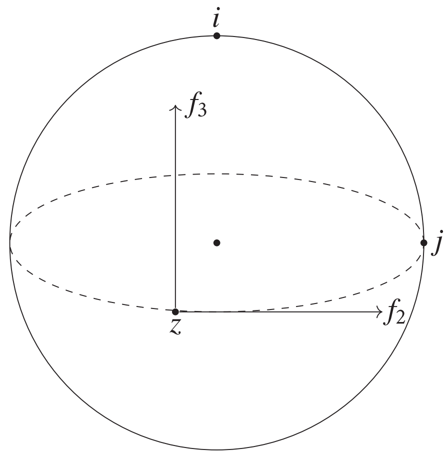

Figure 1: The admissible frame (z, j, i) of E_u.

Finally, suppose that $n = 1$, so that $k = 1$. Then $\Sigma^3 = \mathcal{L}(U)$ is ω_{KE}-Lagrangian and ω_{NK}-Lagrangian. By a result of Storm [30], the surface $U \subset Q^4$ is superminimal. ■

Remark 5.15 If U is an embedded submanifold of Q, then its geodesic circle bundle is embedded in Z. Therefore, in order to characterize those submanifolds Σ of Z which are geodesic circle bundles in Z, we need to assume a priori that Σ is embedded.

5.2.4 Applications

In previous sections, we considered $\mathrm{Re}(\gamma)$-calibrated 3-folds $\Sigma^3 \subset Z$ that are ω_{KE}-isotropic, describing their p_1-horizontal lifts $L^3 \subset M^{4n+3}$ (Theorem 4.31). Now, we are in a position to classify such 3-folds in Z as circle bundle lifts of totally complex surfaces in Q.

Theorem 5.16

(1) *If $U^2 \subset Q^{4n}$ is totally complex and $n \geq 2$, or if U is superminimal and $n = 1$, then $\mathcal{L}(U) \subset Z$ is $\mathrm{Re}(\gamma)$-calibrated and ω_{KE}-isotropic.*

(2) *Conversely, if $\Sigma^3 \subset Z^{4n+2}$ is a compact three-dimensional submanifold that is $\mathrm{Re}(\gamma)$-calibrated and ω_{KE}-isotropic, then $\Sigma = \mathcal{L}(U)$ for some totally complex surface $U^2 \subset Q^{4n}$. Moreover, if $n = 1$, then U is superminimal.*

Proof (a) Let $U^2 \subset Q^{4n}$ be totally complex if $n \geq 2$, or superminimal if $n = 1$. By Theorem 5.11(a), the 3-fold $\mathcal{L}(U) \subset Z$ is ω_{KE}-isotropic. Fix $z \in L$, and let $L \subset M$ denote a p_1-horizontal lift of a neighborhood of z. By Corollary 5.12, L is $(c_\theta I_2 + s_\theta I_3)$-CR isotropic. Therefore, by Theorem 4.31((ii) $\Longrightarrow$ (iv)), $p_1(L) \subset \mathcal{L}(U)$ is $\mathrm{Re}(\gamma)$-calibrated.

(b) Suppose $\Sigma^3 \subset Z$ is a compact three-dimensional submanifold that is $\mathrm{Re}(\gamma)$-calibrated and ω_{KE}-isotropic. By Proposition 4.15(c), Σ is HV-compatible and $\dim(T_z\Sigma \cap \mathsf{V}) = 1$ for all $z \in \Sigma$. Therefore, Lemma 5.13 implies that $U^2 = \tau(\Sigma) \subset Q$ is a two-dimensional surface and that $\tau|_\Sigma : \Sigma \to U$ is an S^1-bundle with geodesic fibers.

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Fix $z \in \Sigma$, and let $u = \tau(z)$. We may write $T_z\Sigma = H_\Sigma \oplus V_\Sigma$ for some 2-plane $H_\Sigma \subset \mathsf{H}$ and line $V_\Sigma \subset \mathsf{V}$. Let $(e_{10}, \ldots, e_{n3}, f_2, f_3)$ be an $\mathrm{Sp}(n)\mathrm{U}(1)$-frame at z, with dual coframe $(\rho_{10}, \ldots, \rho_{n3}, \mu_2, \mu_3)$, such that

$$V_\Sigma = \mathrm{span}(f_2), \qquad \mathrm{vol}_{V_\Sigma} = \mu_2. \tag{5.3}$$

Let $(\beta_1, \beta_2, \beta_3) = (\omega_{\mathsf{H}}, \iota_{f_2}(\mathrm{Re}\,\gamma), \iota_{f_3}(\mathrm{Re}\,\gamma))$ denote the induced hyperkähler triple on H_z, and let (J_1, J_2, J_3) be the corresponding complex structures on H_z.

Now, the S^1-fiber $\tau|_\Sigma^{-1}(u) \subset \Sigma$ is a great circle through z in the twistor 2-sphere Z_u. Let $j \in \tau|_\Sigma^{-1}(u) \cap z^\perp$ be the point on this circle that corresponds to V under the natural isomorphism $\mathsf{V}_z \simeq z^\perp$. Setting $i = z \circ j$, we see that (z, j, i) is an admissible frame of E_u (see Figure 1), that $\tau|_\Sigma^{-1}(u) = \{k \in Z_u : \langle k, i\rangle = 0\}$, and moreover,

$$J_2 = (\tau_*|_{\mathsf{H}_z})^{-1} \circ i \circ \tau_* \qquad J_3 = -(\tau_*|_{\mathsf{H}_z})^{-1} \circ j \circ \tau_*.$$

Using (5.3), we compute

$$\begin{aligned} \mu_2|_{V_\Sigma} \wedge \mathrm{vol}_{H_\Sigma} = \mathrm{vol}_{T_z\Sigma} = \mathrm{Re}(\gamma)|_{T_z\Sigma} &= (\mu_2 \wedge \beta_2 + \mu_3 \wedge \beta_3)|_{T_z\Sigma} \\ &= \mu_2|_{V_\Sigma} \wedge \beta_2|_{H_\Sigma} + \mu_3|_{V_\Sigma} \wedge \beta_3|_{H_\Sigma} \\ &= \mu_2|_{V_\Sigma} \wedge \beta_2|_{H_\Sigma}. \end{aligned}$$

Contracting with f_2 gives $\beta_2|_{H_\Sigma} = \mathrm{vol}_{H_\Sigma}$, which implies that the real 2-plane $H_\Sigma \subset \mathsf{H}_z$ is J_2-invariant. Consequently, $T_uU \subset T_uQ$ is i-invariant.

Repeating the argument at the end of the proof of Theorem 5.14, we observe that $z(T_uU) \subset (T_uU)^\perp$ and $j(T_uU) \subset (T_uU)^\perp$. This proves that U is totally complex and that

$$\tau|_\Sigma^{-1}(u) = \{k \in Z_u : \langle k, i\rangle = 0\} = \{k \in Z_u : k(T_uU) \subset (T_uU)^\perp\} = \mathcal{L}(U)|_u.$$

Finally, suppose that $n = 1$. Since $\Sigma^3 = \mathcal{L}(U) \subset Z^6$ is $\mathrm{Re}(\gamma)$-calibrated, it follows from Proposition 4.16 that Σ is ω_{NK}-Lagrangian. Thus, $\mathcal{L}(U)$ is both ω_{KE}-Lagrangian and ω_{NK}-Lagrangian, so the superminimality of $U^2 \subset Q^4$ follows from Storm's theorem [30]. ■

We can now classify the compact submanifolds of Z that are Lagrangian with respect to both ω_{KE} and ω_{NK}.

Theorem 5.17

(1) *If $U^{2n} \subset Q^{4n}$ is totally complex and $n \geq 2$, or if U is superminimal and $n = 1$, then $\mathcal{L}(U) \subset Z$ is ω_{KE}-Lagrangian and ω_{NK}-Lagrangian.*
(2) *Conversely, if $\Sigma^{2n+1} \subset Z^{4n+2}$ is a compact $(2n+1)$-dimensional submanifold that is both ω_{KE}-Lagrangian and ω_{NK}-Lagrangian, then $\Sigma = \mathcal{L}(U)$ for some (maximal) totally complex $2n$-fold $U^{2n} \subset Q^{4n}$.*

Proof (a) This follows from Theorem 5.11(a).

(b) Suppose $\Sigma^{2n+1} \subset Z$ is a compact submanifold that is both ω_{KE}-Lagrangian and ω_{NK}-Lagrangian. By Proposition 4.15(b), Σ is HV compatible, $\dim(T_z\Sigma \cap \mathsf{H}) = 2n$, and $\dim(T_z\Sigma \cap \mathsf{V}) = 1$ for all $z \in \Sigma$. By Lemma 5.13, $U := \tau(\Sigma) \subset Q$ is a $2n$-dimensional submanifold, and $\tau|_\Sigma : \Sigma \to U$ is an S^1-bundle with geodesic fibers.

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It remains to prove that U is totally complex and that $\tau|_{\Sigma}^{-1}(u) = \mathcal{L}(U)|_u$. For this, note that Corollary 4.27(b) implies that every local p_1-horizontal lift of Σ is $(c_\theta I_2 + s_\theta I_3)$-CR Legendrian for some $e^{i\theta} \in S^1$. The proof now follows exactly as in Theorem 5.14. ∎

6 Characterizations of complex Lagrangian cones

In a hyperkähler cone C^{4n+4}, recall that a $(2k+2)$-dimensional cone $\mathrm{C}(L)$ is $(c_\theta I_2 + s_\theta I_3)$-complex isotropic provided that it satisfies the following three conditions:

$$\omega_1|_{\mathrm{C}(L)} = 0, \qquad (-s_\theta\omega_2 + c_\theta\omega_3)|_{\mathrm{C}(L)} = 0, \qquad (c_\theta I_2 + s_\theta I_3)\text{-complex.}$$

As discussed in Section 3.3, this is equivalent to requiring that the $(2k+1)$-dimensional link L be $(c_\theta I_2 + s_\theta I_3)$-CR isotropic, meaning

$$\alpha_1|_L = 0, \qquad (-s_\theta\alpha_2 + c_\theta\alpha_3)|_L = 0, \qquad (c_\theta I_2 + s_\theta I_3)\text{-CR.}$$

In this short section, we characterize complex isotropic cones $\mathrm{C}(L)^{2k+2} \subset C^{4n+4}$, $1 \leq k \leq n$, in terms of related geometries in M^{4n+3}, Z^{4n+2}, and Q^{4n}.

To begin, we generalize a result of Ejiri and Tsukada [13] – originally established for complex Lagrangian cones (i.e., $k = n$) in the flat model $C^{4n+4} = \mathbb{H}^{n+1}$ – to complex isotropic cones of any dimension $2k+2$ in arbitrary hyperkähler cones C^{4n+4}.

Theorem 6.1 *Let $L^{2k+1} \subset M^{4n+3}$, where $3 \leq 2k+1 \leq 2n+1$. The following conditions are equivalent:*

(1) *$\mathrm{C}(L)$ is $(c_\theta I_2 + s_\theta I_3)$-complex isotropic for some constant $e^{i\theta} \in S^1$.*
(2) *L is $(c_\theta I_2 + s_\theta I_3)$-CR isotropic for some constant $e^{i\theta} \in S^1$.*
(3) *L is locally of the form $p_v^{-1}(V)$ for some horizontal J_{KE}-complex submanifold $V^{2k} \subset Z$ and some $v = (0, c_\theta, s_\theta)$.*
(4) *L is locally of the form $p_v^{-1}(\widetilde{U})$ for some totally complex submanifold $U^{2k} \subset Q$ (resp. superminimal surface if $n = 1$) and some $v = (0, c_\theta, s_\theta)$.*

If, in addition, L is compact and $p_1(L) \subset Z$ is embedded, then the above conditions are equivalent to:

(⋆) *L is a p_1-horizontal lift of $\mathcal{L}(U) \subset Z$ for some totally complex submanifold $U^{2k} \subset Q^{4n}$ (resp. superminimal surface $U^2 \subset Q^4$ if $n = 1$).*

Proof The equivalence (1) $\Longleftrightarrow$ (2) is Proposition 3.8. The equivalence (2) $\Longleftrightarrow$ (3) is Corollary 4.21. The equivalence (3) $\Longleftrightarrow$ (4) follows from Theorem 5.10.
(⋆) $\Longrightarrow$ (2). This is Corollary 5.12.
(2) $\Longrightarrow$ (⋆). This is Theorem 5.14. ∎

Therefore, given a $(c_\theta I_2 + s_\theta I_3)$-complex isotropic cone $\mathrm{C}(L) \subset C$, its link $L \subset M$ can be viewed in two ways. On the one hand, L is a $p_{(1,0,0)}$-horizontal lift of a circle bundle over a totally complex submanifold $U \subset Q$. On the other hand, L is also a $p_{(0,c_\theta,s_\theta)}$-circle bundle over a τ-horizontal lift of a totally complex submanifold $U \subset Q$.

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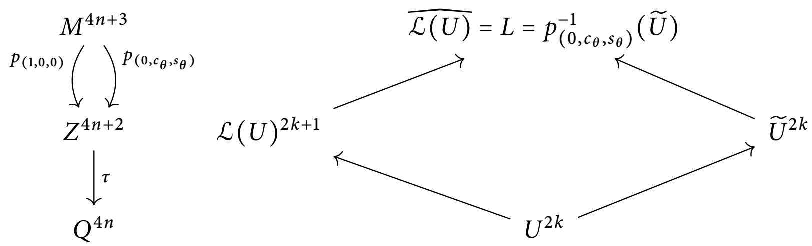
Thus, loosely speaking, the operations of "horizontal lift" and "circle bundle lift" form a commutative diagram of sorts:



For complex Lagrangian cones in C^{4n+4}, we are able to say more.

Theorem 6.2 *Let $L^{2n+1} \subset M^{4n+3}$ be a $(2n+1)$-dimensional submanifold. The following five conditions are equivalent:*

(1) $\mathrm{C}(L)$ *is $(c_\theta I_2 + s_\theta I_3)$-complex Lagrangian for some constant $e^{i\theta} \in S^1$.*
(2) *L is $(c_\theta I_2 + s_\theta I_3)$-CR Legendrian for some constant $e^{i\theta} \in S^1$.*
(3) *L is locally of the form $p_v^{-1}(V)$ for some horizontal J_{KE}-complex submanifold $V^{2n} \subset Z$ and some $v = (0, c_\theta, s_\theta)$.*
(4) *L is locally of the form $p_v^{-1}(\widetilde{U})$ for some totally complex submanifold $U^{2n} \subset Q$ (resp. superminimal surface if $n = 1$) and some $v = (0, c_\theta, s_\theta)$.*
(5) *L is locally a p_1-horizontal lift of a $(2n+1)$-fold $\Sigma^{2n+1} \subset Z$ that is ω_{KE}-Lagrangian and ω_{NK}-Lagrangian.*

If, in addition, L is compact and $p_1(L) \subset Z$ is embedded, then the above conditions are equivalent to:

($\star$) *L is a p_1-horizontal lift of $\mathcal{L}(U) \subset Z$ for some totally complex submanifold $U^{2n} \subset Q^{4n}$ (resp. superminimal surface $U^2 \subset Q^4$ if $n = 1$).*

Proof The equivalence (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ ($\star$) was proven in Theorem 6.1. It remains only to involve condition (5). For this, note that (5) $\Longleftrightarrow$ (2) is the content of Corollary 4.27. Alternatively, (5) $\Longleftrightarrow$ ($\star$) is Theorem 5.17. ∎

Finally, for four-dimensional complex isotropic cones in C^{4n+4}, even more characterizations are available:

Theorem 6.3 *Let $L^3 \subset M^{4n+3}$ be a three-dimensional submanifold. The following six conditions are equivalent:*

(1) $\mathrm{C}(L)$ *is $(c_\theta I_2 + s_\theta I_3)$-complex isotropic for some constant $e^{i\theta} \in S^1$.*
(2) *L is $(c_\theta I_2 + s_\theta I_3)$-CR isotropic for some constant $e^{i\theta} \in S^1$.*
(3) *L is locally of the form $p_v^{-1}(V)$ for some horizontal J_{KE}-complex submanifold $V^2 \subset Z$ and some $v = (0, c_\theta, s_\theta)$.*
(4) *L is locally of the form $p_v^{-1}(\widetilde{U})$ for some totally complex submanifold $U^2 \subset Q$ (resp. superminimal surface if $n = 1$) and some $v = (0, c_\theta, s_\theta)$.*
(5) *L is locally a p_1-horizontal lift of a* $\mathrm{Re}(\gamma)$*-calibrated 3-fold that is ω_{KE}-isotropic.*
(6) *L is* $\mathrm{Re}(\Gamma_1)$*-calibrated.*

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If, in addition, L is compact and $p_1(L) \subset Z$ is embedded, then the above conditions are equivalent to:

(1) L is a p_1-horizontal lift of $\mathcal{L}(U) \subset Z$ for some totally complex submanifold $U^2 \subset Q^{4n}$ (resp. superminimal surface $U^2 \subset Q^4$ if $n = 1$).

Proof Theorem 4.31 gives (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6). Now, as Theorem 6.1 proves (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ ($\star$), we deduce the result. Alternatively, Theorem 5.10 gives (3) $\Longleftrightarrow$ (4), and Theorem 5.16 gives (5) $\Longleftrightarrow$ ($\star$). ■

A Appendix

A.1 Linear algebra of calibrations

Let (V, g) be an n-dimensional oriented real inner product space. Recall that a k-form γ on V is said to have *comass one* if $\gamma(P) \leq 1$ for any oriented orthonormal k-plane P in V, with equality on at least one such P. Equivalently, by writing $P = e_1 \wedge \cdots \wedge e_k$, this means that

$$\gamma(e_1, \ldots, e_k) \leq 1$$

whenever $e_1, \ldots, e_k$ are orthonormal in V, with equality on at least one such set. Throughout this paper, a k-form with comass one will be called a *semi-calibration*. Let $\gamma \in \Lambda^k(V^*)$ be a semi-calibration. An oriented k-plane P is called γ-calibrated if $\gamma(P) = 1$.

It is easy to see that $\gamma \in \Lambda^k(V^*)$ is a semi-calibration if and only if $*\gamma \in \Lambda^{n-k}(V^*)$ is a semi-calibration, where $*$ is the Hodge star operator induced by the inner product and orientation on V. We collect here some results on semi-calibrations that we will need.

Proposition A.1 *Let $\gamma \in \Lambda^k(V^*)$, be a semi-calibration, and let $L \subset V$ be an oriented one-dimensional subspace with oriented orthonormal basis $\{e_1\}$. Write $V = L \oplus L^\perp$, and*

$$\gamma = e_1^\flat \wedge \alpha + \beta,$$

where $\alpha = \iota_{e_1}\gamma \in \Lambda^{k-1}(L^\perp)^*$ and $\beta = \gamma - e_1^\flat \wedge \alpha \in \Lambda^k(L^\perp)^*$.

(1) *If every oriented line in V lies in a γ-calibrated k-plane, then α is a semi-calibration.*
(2) *Suppose (a) holds. Then an oriented $(k-1)$-plane W in $L^\perp$ is α-calibrated if and only if the oriented k-plane $P = L \oplus W$ is γ-calibrated.*
(3) *If every oriented line in V lies in a $(*\gamma)$-calibrated $(n-k)$-plane, then β is a semi-calibration.*

Proof Let W be an oriented $(k-1)$-plane in $L^\perp$, where $W = e_2 \wedge \cdots \wedge e_k$ for some oriented orthonormal bases $e_2, \ldots, e_k$ of W. Then

$$\alpha(W) = \alpha(e_2, \ldots, e_k) = \gamma(e_1, e_2, \ldots, e_k) = \gamma(L \oplus W). \tag{A.1}$$

Since $\gamma(L \oplus W) \leq 1$, the comass of α is at most 1. By hypothesis, there exists a γ-calibrated k-plane P containing L. Let W be the unique oriented $(k-1)$-plane in $L^\perp$

that $P = L \oplus W$. Then $\alpha(W) = \gamma(L \oplus W) = \gamma(P) = 1$, so α is a semi-calibration. This proves (a), and then (b) is immediate from (A.1). For (c), observe that

$$*\gamma = *(e_1^\flat \wedge \alpha + \beta) = *_{L^\perp}\alpha + (-1)^k e_1^\flat \wedge *_{L^\perp}\beta.$$

If every oriented line L lies in a $(*\gamma)$-calibrated $(n-k)$-plane, then (a) holds for $*\gamma$, so $\iota_{e_1}(*\gamma) = (-1)^k *_{L^\perp} \beta$ is a semi-calibration on $L^\perp$, but then so is β. ∎

Proposition A.2 *Let γ be a semi-calibration on V, and suppose we have an orthogonal splitting $V = L \oplus L^\perp$ for some oriented line L, with oriented orthonormal basis $\{e_1\}$. If $\iota_{e_1}\gamma = 0$, then any γ-calibrated k-plane lies in $L^\perp$.*

Proof It is trivial that $\dim(P \cap L^\perp) \geq k-1$. Therefore, we can find an oriented orthonormal basis $v_1, w_2, \ldots, w_k$ of P such that $v_1 = \cos(\theta)e_1 + \sin(\theta)w_1$ and $w_1, \ldots, w_k \in L^\perp$ orthonormal. Then since $\iota_{e_1}\gamma = 0$, we have

$$1 = \gamma(v_1, w_2, \ldots, w_k) = \sin(\theta)\,\gamma(w_1, w_2, \ldots, w_k) \leq \sin(\theta).$$

Thus, $\sin(\theta) = 1$, and $v_1 = w_1 \in P$. ∎

Proposition A.3 *Let (W, g) be a finite-dimensional real inner product space, and suppose we have an orthogonal splitting $W = H \oplus V$, so that the inner product is given by $g = g_H + g_V$. Define a new inner product $\tilde{g}$ on V by $\tilde{g} = t^2 g_H + g_V$. Let γ be a semi-calibration on V such that $\gamma \in \Lambda^m(H^*) \otimes \Lambda^{k-m}(V^*)$. Then $t^m\gamma$ is a semi-calibration on $(W, \tilde{g})$.*

Proof Let $\tilde{e}_1, \ldots, \tilde{e}_k$ be orthonormal for $\tilde{g}$. We can decompose $\tilde{e}_j = h_j + v_j$ where $h_j \in H$ and $v_j \in V$, so

$$\delta_{ij} = \tilde{g}(e_i, e_j) = t^2 g(h_i, h_j) + g(v_i, v_j).$$

Thus, if we define $e_j = th_j + v_j$, then $e_1, \ldots, e_k$ are orthonormal for g. Using the fact that $\gamma \in \Lambda^m(H^*) \otimes \Lambda^{k-m}(V^*)$, we have

$$(t^m\gamma)(\tilde{e}_1, \ldots, \tilde{e}_k) = t^m\gamma(h_1 + v_1, \ldots, h_k + v_k)$$

is a sum of terms, each of which has exactly m of the h_j's and $k-m$ of the v_j's in the argument of $t^m\gamma$. By multilinearity, we can bring one factor of t in to each of the h_j arguments, to get

$$\gamma(th_1 + v_1, \ldots, th_k + v_k) = \gamma(e_1, \ldots, e_k) \leq 1.$$

Thus, $t^m\gamma$ has comass at most one with respect to $\tilde{g}$. But now it is clear that if $P = e_1 \wedge \cdots \wedge e_k$ is γ-calibrated with respect to g, then $\tilde{P} = \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_k$ is $t^m\gamma$-calibrated with respect to $\tilde{g}$, where $\tilde{e}_j = t^{-1}h_j + v_j$ if $e_j = h_j + v_j$. ∎

Proposition A.4 *Let (V, g) and (W, h) be finite-dimensional real inner product spaces, and let $p\colon V \to W$ be a Riemannian submersion. That is, p is a linear surjection that maps $(\operatorname{Ker} p)^\perp \subset V$ isometrically onto W. If $\alpha \in \Lambda^k(W^*)$ is a semi-calibration on (W, h), then $p^*\alpha$ is a semi-calibration on (V, g).*

Proof Let $v_1, \ldots, v_k$ be orthonormal vectors in V. We can orthogonally decompose $v_j = u_j + w_j$ where $u_j \in \operatorname{Ker} p$ and $w_j \in (\operatorname{Ker} p)^\perp$. Using that α is a semi-calibration,

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$p\colon ((\operatorname{Ker} p)^\perp, g) \to (W, h)$ is an isometry, and Hadamard's inequality, we have

$$\begin{aligned}(p^*\alpha)(v_1,\dots,v_k) &= (p^*\alpha)(u_1+w_1,\dots,u_k+w_k) = \alpha(p(w_1),\dots,p(w_k)) \\ &\le |p(w_1)\wedge\cdots\wedge p(w_k)| \le |p(w_1)|\cdots|p(w_k)| = |w_1|\cdots|w_k| \le 1.\end{aligned}$$

Thus, the comass of $p^*\alpha$ is at most one. Let $L \subset W$ be an oriented k-plane calibrated by α, with oriented orthonormal basis $e_1,\dots,e_k$. For $1 \le j \le k$, let w_j be the unique vector in $(\operatorname{Ker} p)^\perp$ such that $p(w_j) = e_j$. Then it is clear that $w_1 \wedge \cdots \wedge w_k \subset V$ is calibrated by $p^*\alpha$. ■

Proposition A.5 *Let (V, g, ω, I) be a Hermitian vector space of real dimension $2n$, where I is the complex structure and $\omega = g(I\cdot,\cdot)$ is the associated real $(1,1)$-form. Let $\gamma \in \Lambda^k(V^*)$ be of type $(k,0)+(0,k)$, where $k \le n$. If $P \subset V$ is an oriented k-plane on which γ attains its maximum, then P is ω-isotropic. That is, $\omega|_P = 0$.*

Proof Let $P \subset V$ be an oriented k-plane, and write $k = 2m+1$ if k is odd, and $k = 2m$ if k is even. By [19, Lemma 7.18], which actually works for any k, there exists an orthonormal basis $(e_1, Ie_1, \dots, e_n, Ie_n)$ of V and constants $\theta_1,\dots,\theta_m \in [0, 2\pi)$ such that

$$\begin{aligned}P &= e_1 \wedge (\sin(\theta_1)Ie_1 + \cos(\theta_1)e_2) \wedge \cdots \wedge (\sin(\theta_m)Ie_{2m-1} + \cos(\theta_m)e_{2m}) \wedge e_{2m+1} \\ &\quad (\text{for } k = 2m+1), \\ P &= e_1 \wedge (\sin(\theta_1)Ie_1 + \cos(\theta_1)e_2) \wedge \cdots \wedge (\sin(\theta_m)Ie_{2m-1} + \cos(\theta_m)e_{2m}) \\ &\quad (\text{for } k = 2m).\end{aligned}$$

Since γ is of type $(k,0)+(0,k)$, we have $\iota_{e_i}(\iota_{Ie_i}\gamma) = 0$. Therefore, we have

$$\gamma(P) = \cos(\theta_1)\cdots\cos(\theta_m)\,\gamma(e_1,\dots,e_k).$$

Since γ attains its maximum at P, it follows that $\theta_1 = \theta_2 = \cdots = \theta_m = 0$. Therefore, $P = e_1 \wedge \cdots \wedge e_k$. In particular, if $v \in P$, then $Iv \in P^\perp$. Hence, P is ω-isotropic. ■

Theorem A.6 *Let $(V, g, \omega_1, \omega_2, \omega_3, I_1, I_2, I_3)$ be a quaternionic-Hermitian vector space of real dimension $4n$, where $\omega_p = g(I_p\cdot,\cdot)$ is the associated real 2-form of I_p-type $(1,1)$. Let $\sigma = \omega_2 + i\omega_3$. It is easy to check that σ is of I_1-type $(2,0)$. Let $\Theta_{2k} = \operatorname{Re}(\frac{1}{k!}\sigma^k) \in \Lambda^{2k}(V^*)$. Then Θ_{2k} has comass one.*

Proof We prove this by induction on k, for any n. The case $k = 1$ is clear, because then $\Theta_2 = \omega_2$. Note also that if $\Theta_{2k} = \operatorname{Re}(\frac{1}{k!}\sigma^k)$ has comass one, then so does $\operatorname{Re}(e^{-i\theta}\frac{1}{k!}\sigma^k)$ for any $e^{i\theta} \in S^1$, since this just corresponds to rotating the complex structures I_2, I_3 by θ, and thus again corresponds to a quaternionic-Hermitian structure. Thus, we can assume that $k \ge 2$ and that both $\operatorname{Re}(\frac{1}{(k-1)!}\sigma^{k-1})$ and $\operatorname{Im}(\frac{1}{(k-1)!}\sigma^{k-1})$ have comass one for any quaternionic dimension n.

Let P be an oriented $2k$-plane on which Θ_{2k} attains its maximum. Since Θ_{2k} is of I_1-type $(2k,0)+(0,2k)$, we can apply Proposition A.5 to deduce that P is I_1-isotropic. In particular, P does not contain any I_1-complex lines. Let e_1 be a unit vector in P. Complete e_1 to a quaternionic orthonormal basis

$$\{e_1, I_1e_1, I_2e_1, I_3e_1, \dots, e_n, I_1e_n, I_2e_n, I_3e_n\},$$

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so that

$$\omega_1 = \sum_{j=1}^{n} (e_j \wedge I_1 e_j + I_2 e_j \wedge I_3 e_j),$$

and similarly for ω_2, ω_3 by cyclically permuting 1, 2, 3 above. In particular, we have

$$\iota_{e_1} \sigma = I_2 e_1 + i I_3 e_1. \tag{A.2}$$

Write $P = e_1 \wedge Q$ for an oriented $(2k-1)$-plane, so

$$\Theta_{2k}(P) = \Theta_{2k}(e_1 \wedge Q) = (\iota_{e_1} \Theta_{2k})(Q). \tag{A.3}$$

Moreover, we have

$$Q \subset (\mathrm{span}(e_1, I_1 e_1))^{\perp} = W \oplus \widetilde{V},$$

where

$$W = \mathrm{span}(I_2 e_1, I_3 e_1) \quad \text{is an } I_1\text{-complex line},$$

and

$$\widetilde{V} = \mathrm{span}(e_2, I_1 e_2, I_2 e_2, I_3 e_2, \dots, e_n, I_1 e_n, I_2 e_n, I_3 e_n\}$$

is a quaternionic-Hermitian subspace of real dimension $4(n-1)$. In particular, our induction hypothesis tells us that both $\mathrm{Re}(\frac{1}{(k-1)!}\sigma^{k-1})$ and $\mathrm{Im}(\frac{1}{(k-1)!}\sigma^{k-1})$ have comass one on $\widetilde{V}$.

We observe from $Q + \widetilde{V} \subset W \oplus \widetilde{V}$ that

$$\begin{aligned} \dim(Q \cap \widetilde{V}) &= \dim Q + \dim \widetilde{V} - \dim(Q + \widetilde{V}) \\ &\geq (2k-1) + (4n-4) - (4n-2) = 2k-3, \end{aligned}$$

so we can write $Q = u_2 \wedge u_3 \wedge v_4 \wedge \cdots \wedge v_{2k}$ for an oriented orthonormal basis $\{u_2, u_3, v_4, \dots, v_{2k}\}$ of Q, where $v_4, \dots, v_{2k} \in \widetilde{V}$. We also have

$$u_2 = \cos(\phi) w_2 + \sin(\phi) v_2, \qquad u_3 = \cos(\psi) w_3 + \sin(\psi) v_3,$$

for some unit vectors $w_2, w_3 \in W$ and $v_2, v_3 \in \widetilde{V}$. Abbreviating $R = v_4 \wedge \cdots \wedge v_{2k}$, $\cos(\phi) = c_\phi$ and similarly, we have

$$\begin{aligned} Q &= u_2 \wedge u_3 \wedge R = (c_\phi w_2 + s_\phi v_2) \wedge (c_\psi w_3 + s_\psi v_3) \wedge R \\ &= c_\phi c_\psi w_2 \wedge w_3 \wedge R + c_\phi s_\psi w_2 \wedge v_3 \wedge R + s_\phi c_\psi v_2 \wedge w_3 \wedge R + s_\phi s_\psi v_2 \wedge v_3 \wedge R. \end{aligned}$$

From (A.3) and the above, we get

$$\Theta_{2k}(P) = (\iota_{e_1} \alpha)(c_\phi c_\psi w_2 \wedge w_3 \wedge R + c_\phi s_\psi w_2 \wedge v_3 \wedge R + s_\phi c_\psi v_2 \wedge w_3 \wedge R + s_\phi s_\psi v_2 \wedge v_3 \wedge R). \tag{A.4}$$

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Since $\iota_{e_1}\Theta_{2k}$ is of I_1-type $(2k-1,0)+(0,2k-1)$, the first term in (A.4) must vanish because it contains the I_1-complex line $w_2 \wedge w_3$. Moreover, from (A.2), we have

$$\begin{aligned}\iota_{e_1}\Theta_{2k} &= \iota_{e_1}\mathrm{Re}\left(\frac{1}{k!}\sigma^k\right) = \mathrm{Re}\left((\iota_{e_1}\sigma)\wedge\frac{1}{(k-1)!}\sigma^{k-1}\right)\\ &= I_2e_1\wedge\mathrm{Re}\left(\frac{1}{(k-1)!}\sigma^{k-1}\right) - I_3e_1\wedge\mathrm{Im}\left(\frac{1}{(k-1)!}\sigma^{k-1}\right).\end{aligned}$$

Using the orthogonality of W and $\widetilde{V}$ and the above, the fourth term in (A.4) must also vanish, and we are left with

$$\begin{aligned}\Theta_{2k}(P) &= c_\phi s_\psi\, g(I_2e_1,w_2)\,\mathrm{Re}\left(\frac{1}{(k-1)!}\sigma^{k-1}\right)(v_3\wedge R)\\ &\quad + s_\phi c_\psi\, g(I_3e_1,w_3)\,\mathrm{Im}\left(\frac{1}{(k-1)!}\sigma^{k-1}\right)(v_2\wedge R).\end{aligned}$$

Applying the induction hypothesis and Cauchy–Schwarz, we deduce that

$$\Theta_{2k}(P) \le c_\phi s_\psi + s_\phi c_\psi = \sin(\phi+\psi) \le 1,$$

so Θ_{2k} has comass at most one. But letting $v_3\wedge\cdots\wedge v_{2k}$ be a calibrated $(2k-2)$-plane for $\mathrm{Re}(\frac{1}{(k-1)!}\sigma^{k-1})$ and choosing

$$\begin{aligned} u_2 &= I_2e_1 \in W, &&\text{so that } \cos(\phi)=1, \sin(\phi)=0, \text{and } g(I_2e_1,w_2)=1,\\ u_3 &= v_3 \in \widetilde{V}, &&\text{so that } \cos(\psi)=0, \sin(\psi)=1,\end{aligned}$$

gives $\Theta_{2k}(P)=1$. Thus, the comass of Θ_{2k} is exactly one. ■

Remark A.7 The case $k=2$ of Theorem A.6 is proved in [9, Theorem 2.38], where they also prove that a Θ_4-calibrated 4-plane is contained in a quaternionic 2-plane in V. It is likely that this fact remains true for general k. That is, a Θ_{2k}-calibrated $2k$-plane in V is contained in a quaternionic k-plane. However, we do not have need for this fact.

A.2 Riemannian cones and homogeneous forms

Let (M,g_M) be a Riemannian manifold. Let $C=\mathrm{C}(M)=(0,\infty)\times M$, and let r denote the standard coordinate on $(0,\infty)$. The *cone metric* g_C on C induced by g_M is defined to be

$$g_C = dr^2 + r^2 g_M. \tag{A.5}$$

The codimension one submanifold $\{1\}\times M \cong M$ is called the *link* of the cone. We have a projection map $\pi\colon C\to M$ given by $\pi(r,x)=x$. Given differential forms on the link M, we can regard them as forms on the cone C by pulling back by $\pi\colon C\to M$. We omit the explicit pullback notation.

Definition A.8 Consider the vector field

$$R = r\frac{\partial}{\partial r} \tag{A.6}$$

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on the cone C. The flow F_s of R is given by $(r, p) \mapsto (e^s r, p)$. For this reason, R is called the *dilation vector field* on the cone.

It follows that $\mathscr{L}_R g_C = 2g_C$. We say that g_C is *homogeneous of degree* 2 under dilations.

Definition A.9 Let $\gamma \in \Omega^k(C)$. We say that γ is *conical* if γ is *homogeneous of degree* k, or equivalently if $\mathscr{L}_R \gamma = k\gamma$.

Proposition A.10 *Let $\gamma \in \Omega^k(C)$ be a* closed *form which is homogeneous of degree k. Then, in fact,*

$$\gamma = dr \wedge (r^{k-1}\alpha_0) + \frac{r^k}{k}\hat{d}\alpha_0 = d\Big(\frac{r^k}{k}\alpha_0\Big),$$

where $\alpha_0 = (\iota_R \gamma)|_M \in \Omega^{k-1}(M)$.

Proof Write $\gamma = dr \wedge \alpha + \beta$ for some $(k-1)$-form α and k-form β on C such that $\iota_{\frac{\partial}{\partial r}} \alpha = \iota_{\frac{\partial}{\partial r}} \beta = 0$. That is, α and β have no dr factor, so they can be considered as forms on M depending on a parameter r, pulled back to C by π.

From $\gamma = dr \wedge \alpha + \beta$, and denoting by $\hat{d}$ the exterior derivative on M, we have

$$0 = d\gamma = -dr \wedge \hat{d}\alpha + dr \wedge \beta' + \hat{d}\beta,$$

and thus

$$\beta' = \hat{d}\alpha \quad \text{and} \quad \hat{d}\beta = 0. \tag{A.7}$$

But from $\mathscr{L}_R \gamma = k\gamma$, since $d\gamma = 0$, we have $k\gamma = d(\iota_R \gamma)$. Hence, since $\iota_R \gamma = r\alpha$, we obtain

$$k(dr \wedge \alpha + \beta) = k\gamma = d(r\alpha) = dr \wedge \alpha + r dr \wedge \alpha' + r\hat{d}\alpha.$$

Comparing the two sides above gives

$$k\alpha = \alpha + r\alpha' \quad \text{and} \quad k\beta = r\hat{d}\alpha. \tag{A.8}$$

The first equation in (A.8) gives $r\alpha' = (k-1)\alpha$, so $\alpha = r^{k-1}\alpha_0$ where α_0 is independent of r. Then the second equation gives $k\beta = r\hat{d}(r^{k-1}\alpha_0) = r^k \hat{d}\alpha_0$, so $\beta = \frac{r^k}{k}\hat{d}\alpha_0$. Note that the two equations in (A.7) are now automatically satisfied. Since $\iota_R \gamma = r\alpha = r^k \alpha_0$, we therefore conclude that

$$\gamma = dr \wedge (r^{k-1}\alpha_0) + \frac{r^k}{k}\hat{d}\alpha_0 = d\Big(\frac{r^k}{k}\alpha_0\Big),$$

where $\alpha_0 = (r^k \alpha_0)|_M = (\iota_R \gamma)|_M$. ∎

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