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ON THE β-CONSTRUCTION IN K-THEORY

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Introduction. The β-construction assigns to each complex representation φ of the compact Lie group G a unique element $\beta(\varphi)$ in $\tilde{K}^1(G)$. For the details of this construction the reader is referred to [**1**] or [**5**]. The purpose of the present paper is to determine some of the properties of the element $\beta(\varphi)$ in terms of the invariants of the representation φ. More precisely, we consider the following question. Let G be a simple, simply-connected compact Lie group and let f: $S^3 \rightarrow G$ be a Lie group homomorphism. Then $\tilde{K}^1(S^3) \simeq \mathbf{Z}$ with generator $x = \beta(\varphi_1)$, φ_1 the fundamental representation of S^3, so that if φ is a representation of $G, f^*(\varphi) = n(\varphi)x$, where $n(\varphi)$ is an integer depending on φ and f. The problem is to determine $n(\varphi)$.

Since G is simple and simply-connected we may assume that ch_2, the component of the Chern character in dimension 4 takes its values in $H^4(SG, \mathbf{Z}) \cong \mathbf{Z}$. Let u be a generator of $H^4(SG, \mathbf{Z})$ so that $\mathrm{ch}_2(\beta(\varphi)) = m(\varphi)u$, $m(\varphi)$ an integer depending on φ. It turns out that the integers $n(\varphi)$ and $m(\varphi)$ are closely related to an invariant of the representation φ studied by Dynkin in [**4**] and called by him the index of φ, $l(\varphi)$. In section 2 it is shown that $n(\varphi)$ is a fixed integral multiple, depending on f, of $l(\varphi)$, and in section 3 $m(\varphi)$ is shown to be $\pm l(\varphi)$.

The above may be interpreted as giving information about the question of homotopy of group representations. If we say that two representations φ_1, φ_2 are stably homotopic ($\varphi_1 \sim \varphi_2$) if $i_1 \circ \varphi_1$ and $i_2 \circ \varphi_2$ are homotopic as maps $G \rightarrow \mathrm{Aut}(\mathbf{C}^N)$, N large, i_1, i_2 suitable inclusions, then $\varphi_1 \sim \varphi_2$ if and only if $\beta(\varphi_1) = \beta(\varphi_2)$. It is a consequence of a result of Hodgkin [**5**] that given an arbitrary φ, $\varphi \sim n_1\lambda_1 + \ldots + n_k\lambda_k$, $\lambda_1, \ldots, \lambda_k$ the fundamental representations of G and $n_1, \ldots, n_k$ suitable integers. In particular inequivalent representations may be stably homotopic. In these terms the above says that a necessary condition that two representations be stably homotopic is that their indices be equal. Since a theorem of Dynkin (1.4 below) gives the index of an irreducible φ as a simple expression involving the dimensions of φ and G and the highest weight of φ, this criterion is a useful one for studying the question of stable homotopy.

The contents of this paper are arranged in the following way. Section 1 develops the properties of the index in the context of representations and homomorphisms of complex simple Lie algebras. This material is essentially all contained in [**4**]. In section 2 we establish the connection between the index and the β-construction. This section also discusses the question of stable homotopy

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of representations. The final section gives the relationship between the index and the Chern character.

1. The Index. Let $\mathfrak{g}$ be a complex simple Lie algebra. Then $\mathfrak{g}$ admits an invariant inner product (e.g. the Killing form) and further any two such differ by a scalar factor. Take $\mathfrak{h} \subset \mathfrak{g}$ to be a fixed Cartan subalgebra and let $R \subset \mathfrak{h}^*$ be the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. An inner product on $\mathfrak{h}$ determines one on $\mathfrak{h}^*$ by duality; it follows that we obtain a unique invariant inner product, which will be denoted as $\mathfrak{g}(\cdot,\cdot)$ by imposing the normalization condition $\mathfrak{g}(\mu, \mu) = 2$ where $\mu \in R$ is the maximal root with respect to some ordering of R. Given the form $\mathfrak{g}(\cdot,\cdot)$ and a root $\alpha \in R$ we denote as h_α the element dual to α, that is, the unique element $h_\alpha \in \mathfrak{h}$ such that $\mathfrak{g}(h_\alpha, X) = \alpha(X)$ for all $X \in \mathfrak{h}$.

Now let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be simple Lie algebras and let $f: \mathfrak{g}_1 \to \mathfrak{g}_2$ be a homomorphism of Lie algebras. Then the formula $(x, y) = \mathfrak{g}_2(f(x), f(y))$ defines an invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}_1$. By the above remarks, $(\cdot,\cdot) = j(f)\mathfrak{g}_1(\cdot,\cdot)$ for a fixed scalar $j(f)$. Following Dynkin [**4**] we define the index of f to be the scalar $j(f)$. It is clear that $j(f) = \frac{1}{2}\mathfrak{g}_2(f(h_\mu), f(h_\mu))$, μ the maximal root of g_1 and that if $g: \mathfrak{g}_2 \to \mathfrak{g}_3$ is a second homomorphism then $j(g \circ f) = j(f)j(g)$. More difficult is the fact ([**4**], Theorem 2.2) that $j(f)$ is always integral. Although not central to the present work, 3.3 below leads easily to an independent proof of this fact. For later reference we also record the following obvious proposition.

PROPOSITION 1.1. *Let μ be the maximal root of g with respect to some ordering of R and let X_μ, $X_{-\mu}$ be root vectors for μ and $-\mu$ respectively. Then the subalgebra spanned by $\{h_\mu, X_\mu, X_{-\mu}\}$ is isomorphic to A_1 and if $f: A_1 \to \mathfrak{g}$ denotes the corresponding inclusion, then $j(f) = 1$.*

Now let $\varphi: \mathfrak{g} \to \mathrm{End}(\mathbf{C}^n)$ be a representation of $\mathfrak{g}$. The trace form $\mathrm{Tr}_\varphi(\cdot,\cdot)$ defined by $\mathrm{Tr}_\varphi(x, y) = \mathrm{Tr}(\varphi(x) \circ \varphi(y))$ is again an invariant inner product on $\mathfrak{g}$ so that once again $\mathrm{Tr}_\varphi(\cdot,\cdot) = l(\varphi)\mathfrak{g}(\cdot,\cdot)$ where $l(\varphi)$ is a scalar called by Dynkin the index of φ. As immediate consequences of the definition of $l(\varphi)$ we have the following propositions.

PROPOSITION 1.2. *If $f: \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism and $\varphi: \mathfrak{g}_2 \to \mathrm{End}(\mathbf{C}^n)$ a representation, then $l(\varphi \circ f) = j(f)l(\varphi)$.*

PROPOSITION 1.3. *If $\varphi: \mathfrak{g} \to \mathrm{End}(\mathbf{C}^n)$, $\psi: \mathfrak{g} \to \mathrm{End}(\mathbf{C}^m)$ are two representations, then $l(\varphi + \psi) = l(\varphi) + l(\psi)$ and $l(\varphi \otimes \psi) = ml(\varphi) + nl(\psi)$ (for the latter property one requires the fact that $\mathfrak{g}$ is simple).*

The following theorem [**4**, Theorem 2.5] gives an elegant expression for $l(\varphi)$ in terms of the invariants of φ. For completeness we include a proof.

THEOREM 1.4. *Let φ be an irreducible representation of the simple Lie algebra $\mathfrak{g}$. Then $l(\varphi) = (\dim \varphi/\dim \mathfrak{g})\mathfrak{g}(\lambda, \lambda + 2\delta)$, where λ is the highest weight of φ and δ is $\frac{1}{2}$ the sum of the positive roots.*

Proof. Let l be the dimension of $\mathfrak{h}$ and choose a basis $\{H_i\}_{i=1}^{l}$ of $\mathfrak{h}$ orthonormal with respect to $\mathfrak{g}(\cdot,\cdot)$. We now choose a root vector X_α corresponding to each $\alpha \in R$, subject to the condition (to be used later) that $[X_\alpha, X_{-\alpha}] = h_\alpha \mathfrak{g}(X_\alpha, X_{-\alpha})$. Then $\{H_i\}_{i-1}^{l} \cup \{X_\alpha\}_{\alpha \in R}$ is a basis for $\mathfrak{g}$. Recalling that $g(X_\alpha, X_\beta) = 0$ unless $\alpha + \beta = 0$ and that $\mathfrak{g}(H_i, X_\alpha) = 0$ all i, α, it follows that

$$\left\{\frac{H_i}{l(\varphi)}\right\}_{i=1}^{l} \cup \left\{\frac{X_{-\alpha}}{l(\varphi)\mathfrak{g}(X_\alpha, X_{-\alpha})}\right\}_{\alpha \in R}$$

is dual to the above basis with respect to $\mathrm{Tr}_\varphi(\cdot,\cdot)$. ($l(\varphi) \neq 0$ since $\mathfrak{g}$ simple, $\varphi \neq 0 \Rightarrow \mathrm{Tr}_\varphi(\cdot,\cdot)$ non-degenerate.) Since $\mathfrak{g}$ is simple and φ is irreducible, it is a standard fact (cf. [**6**, III.4]) that the Casimir operator

$$\Phi = \frac{1}{l(\varphi)}\left[\sum_{i=1}^{l} \varphi(H_i)^2 + \sum_{\alpha \in R} \frac{\varphi(X_\alpha)\varphi(X_{-\alpha})}{\mathfrak{g}(X_\alpha, X_{-\alpha})}\right]$$

is a scalar multiple of the identity and that $\mathrm{Tr}\,\Phi = \dim \mathfrak{g}$. On the other hand if v is a vector of weight λ the relations $\varphi(H_i)v = \lambda(H_i)v$ and

$$\frac{\varphi(X_\alpha)\varphi(X_{-\alpha})}{\mathfrak{g}(X_\alpha, X_{-\alpha})} v = \begin{cases} 0, & \alpha < 0 \\ \dfrac{\lambda([X_\alpha, X_{-\alpha}])}{\mathfrak{g}(X_\alpha, X_{-\alpha})} v = \mathfrak{g}(\lambda, \alpha)v, & \alpha > 0 \end{cases}$$

show that $\Phi v = (1/l(\varphi))\mathfrak{g}(\lambda, \lambda + 2\delta)v$ and thus that

$$\mathrm{Tr}\,\Phi = (\mathfrak{g}(\lambda, \lambda + 2\delta)/l(\varphi))\dim \varphi.$$

Theorem 1.4 follows.

Dynkin [**4**, table 5, p. 135] has used 1.4 to compute $l(\varphi)$ when φ is a fundamental representation of any of the simple algebras. In particular if $\mathfrak{g} = A_n$ and φ is the representation of A_n as $\mathrm{sl}(n + 1, \mathbf{C})$, then

(1.5) $$l(\varphi) = 1 \quad \text{and} \quad l(\Lambda^k\varphi) = \binom{n-1}{k-1}.$$

COROLLARY 1.6. *Let φ be any representation of the simple Lie algebra $\mathfrak{g}$. Then*

$$l(\Lambda^k\varphi) = \binom{\dim \varphi - 2}{k-1} l(\varphi).$$

The next corollary is a consequence of 2.2 below together with the properties of the Adams operations in $\tilde{K}^0(S^4)$. Let φ be a representation of $\mathfrak{g}$ and let $\psi^k(\varphi)$ be the virtual representation defined in a fashion analogous to the definition of the Adams operations in K-theory, i.e., let $P_k(\sigma_1, \ldots, \sigma_k)$ be the Newton polynomial expressing $x_1^k + \ldots + x_k^k$ in terms of the elementary symmetric functions in $x_1, \ldots, x_k$ and define $\psi^k(\varphi) = P_k(\varphi, \Lambda^2\varphi, \ldots, \Lambda^k\varphi)$.

COROLLARY 1.7. $l(\psi^k(\varphi)) = k^2 l(\varphi)$.

2. The relation of the index to K-theory. Let G be a simple and simply-connected compact Lie group and let $f: S^3 \to G$, $\varphi: G \to \mathrm{Aut}(\mathbf{C}^n)$, be a homomorphism and representation respectively. We denote as $\tilde{f}$, $\tilde{\varphi}$ the associated homomorphism and representation of (complex) Lie algebras. The β-construction associates to each such φ an element $\beta(\varphi) \in \tilde{K}^1(G)$. Recall [**5**, §4] that the β-construction has the properties

(2.1) $\beta(\varphi + \psi) = \beta(\varphi) + \beta(\psi)$ and $\beta(\varphi \otimes \psi) = \dim \psi\beta(\varphi) + \dim \varphi\beta(\psi)$.

Recall also that $\tilde{K}^1(S^3) \simeq \mathbf{Z}$ with generator $x = \beta(\varphi_1)$ where φ_1 is the representation of S^3 corresponding to the representation of A_1 as $\mathrm{sl}(2, \mathbf{C})$. The main result of this section is the following theorem.

THEOREM 2.2. $f^*\beta(\varphi) = l(\tilde{\varphi} \circ \tilde{f})x = j(\tilde{f})l(\tilde{\varphi})x$.

Proof. $\varphi \circ f$ is a representation of S^3 so that $f^*\beta(\varphi) = \beta(\varphi \circ f)$; therefore by 1.2 it is enough to show that if ψ is any representation of S^3, $\tilde{\psi}$ the associated representation of A_1, then $\beta(\psi) = l(\tilde{\psi})x$. Since $l(\tilde{\varphi}_1) = 1$ (1.5) and since any such ψ may be written as a polynomial in φ_1, 2.2 now follows from 1.3 and 2.1.

Let φ: $G \to \mathrm{Aut}(\mathbf{C}^n)$, ψ: $G \to \mathrm{Aut}(\mathbf{C}^m)$ be two representations and define φ to be stably homotopic to ψ, written $\varphi \sim \psi$, if $i_1 \circ \varphi$ and $i_2 \circ \psi$ are homotopic as maps, where i_1: $\mathrm{Aut}(\mathbf{C}^n) \to \mathrm{Aut}(\mathbf{C}^N)$ and i_2: $\mathrm{Aut}(\mathbf{C}^m) \to \mathrm{Aut}(\mathbf{C}^N)$ are induced by inclusion and N is suitably large. From the characterization of the β-construction given in [**5**, §4] it is clear that $\varphi \sim \psi$ if and only if $\beta(\varphi) = \beta(\psi)$. The following is thus a corollary of Theorems 2.2 and 1.4.

COROLLARY 2.3. *Let φ, ψ be two irreducible representations of G with highest weights λ, λ' respectively. Then a necessary condition that φ be stably homotopic to ψ is that* $\dim \varphi(\lambda, \lambda + 2\delta) = \dim \psi(\lambda', \lambda' + 2\delta)$ *(here $(\cdot,\cdot)$ denotes any invariant inner product on $\mathfrak{g}$).*

Remark. The above condition is clearly not sufficient. Since every representation of S^3 is self-conjugate the above cannot distinguish a representation from its dual. However if φ is the representation of $\mathrm{SU}(n + 1)$ defined by φ: $\mathrm{SU}(n + 1) \subset \mathrm{Aut}(\mathbf{C}^n)$ then $\Lambda^k\varphi$ and $\Lambda^{n-k+1}\varphi$ are dual but not (unless $k = n - k + 1$) stably homotopic. I do not know an example of two distinct, non-conjugate, irreducible representations φ, ψ with $l(\varphi) = l(\psi)$ nor any example of two distinct irreducible representations φ, ψ with $\varphi \sim \psi$.

Corollary 2.3 together with the remarks made above does give the following result concerning the geometry of the Coxeter-Stiefel diagram.

COROLLARY 2.4. *Let φ be an irreducible representation of the simple Lie algebra $\mathfrak{g}$ and let λ be the highest, λ' the lowest, weight of φ. Then*

$$(\lambda, \lambda + 2\delta) = (-\lambda', -\lambda' + 2\delta).$$

Proof. $-\lambda'$ is the highest weight of the dual (contragredient) representation.

3. The relationship between the Chern character and the index. To each $x \in \tilde{K}^1(G)$ the Chern character associates an element $\mathrm{ch}_2 x \in H^4(SG, \mathbf{Q})$.

We consider $H^4(SG, \mathbf{Z})$ as a subgroup of $H_4(SG, \mathbf{Q})$ under the coefficient homomorphism $\mathbf{Z} \subset \mathbf{Q}$ and consider $H^4(SG, \mathbf{Z})$ as those $u \in H^4(SG, \mathbf{Q})$ for which $\langle u, v\rangle \in \mathbf{Z}$ whenever $v \in H_4(SG, \mathbf{Z})$. If G is simple and simply-connected $H_4(SG, \mathbf{Z}) \simeq \mathbf{Z}$ has all its elements spherical. Since the Chern character takes integral values on spherical cycles (cf. [**2**]) we may, by the above, consider ch_2 as a map $\mathrm{ch}_2\colon \tilde{K}^1(G) \to H^4(SG, \mathbf{Z})$. We will need the following proposition.

PROPOSITION 3.1. *There is a homomorphism* $f\colon S^3 \to G$ *such that* $[f]$ *generates* $\pi_3(G) \simeq \mathbf{Z}$ *and such that* $j(\tilde{f}) = 1$. ($[f]$ *denotes the homotopy class of* f.)

Proof. This is a consequence of Proposition 10.2 Chapter III of [**3**]; in particular, the statement about $j(\tilde{f})$ follows from the proof of part B of this proposition together with 1.1 above.

Now let x be the generator of $\tilde{K}^1(S^3)$ described in §2 and let ι be the generator of $H_4(S^4, \mathbf{Z})$ with $\langle \mathrm{ch}x, \iota\rangle = 1$. Let $v = f_*(\iota)$ generate $H_4(SG, \mathbf{Z})$ (f the homomorphism of 3.1) and choose as generator of $H^4(SG, \mathbf{Z})$ the element u such that $\langle u, v\rangle = 1$.

THEOREM 3.2. (a) *Let* φ *be a representation of* G. *Then*

$$\mathrm{ch}_2\beta(\varphi) = l(\varphi)u.$$

(b) *The image of* $\mathrm{ch}_2\colon \tilde{K}^1(G) \to H^4(SG, \mathbf{Z})$ *is generated by* ku *where*

$$k = \begin{cases} 1, & G = \mathrm{SU}(n) \text{ or } \mathrm{Sp}(n) \\ 2, & G = \mathrm{Spin}\, n,\ n = 5,\ n \geqq 7, \text{ or } G_2 \\ 6, & G = F_4 \text{ or } E_6 \\ 12, & G = E_7 \\ 20, & G = E_8. \end{cases}$$

Proof. $\mathrm{ch}_2\beta(\varphi) = m(\varphi)u$ where $m(\varphi) = \langle \mathrm{ch}_2\beta(\varphi), v\rangle$. But $\langle \mathrm{ch}_2\beta(\varphi), v\rangle = \langle \mathrm{ch}_2\beta(\varphi \circ f), \iota\rangle = l(\varphi)\langle \mathrm{ch}_2 x, \iota\rangle = l(\varphi)$.

Part (b) follows from the computations of Dynkin ([**4**], Table 5).

Similar computations now yield the following.

COROLLARY 3.3. *Let* $g\colon S^3 \to G$ *be any homomorphism. Then in* $\pi_3(G)$, $[g] = j(\tilde{g})[f]$, f *as above.*

Remark. As noted earlier this corollary leads to a proof of the fact that $j(f)$ is always integral. This proof makes no use of 1.4 but does require several deep topological results, in particular 3.1 and the integrality of the Chern character.

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