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THE MINIMAL PRIME SPECTRUM OF A COMMUTATIVE RING

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0. Introduction. We call a topological space X *minspectral* if it is homeomorphic to the space $\mathscr{M}(A)$ of minimal prime ideals of a commutative ring A in the usual (hull-kernel or Zariski) topology (see [**2**, p. 111]). Note that if A has an identity, $\mathscr{M}(A)$ is a subspace of Spec A (as defined in [**1**, p. 124]). It is well known that a minspectral space is Hausdorff and has a clopen basis (and hence is completely regular). We give here a topological characterization of the minspectral spaces, and we show that all minspectral spaces can actually be obtained from rings with identity and that open (but not closed) subspaces of minspectral spaces are minspectral (Theorem 1, Proposition 5). In the metrizable case, we prove, surprisingly, that a minspectral space has a metric in which it is complete (is an absolute G_δ; see [**4**, p. 207, K]), and we give an analogous result in the general case. If X is metrizable and Ind $X = 0$ (disjoint closed sets have disjoint clopen neighbourhoods) we show that X is minspectral if and only if it has a metric in which it is complete. If X is separable and metrizable, ind $X = 0$ (X has a basis of clopen sets) implies that Ind $X = 0$, but this is false in general, even if X has a metric in which it is complete [**5**].

1. The characterization. By an *m-subbasis* (respectively, an *m-basis*) $\mathfrak{B}$ for a Hausdorff space X we mean a subbasis (respectively, a basis) for the open sets such that each subset of $\mathfrak{B}$ with the FIP (finite intersection property) intersects. Thus, $\mathfrak{B}$ is an m-subbasis if and only if $\mathfrak{B}$ is a subbasis and at the same time a subbasis for the closed sets of a (usually different) quasicompact topology on X (we reserve the term *compact* for quasicompact Hausdorff spaces); this is immediate from the Alexander Subbasis Theorem [**4**, p. 139]. We call a basis $\mathfrak{B}$ *full* if $\emptyset, S \in \mathfrak{B}$ and $\mathfrak{B}$ is closed under finite union and intersection.

If R is a commutative ring with identity, let $\mathscr{Q}(R)$ be the set of quasicompact open subspaces of Spec R.

PROPOSITION 1. *The following conditions on a basis $\mathfrak{B}$ for a Hausdorff space X are equivalent.*

(1) *$\mathfrak{B}$ is a full m-basis.*

(2) *There is a commutative ring with identity R and a surjective homeomorphism $h: X \to \mathscr{M}(R) \subset \operatorname{Spec} R$ such that h induces a bijection of $\mathfrak{B}$ onto $\mathscr{C} = \{Q \cap \mathscr{M}(R): Q \in \mathscr{Q}(R)\}$.*

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Proof. Suppose that R, h are given. It suffices to show that the given basis for $\mathscr{M}(R)$ is a full m-basis. It is well known that $\mathscr{Q}(R)$ is a full basis for Spec R, and so $\mathscr{C}$ is a full basis for $\mathscr{M}(R)$. Now let $\{Q_\lambda\}_{\lambda\in\Lambda}$ be a family of sets in $\mathscr{Q}(R)$ such that $\{Q_\lambda \cap \mathscr{M}(R)\}_{\lambda\in\Lambda}$ has the FIP. Then $\{Q_\lambda\}_{\lambda\in\Lambda}$ has the FIP, and by [**3**, Theorem 1], $\bigcap_\lambda Q_\lambda \neq \emptyset$. Let $p \in \bigcap_\lambda Q_\lambda$. Choose $p' \subset p$ minimal. Since Q_λ is open, $p \in Q_\lambda$ implies that $p' \in Q_\lambda$, and so $p' \in Q_\lambda \cap \mathscr{M}(R)$. Hence, $p' \in \bigcap_\lambda(Q_\lambda \cap \mathscr{M}(R))$, as required.

Now suppose that X and a full m-basis $\mathfrak{B}$ are given. Let $W = \{0, 1\}$ with the topology given by the set of open sets $\mathscr{T} = \{\emptyset, \{0\}, \{0, 1\}\}$. We assume that one copy W_B of W is given for each $B \in \mathfrak{B}$, and we let f_B be the continuous map from X to W defined by $f_B(x) = 0$ if $x \in B$, $f_B(x) = 1$ if $x \notin B$. We then get a continuous map $f = \Pi_{B\in\mathfrak{B}} f_B$ from X to $\Pi_{B\in\mathfrak{B}} W_B = P$. We give P two topologies: the product topology coming from the topology $\mathscr{T}$ specified before, which we call the *weak* topology, and the product topology obtained by letting each W_B have the discrete topology, which we call the *strong* topology. If no specification is made, weak is assumed.

Let Y be the strong closure of $f(X)$ in P. Topologize Y by giving it the inherited weak topology from P. By [**3**, Proposition 9], Y is *spectral*, i.e., there is a commutative ring R with identity such that Y is homeomorphic to Spec R. We write p_y for the prime of R corresponding to a given $y \in Y$. Given $y, y' \in Y$, the following are equivalent: (1) $y' \in \mathrm{Cl}_Y\{y\}$, (2) for every open (respectively, quasicompact open) neighbourhood U of y', $y \in U$, and (3) $p_y \subset p_{y'}$. The subspace $\mathscr{M}(R)$ corresponds to

$$Y_0 = \{y \in Y: y \in \mathrm{Cl}\{y'\} \text{ implies } y' = y\}.$$

Now, since $\mathfrak{B}$ is a basis for the Hausdorff space X, the maps f_B separate points and closed sets, and hence f is an embedding of X into P (with the weak topology). We now want to show that $f(X) = Y_0$. We first prove that for each $y \in Y$ there is an $x \in X$ such that $y \in \mathrm{Cl}_Y\{f(x)\}$. Let $y \in Y$ be given. Let $\mathfrak{B}_y = \{B \in \mathfrak{B}: f_B(y) = 0\}$. A basis for the weak neighbourhoods of y in P is given by the sets of the form $U(\mathfrak{D}) = \{z \in P: f_D(z) = 0$ for each $D \in \mathfrak{D}\}$, where $\mathfrak{D}$ runs through the finite subsets of $\mathfrak{B}_y$. It suffices to show that $\bigcap_{\mathfrak{D}}(U(\mathfrak{D}) \cap f(X)) \neq \emptyset$; for if $f(x)$ is in the intersection, then $y \in \mathrm{Cl}\{f(x)\}$. This is equivalent to showing that $\bigcap_{\mathfrak{D}} f^{-1}(U(\mathfrak{D})) \neq \emptyset$. But $f^{-1}(U(\mathfrak{D})) = \bigcap_{D\in\mathfrak{D}} D$. Since each $D \in \mathfrak{B}$ and a family of sets in $\mathfrak{B}$ with the FIP intersects, we need only show that if $\mathfrak{D}_1, \ldots, \mathfrak{D}_k \subset \mathfrak{B}_y$ then $\bigcap_i(\bigcap_{D\in\mathfrak{D}_i} D) \neq \emptyset$, where $1 \leqq i \leqq k$. But y is in the strong closure of $f(X)$ in P and the set $\bigcap_i(\bigcap_{D\in\mathfrak{D}_i} D) = \bigcap(\bigcap_i \mathfrak{D}_i) = \{z \in P: f_D(z) = 0$ for each $D \in \bigcup_i \mathfrak{D}_i\}$ is an open neighbourhood of y in the strong topology on P; hence, it meets $f(X)$. But this says precisely that the intersection of the sets in $\bigcup_i \mathfrak{D}_i$ is not empty.

We can now prove that $f(X) = Y_0$. Given $y \in Y_0$, choose $x \in X$ such that $y \in \mathrm{Cl}_Y\{f(x)\}$. By definition of Y_0, $y = f(x)$. Thus, $Y_0 \subset f(X)$. Now let $x \in X$ be given. For some $y \in Y_0$, $f(x) \in \mathrm{Cl}_Y\{y\}$. We also know that

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$y = f(x')$ for some $x' \in X$. Since f induces a homeomorphism of X onto $f(X)$, and $f(x) \in \mathrm{Cl}_Y\{f(x')\} \Rightarrow f(x) \in \mathrm{Cl}_{f(X)}\{f(x')\}$, we must have $x \in \mathrm{Cl}_X\{x'\}$. Since X is Hausdorff, $x = x'$ and $f(x) = y \in Y$.

Thus, f induces a homeomorphism of X onto Y_0. It remains to show that this establishes a bijective correspondence between $\mathfrak{B}$ and $\{Q \cap Y_0: Q$ quasicompact open in $Y\}$. To see this, first note that for a given $B \in \mathfrak{B}$, $f(B) = \{y \in Y_0: f_B(y) = 0\} = (Q_B \cap Y) \cap Y_0$, where $Q_B = \{z \in P: f_B(z) = 0\}$ is quasicompact open in P, and $Q_B \cap Y$ is quasicompact open in Y. The sets $Q_{B_1} \cap \ldots \cap Q_{B_k} \cap Y$ are a basis for Y, and hence every quasicompact open subset of Y is a finite union of sets of this form. The inverse image of such a set is a finite union of finite intersections of sets $B_i \in \mathfrak{B}$, and, since $\mathfrak{B}$ is full, is in $\mathfrak{B}$.

If $\mathfrak{B}$ is a subbasis for X, call the topology which has $\mathfrak{B}$ as a subbasis for its closed sets the *dual topology on X determined by* $\mathfrak{B}$. Any subbasis $\mathfrak{B}$ generates a least full basis containing it, consisting of $\emptyset$, X, and the finite unions of finite intersections of sets in $\mathfrak{B}$. This full basis and $\mathfrak{B}$ obviously determine the same dual topology.

PROPOSITION 2. *Let X be a Hausdorff space. Then:*

(1) *If $\mathfrak{B}$ is an m-subbasis, the full basis generated by $\mathfrak{B}$ is an m-basis.*

(2) *If $\mathfrak{B}$ is an m-subbasis, any subset of $\mathfrak{B}$ which is a subbasis is an m-subbasis.*

(3) *If $\mathfrak{B}$ is an m-subbasis (respectively m-basis, full m-basis) for X, and $Y \subset X$ is closed in the dual topology determined by $\mathfrak{B}$, then $\{B \cap Y: B \in \mathfrak{B}\}$ is an m-subbasis (respectively m-basis, full m-basis) for Y.*

(4) *If $\mathfrak{B}$ is an m-basis for X and $U \subset X$ is open, then $\{B \in \mathfrak{B}: B \subset U\}$ is an m-basis for U.*

(5) *If $\mathfrak{B}$ is an m-subbasis for X, each set in $\mathfrak{B}$ is clopen.*

Proof. (1) The dual topologies involved are equal, and hence quasicompact or not, alike.

(2) This is obvious.

(3) Y is quasicompact in the inherited dual topology determined by $\{B \subset Y: B \in \mathfrak{B}\}$.

(4) This is obvious.

(5) Suppose that $p \in \mathrm{Cl}_X B$. Let $\mathfrak{B}_p = \{C \in \mathfrak{B}: p \in C\}$. For each finite subset $\{C_1, \ldots, C_k\}$ of $\mathfrak{B}_p$, $(C_1 \cap \ldots \cap C_k) \cap B \neq \emptyset$, since $C_1 \cap \ldots \cap C_k$ is a neighbourhood of p. Hence, $(\bigcap_{C \in \mathfrak{B}_p} C) \cap B \neq \emptyset$. But $\bigcap_{C \in \mathfrak{B}_p} C = \{p\}$. Therefore, $p \in B$.

THEOREM 1. *The following conditions on a Hausdorff space X are equivalent.*

(1) *X is minspectral, i.e., X is homeomorphic to the space of minimal primes $\mathscr{M}(A)$ for some commutative ring A.*

(2) *X is homeomorphic to the space of minimal primes in some commutative ring R with identity.*

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(3) *X has an m-subbasis.*
(4) *X has a full m-basis.*

Proof. (2) ⇔ (4) and (4) ⇔ (3) are obvious from Propositions (1) and (2), respectively, while (2) ⇒ (1) is clear. Now assume (1). By [**2**, Corollary 2.11, p. 115], X is the result of deleting one point from the space of minimal primes of a commutative ring R with identity, and hence is open in a space satisfying (2). It is consequently open in a space satisfying (4), and therefore satisfies (4) itself.

2. Building minspectral spaces. In this section, all spaces are assumed Hausdorff.

PROPOSITION 3. *Locally compact totally disconnected spaces are minspectral.*

Proof. The compact open subsets are an m-basis.

A metric space (X, d) is called *ultrametric* (and d is called an *ultrametric*) if for all $x, y, z \in X$, $d(x, y) \leqq \max\{d(x, z), d(y, z)\}$ with equality unless $d(x, z) = d(y, z)$.

PROPOSITION 4. *If X has an ultrametric d in which it is complete, X is minspectral.*

Proof. The open balls of radius 2^{-n}, n a nonnegative integer, form an m-basis. To see this, notice that two such balls are either disjoint, or else one is contained in the other. Given a family with the FIP, consider two cases. If the radii are bounded away from zero, let B be a ball with minimum radius in the family. It is evident that B *is* the intersection. If not, let $\{B_i\}$ be a decreasing sequence of balls in the family with radius approaching zero. The B_i's intersect in a single point, p, by completeness. Every B in the family contains every B_i of smaller radius, and hence contains p.

If Ind $X = 0$ and X is metrizable the condition is also necessary (see § 4).

PROPOSITION 5. *Arbitrary products and arbitrary topological unions of minspectral spaces are minspectral. Open subspaces of minspectral spaces are minspectral.*

Proof. Choose an m-basis for each factor of the product. The subproducts in which all but finitely many factors equal the whole space and each of the finitely many comes from the m-basis for that space constitute an m-basis for the product. It is clear that they give a basis, and that they are a subbasis for the closed sets when the product of the dual topologies is imposed. By the Tychonoff Theorem, the product of the dual topologies is quasicompact.

As for unions, it is clear that the union of m-bases for the various spaces in the union will be an m-basis for the union.

The last statement is part of Proposition 2.

PROPOSITION 6. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of minspectral spaces, let $\mathfrak{B}_\lambda$ be a full m-basis for X_λ, each $\lambda \in \Lambda$, let Y be a minspectral space with m-basis $\mathscr{C}$, and suppose that Y is contained as a closed subset in each X_λ, but that the sets $\{X_\lambda - Y: \lambda \in \Lambda\}$ are mutually disjoint. Assume, moreover, that for each $C \in \mathscr{C}$, each $\lambda \in \Lambda$, and each X_λ-open neighbourhood U of C, there is a $B \in \mathfrak{B}_\lambda$ such that $B \subset U$ and $B \cap Y = C$ (if Y is locally compact, we can choose $\mathscr{C}$ so that each C is compact, and the existence of such a B follows automatically). Finally, suppose that for all but possibly finitely many $\lambda \in \Lambda$, Y is a retract of an X_λ-open neighbourhood U_λ; say r_λ is the retraction. Topologize $\cup_\lambda X_\lambda$ in the obvious way, i.e., first take the topological union, and then form a quotient space in which the copies of Y are identified. Then the union is a minspectral space.*

Proof. We first prove the parenthetical remark. By passing to a possibly smaller U, we can assume that $U \cap Y = C$. For each $c \in C$, we can choose $B_c \in \mathfrak{B}_\lambda$ such that $c \in B_c \subset U$. Since C is compact, some finite set of B_c's covers C, and their union is the required B.

The theorem itself reduces to two cases: the case where Y is a retract of an X_λ-open neighbourhood for all $\lambda \in \Lambda$, and the case where there are only two spaces, for we can "add on" the exceptional spaces one at a time. In the first case we get an m-basis for the union which consists of all sets in each $\mathfrak{B}_\lambda$ which are contained in $X_\lambda - Y$, and for each $C \in \mathscr{C}$, all sets of the form $\cup_\lambda B_\lambda$, where for each λ, $B_\lambda \in \mathfrak{B}_\lambda$ and $B_\lambda \subset r_\lambda^{-1}(C)$ (an open neighbourhood of C in U_λ). Consider a family $\mathfrak{D}$ of sets in the specified basis with the FIP. If one of these sets is contained in $X_\lambda - Y$ for some λ, the intersections of the sets in the family with this one form a subfamily of $\mathfrak{B}_\lambda$ with the FIP, and the intersection is nonempty. Otherwise, $\{D \cap Y: D \in \mathfrak{D}\}$ is a family of sets in $\mathscr{C}$ and has the FIP. For suppose $p \in D_1 \cap \ldots \cap D_k$. Then $p \in U_\lambda$ for some λ and $r_\lambda(p) \in (D_1 \cap Y) \cap \ldots \cap (D_k \cap Y)$.

If there are only two spaces X_1, X_2, we get an m-basis for $X_1 \cup X_2$ which consists of all sets in each $\mathfrak{B}_\lambda$ which are contained in $X_\lambda - Y$, $\lambda = 1, 2$, and all sets of the form $B_1 \cup B_2$, where $B_\lambda \in \mathfrak{B}_\lambda$ and $B_1 \cap Y$, $B_2 \cap Y \in \mathscr{C}$. The verification that this is an m-basis is completely straightforward.

Note that if Y consists of just one point (or even finitely many), the retraction hypothesis is automatic, so that a union of minspectral spaces with basepoint, identifying the basepoints, is always minspectral.

Since discrete spaces are locally compact, any product of discrete spaces is minspectral. The space of irrationals is homeomorphic to a countable infinite product of countable infinite discrete spaces. Hence, the space of irrationals is minspectral. We shall prove later that a minspectral metrizable space has a metric in which it is complete, so that the space $\mathbf{Q}$ of rationals is not. It is possible to embed $\mathbf{Q}$ as a closed set in a product of $2^{\aleph_0}$ copies of the irrationals (see § 5), and hence in a product of $2^{\aleph_0}$ copies of a countable infinite discrete space. Hence, closed subspaces of minspectral spaces need not be minspectral.

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3. G_δ and related properties. Minspectral spaces have a property which in the metrizable case implies that they are absolute G_δ's. To formulate the appropriate notion, we first need to introduce some notation and terminology. In the sequel, we assume that all given spaces are completely regular Hausdorff spaces.

If X is any space, we write Δ_X (or simply Δ) for the diagonal in $X \times X$, and if $Y \subset X$ let $\mathscr{N}_X(Y)$ (or simply $\mathscr{N}(Y)$) be the family of X-open neighbourhoods of Y. If $x \in X - \mathrm{Cl}_X Y$, we call $\{x\} \times \mathrm{Cl}_X Y$ and $\mathrm{Cl}_X \mathscr{Y} \times \{x\}$ the slits of x and Y. A directed family $\mathfrak{U} \subset \mathscr{N}_{X\times X}(\Delta)$ will be called *Y-admissible* if (1) for each $x \notin \mathrm{Cl}_X Y$ there is a $U \in \mathfrak{U}$ such that

$$\pi_2(U \cap (\{x\} \times \mathrm{Cl}_X Y)) \cap \pi_1(U \cap (\mathrm{Cl}_X Y \times \{x\})) = \emptyset,$$

where π_1, π_2 are the product projections $X \times X \rightarrow X$, or, in other words, there is a $U \in \mathfrak{U}$ such that for all $y \in \mathrm{Cl}_X Y$ and for each $x \notin \mathrm{Cl}_X Y$ either (x, y) or (y, x) is not in U, and (2) for all $y \in Y$ and for all $x \in \mathrm{Cl}_X Y - Y$, there is a $U \in \mathfrak{U}$ such that either $(x, y) \notin U$ or $(y, x) \notin U$.

We say that $Y \subset X$ is a G_ϵ in X if there is an order-preserving function ϕ: $\mathscr{N}_{X\times X}(\Delta) \rightarrow \mathscr{N}_X(Y)$ (which we call a G_ϵ-map for Y) such that for each Y-admissible $\mathfrak{U} \subset \mathscr{N}_{X\times X}(\Delta)$, $\bigcap_{U\in\mathfrak{U}}\phi(U) = Y$.

Let $\mathscr{S}_X$ be the family of symmetric neighbourhoods of Δ in $X \times X$. $\mathfrak{U} \subset \mathscr{S}_X$ is Y-admissible if and only if for all $x \in \mathrm{Cl}\, Y$ there is a $U \in \mathfrak{U}$ such that U is disjoint from one ($\Rightarrow$ both) slits determined by x and Y, and for all $y \in Y$ and $x \in \mathrm{Cl}\, Y - Y$, there is a $U \in \mathfrak{U}$ such that $(x, y) \notin U$ ($\Leftrightarrow (y, x) \notin U$). Suppose that ϕ: $\mathscr{S}_X \rightarrow \mathscr{N}_X(Y)$ is order-preserving and that for each Y-admissible family $\mathfrak{U} \subset \mathscr{S}_X$, $\bigcap_{U\in\mathfrak{U}}\phi(U) = Y$. Let ρ be the retraction of $\mathscr{N}_{X\times X}(\Delta)$ onto $\mathscr{S}_X$ defined by $\rho(U) = \{(x, x')\colon (x, x') \text{ and } (x', x) \text{ are both in } U\}$. ρ is order-preserving, $\rho(U) \subset U$ for all U, and ρ takes Y-admissible families into Y-admissible families. It easily follows that $\phi\rho$ is a G_ϵ-map for Y. Conversely, if ϕ is a G_ϵ-map for Y, then $\phi|\mathscr{S}_X$ retains the properties listed above.

Hence, in asking whether there is a G_ϵ-map for Y, we might as well restrict attention to $\mathscr{S}_X$. We refer to maps defined only on $\mathscr{S}_X$ as G_ϵ-maps, too. We rarely bother to specify which kind we mean, since it is generally irrelevant. The context should make it clear if one kind as opposed to the other is required.

The condition that $\mathfrak{U} \subset \mathfrak{S}_X$ be Y-admissible asserts that certain points and slits, disjoint from Δ, must be disjoint from some set in $\mathfrak{U}$. Hence $\mathfrak{U} \subset \mathscr{S}_X$ is Y-admissible for all $Y \subset X$ if and only if for any slit $\{x\} \times \mathrm{Cl}\, Y$, $x \notin \mathrm{Cl}\, Y$, some $U \in \mathfrak{U}$ is disjoint from it. In particular, if $\mathfrak{U}$ is a neighbourhood basis for Δ, $\mathfrak{U}$ is Y-admissible for all $Y \subset X$.

PROPOSITION 7. *The G_ϵ sets in X are closed under finite union and intersection and contain all open sets and all closed sets. If Δ has a countable neighbourhood basis, and, in particular, if X is a metrizable space, then every G_ϵ in X is a G_δ.*

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Proof. Let ϕ_1, ϕ_2 be G_ϵ-maps for Y_1, Y_2. Then $U \mapsto \phi_1(U) \cup \phi_2(U)$ (respectively, $U \mapsto \phi_1(U) \cap \phi_2(U)$) is a G_ϵ-map for $Y_1 \cup Y_2$ (respectively, $Y_1 \cap Y_2$).

If Y is open in X, ϕ: $U \to Y$, all U, is a G_ϵ-map for Y.

If Y is closed in X, we define ϕ as follows: $\phi(U) = \bigcup\{V$ open in X: $V \cap Y \neq \emptyset$ and $V \times V \subset U\}$. Now suppose that $\mathfrak{U}$ is Y-admissible. We must show that the intersection of the sets $\phi(U)$ is Y. Suppose that $x \in X - Y$. Choose $U \in \mathfrak{U}$ disjoint from one of the slits of x and Y. Then it is easy to see that $x \notin \phi(U)$.

If $\mathfrak{U} = \{U_1, U_2, \ldots\}$ is a neighbourhood basis for Δ, and ϕ is a G_ϵ-map for Y, then $Y = \bigcap_n \phi(U_n)$.

We call Y an *absolute* G_ϵ if for any space X in which it is embedded, Y is a G_ϵ in X. Evidently, if a metrizable space is an absolute G_ϵ it is an absolute G_δ, and has a metric in which it is complete.

PROPOSITION 8. *A minspectral space Y is an absolute G_ϵ.*

Proof. Let $\mathfrak{B}$ be an m-basis for Y. Define ϕ: $\mathscr{S}_X \to \mathscr{N}(Y)$ thus: for each U, let $\phi_1(U) = \bigcup\{V$ open in X: $V \cap Y \in \mathfrak{B}$ and $V \times V \subset U\}$, and let ϕ_2 be any G_ϵ-map for Cl Y. Then let ϕ be defined by $\phi(U) = \phi_1(U) \cap \phi_2(U)$, all U. We shall show that ϕ is a G_ϵ-map for Y. First note that if $y \in Y$, we can choose V_1 containing y and open in X such that $V_1 \times V_1 \subset U$. We can choose $B \in \mathfrak{B}$ containing y and contained in V_1. Finally, we can choose $V \subset V_1$ open such that $V \cap Y = B$. We then have $y \in V$, $V \cap Y = B$, and $V \times V \subset U$, so $y \in \phi(U)$ for all U.

It is clear that ϕ is order-preserving. Now suppose that $\mathfrak{U}$ is Y-admissible and that $x \in \bigcap_{U \in \mathfrak{U}} \phi(U)$. Suppose that $x \notin Y$. The presence of ϕ_2 guarantees that $x \in$ Cl Y. For each U, we can choose V_U open in X such that $x \in V_U$, $V_U \cap Y = B_U \in \mathfrak{B}$, and $V_U \times V_U \subset U$. Given $U_1, \ldots, U_n \in \mathfrak{U}$, $\bigcap B_{U_i} \neq \emptyset$, because $\bigcap V_{U_i}$ is a neighbourhood of x in X, $x \in$ Cl Y, and therefore $\emptyset \neq Y \cap (\bigcap V_{U_i}) = \bigcap B_{U_i}$. Since $\{B_U\}_{U \in \mathfrak{U}}$ has the FIP, we can choose $y \in \bigcap_{U \in \mathfrak{U}} B_U$. Since $y \in Y$, $x \in$ Cl $Y - Y$, for some $U \in \mathfrak{U}$, $(x, y) \notin U$. Since $x, y \in V_U$, this contradicts $V_U \times V_U \subset U$.

COROLLARY. *If Y is metrizable and minspectral, then Y is an absolute G_δ; equivalently, Y has a metric in which it is complete.*

4. The metrizable case. We have very satisfactory results in the metrizable case if Ind $X = 0$. We first collect the basic facts about metrizable spaces X with Ind $X = 0$. Call a metric space isosceles if for all $x, y, z \in X$ at least two of the distances $d(x, y)$, $d(x, z)$, $d(y, z)$ are equal.

PROPOSITION 9. *The following conditions on a space X are equivalent.*

(1) *There is a metric d for X such that (X, d) is ultrametric (i.e., X is ultrametrizable).*

(2) *There is a metric d for X such that (X, d) is isosceles.*

(3) *X is metrizable and* Ind $X = 0$ (*disjoint closed sets have disjoint clopen neighbourhoods*).

(4) *X is metrizable and every open set is σ-clopen* (*a countable union of clopen sets*).

(5) *X has a σ-discrete basis consisting of clopen sets.*

(6) *X can be embedded in a countable product of discrete spaces.*

Proof. We show $(6) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$. Consider a countable product $\Pi_n X_n$ of discrete spaces, $n = 1, 2, \ldots$. Given x, y in the product, say $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$, $x \neq y$, define $d(x, y) = 2^{-n}$, where $n = \min\{k: x_k \neq y_k\}$. It is easy to see that d is an ultrametric, and its restrictions give ultrametrics for the subspaces of the product. Thus, $(6) \Rightarrow (1)$. $(1) \Rightarrow (2)$ is obvious. Now, if (X, d) is isosceles and $Y, Z \subset X$ are closed and disjoint, it is easy to check that $\{x \in X: d(x, Y) < d(x, Z)\}$ is a clopen neighbourhood of Y disjoint from Z. Hence, $(2) \Rightarrow (3)$. Assume (3), and let U be any open set in X. For each n, let U_n be the complement of the union of the open balls of radius $1/n$ centered at points not in U, and let V_n be a clopen set containing U_n and disjoint from $X - U$. Then $U = \cup_n V_n$. Now assume (4). Since X is metrizable, it has a σ-discrete basis $\mathfrak{B} = \cup_n \mathfrak{B}_n$, where each $\mathfrak{B}_n$ is a discrete family of open sets. For each open set B in X, let $B = \cup_m B_m$ be an expression for B as a union of clopen sets, where m runs through the positive integers. For each n, m let $\mathfrak{B}_{n,m} = \{B_m: B \in \mathfrak{B}_n\}$. Then the union of the $\mathfrak{B}_{n,m}$ is clearly a σ-discrete basis consisting of clopen sets.

Finally, let $\mathfrak{B}_n$ be a discrete family of clopen sets for each positive integer n such that $\cup_n \mathfrak{B}_n$ is a basis. Let $C_n = X - (\cup_{B \in \mathfrak{B}_n} B)$. Then C_n is clopen, since $\mathfrak{B}_n$ is a locally finite family of clopen sets, and $\mathscr{C}_n = \mathfrak{B}_n \cup \{C_n\}$ is a partition of X into clopen sets, and hence corresponds to a mapping g_n of X onto a discrete space (the quotient). Since $\mathfrak{B}$ is a basis, the mappings g_n separate points and closed sets, so that $\Pi_n g_n$ embeds X in a countable product of discrete spaces.

We need a couple of topological lemmas now.

LEMMA 1. *Let $\{Y_\lambda\}_{\lambda \in \Lambda}$ be a family of subspaces of X. Let $Y = \cap_\lambda Y_\lambda$. Let $f_\lambda: Y \to Y_\lambda$ be the inclusion map. Then $f = \Pi_\lambda f_\lambda$ embeds Y as a closed set in $\Pi_\lambda Y_\lambda$.*

The proof is left to the reader.

LEMMA 2. *Let X be a countable product of discrete spaces and $U \subset X$ an open set. Then U is a disjoint* (*topological*) *union of countable products of discrete spaces, and conseqeuntly can be embedded as a closed set in a countable product of discrete spaces.*

Proof. Let $X = \Pi_n X_n$, $n = 1, 2, \ldots$. For each initial segment $\{1, \ldots, k\}$ of the positive integers and each selection of points $x_i \in X_i$, $1 \leqq i \leqq k$, the set $Y(x_1, \ldots, x_k) = \{x_1\} \times \ldots \times \{x_k\} \times \Pi_{n>k}X_n$ is clopen, and these sets are a basis. Consider the set S of sequences $(x) = (x_1, \ldots, x_k)$ such that $Y(x) \subset U$ but $Y(x_1, \ldots, x_{k-1}) \not\subset U$ (if the sequence has no terms, $Y(x) = X$). It is easy to see that $U = \bigcup_{(x)\in S} Y(x)$ expresses U as a disjoint (topological) union of countable products of discrete spaces. The rest is left to the reader.

THEOREM 2. *The following conditions on a space X are equivalent.*

(1) *X can be embedded as a closed set in a countable product of discrete spaces.*

(2) *There is a metric d for X such that (X, d) is a complete ultrametric space.*

(3) *There is a metric d for X such that (X, d) is a complete isosceles metric space.*

(4) *X is metrizable,* Ind $X = 0$, *and X is minspectral.*

(5) *X is metrizable,* Ind $X = 0$, *and X has some metric in which it is complete.*

(6) *X is metrizable,* Ind $X = 0$, *and X is an absolute* G_δ.

(7) *X is a* G_δ *in a countable product of discrete spaces.*

Proof. The metric for a countable product of discrete spaces described at the beginning of the proof of Proposition 9 is easily seen to be complete. Thus, $(1) \Rightarrow (2)$. $(2) \Rightarrow (3)$, $(2) \Rightarrow (4) \Rightarrow (5) \Leftrightarrow (6) \Rightarrow (7)$, and $(3) \Rightarrow (5)$ are known. It suffices to show $(7) \Rightarrow (1)$. By Lemma 1, X can be embedded as a closed set in a countable product of spaces each of which is open in a countable product of discrete spaces. By Lemma 2, each of the factors can be embedded as a closed set in a countable product of discrete spaces. This completes the proof.

COROLLARY. *Let X be a metrizable space such that* Ind $X = 0$. *Then X is minspectral if and only if it has some metric in which it is complete. In particular, if X is a separable metric space then X is minspectral if and only if X is zero dimensional* (ind X = Ind X *in this case*) *and X has some metric in which it is complete. This is the case if and only if X is homeomorphic to a closed subspace of the irrationals.*

Proof. The last statement follows because in the separable case we need only to consider countable products of *countable* discrete spaces.

5. Closed sets and open questions.

By virtue of Lemma 1 we have

PROPOSITION 10. *Every subspace of a minspectral space can be embedded as a closed subspace of a minspectral space.*

Proof. We can represent the subspace as an intersection of open ($\Rightarrow$ minspectral) subspaces by deleting one point of the complement at a time.

The result of deleting one point from the irrationals is a space homeomorphic to the irrationals. Thus, as asserted earlier, the rationals can be embedded in the product of $2^{\aleph_0}$ copies of the irrationals as a closed set.

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The question of whether a closed G_δ in a minspectral space must be minspectral is open, however, even if the space is metrizable. It is possible that for every minspectral metrizable space X, Ind $X = 0$; it is also possible that every absolute G_δ metric space X with ind $X = 0$ is minspectral. These are the extremes. Either implies that closed subspaces of metrizable minspectral spaces are minspectral. (Note: the closed G_δ's are precisely the zero sets; in the metrizable case they are all the closed sets.) In looking at this problem it does not matter whether we consider only closed G_δ's or all G_δ's, by an easy application of Lemma 1.

It is natural to conjecture that an absolute G_ϵ in a space X with Ind $X=0$ is minspectral. This question is also open.

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